

Homogenization of Hamilton-Jacobi equations with Neumann and Dirichlet boundary conditions

Kazuo Horie (堀江 和男)
Saitama University (埼玉大学)

Introduction.

We describe some results which we have obtained jointly with Prof. H. Ishii in [HI].

We consider the limiting behavior, as $\varepsilon \rightarrow 0$, of solutions of the following boundary value problems for Hamilton-Jacobi equations,

$$(N)_\varepsilon \quad \begin{cases} H\left(Du^\varepsilon(x), u^\varepsilon(x), x, \frac{x}{\varepsilon}\right) = 0 & \text{in } \Omega_\varepsilon, \\ B\left(Du^\varepsilon(x), u^\varepsilon(x), x, \frac{x}{\varepsilon}\right) = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

and

$$(D)_\varepsilon \quad \begin{cases} H\left(Du^\varepsilon(x), u^\varepsilon(x), x, \frac{x}{\varepsilon}\right) = 0 & \text{in } \Omega_\varepsilon, \\ u^\varepsilon(x) = b\left(x, \frac{x}{\varepsilon}\right) & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \varepsilon\Omega$ and Ω is a periodic domain of $\mathbf{R}^N$. In homogenization theory, one of important issues is the treatment of “domain with small holes” and many mathematicians studied this problem in linear cases ([A] and its references). We want to study the case of Hamilton-Jacobi equations via the viscosity solutions approach.

The study of homogenization based on viscosity solutions was initiated by [LPV] and then developed by [E1] and [E2]. In those papers, they considered equations of the following type

$$(1) \quad u^\varepsilon(x) + |Du^\varepsilon(x)|^2 = V\left(\frac{x}{\varepsilon}\right) \quad \text{in } \mathbf{R}^N,$$

and examined the asymptotics of u^ε as $\varepsilon \rightarrow 0$ (or the corresponding evolution equations).

Thanks to [LPV], [E1] and [E2], we have a rather good comprehension of homogenization of (1) in the case $N = 1$. Assume that $V \in C(\mathbf{R})$, $V(y+1) = V(y)$ and $\min_{\mathbf{R}} V = 0$. Then, according to [LPV], the solution of (1) converges uniformly on $\mathbf{R}$ to the solution of the PDE

$$(2) \quad u(x) + \overline{H}(Du(x)) = 0 \quad \text{in } \mathbf{R}.$$

Here $\overline{H}$ is the function on $\mathbf{R}$ defined by

$$\overline{H}(p) = \begin{cases} 0 & \text{if } |p| \leq \int_0^1 V^{\frac{1}{2}}(y) dy, \\ \lambda(p) \geq 0 \text{ is a solution of } |p| = \int_0^1 (V(y) + \lambda)^{\frac{1}{2}} dy & \text{if } |p| \geq \int_0^1 V^{\frac{1}{2}}(y) dy. \end{cases}$$

In this example $\overline{H}$ is given explicitly, but in general situations it is determined through the “cell problem” (see [LPV], [E1] or [E2]).

Through this paper, we will deal only with viscosity solutions and omit giving here the definitions (for example, see [CIL]).

1. Main results.

We give the list of assumptions of Ω , H , B and b .

($\Omega 1$) Ω is a (connected) domain of $\mathbf{R}^N$.

($\Omega 2$) $\partial\Omega \in C^1$.

($\Omega 3$) $\Omega + e_i = \Omega$ for all $1 \leq i \leq N$, where $\{e_1, \dots, e_N\}$ denote the standard basis of $\mathbf{R}^N$.

(H1) For each $R > 0$,

$$H \in BUC(B(0, R) \times [-R, R] \times \mathbf{R}^N \times \overline{\Omega}),$$

where $B(z, R)$ denotes the closed ball with radius R and center z .

(H2) $H(p, u, x, y + e_i) = H(p, u, x, y)$ for all $1 \leq i \leq N$.

(H3) For some $\gamma > 0$, the function $u \mapsto H(p, u, x, y) - \gamma u$ is nondecreasing.

(H4) For each $u \in \mathbf{R}$,

$$\lim_{r \rightarrow \infty} \inf \{ H(p, u, x, y) \mid |p| > r, x \in \mathbf{R}^N, y \in \overline{\Omega} \} = \infty.$$

(B1) For each $R > 0$,

$$B \in BUC(B(0, R) \times [-R, R] \times \mathbf{R}^N \times \partial\Omega).$$

(B2) $B(p, u, x, y + e_i) = B(p, u, x, y)$ for all $1 \leq i \leq N$.

(B3) The function $u \mapsto B(p, u, x, y)$ is nondecreasing.

(B4) For some $\nu > 0$, the function $t \mapsto B(p + tn(y), u, x, y) - \nu t$ is nondecreasing, where $n(y)$ denotes the unit normal vector at $y \in \partial\Omega$ outward to Ω .

(b1) $b \in BUC(\mathbf{R}^N \times \partial\Omega)$.

(b2) $b(x, y + e_i) = b(x, y)$ for all $1 \leq i \leq N$.

We state an existence result for solutions of $(N)_\varepsilon$ and some of their properties.

Proposition 1. *Assume that $(\Omega 1)$ -($\Omega 3$), $(H1)$ -($H4$) and $(B1)$ -($B4$) hold. Then, $(N)_\varepsilon$ has a unique bounded Lipschitz continuous solution u^ε . It satisfies*

$$(1.1) \quad \sup_{0 < \varepsilon < 1} \|u^\varepsilon\|_{C(\overline{\Omega}_\varepsilon)} < \infty$$

and

$$(1.2) \quad \sup_{0 < \varepsilon < 1} \|Du^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} < \infty.$$

To determine the effective Hamiltonian associated with the problem $(N)_\varepsilon$, we consider the following “cell problem”.

Proposition 2. (*Cell problem*) Assume that $(\Omega 1)$ – $(\Omega 3)$, $(H 1)$ – $(H 4)$ and $(B 1)$ – $(B 4)$ hold. Then, for each $(p, u, x) \in \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N$, there exists a unique number $\lambda_1 \in \mathbf{R}$ such that the problem

$$(CPN) \quad \begin{cases} H(p + D_y v_1(y), u, x, y) = \lambda_1 & \text{in } \Omega, \\ B(p + D_y v_1(y), u, x, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

has a bounded solution $v_1 \in C^{0,1}(\overline{\Omega})$.

We put $\lambda_1 = \tilde{H}(p, u, x)$ and call $\tilde{H}$ the effective Hamiltonian associated with the problem $(N)_\varepsilon$. This Hamiltonian $\tilde{H}$ satisfies the following properties.

Proposition 3. Assume that $(\Omega 1)$ – $(\Omega 3)$, $(H 1)$ – $(H 4)$ and $(B 1)$ – $(B 4)$ hold. Then:
(1) For each $R > 0$,

$$\tilde{H} \in BUC(B(0, R) \times [-R, R] \times \mathbf{R}^N).$$

(2) For some $\gamma > 0$, the function $u \mapsto \tilde{H}(p, u, x, y) - \gamma u$ is nondecreasing.

Theorem 1. The Hamilton-Jacobi equation

$$(1.3) \quad \tilde{H}(Du(x), u(x), x) = 0 \quad \text{in } \mathbf{R}^N,$$

has a unique solution $u \in BUC(\mathbf{R}^N)$ and

$$(1.4) \quad \lim_{\varepsilon \searrow 0} \sup_{x \in \Omega_\varepsilon} |u^\varepsilon(x) - u(x)| = 0.$$

Proposition 4. Assume that $(\Omega 1)$ – $(\Omega 3)$, $(H 1)$ – $(H 4)$ and $(B 1)$ – $(B 4)$ hold. Then, $(D)_\varepsilon$ has a unique bounded Lipschitz continuous solution u^ε . It satisfies

$$(1.5) \quad \sup_{0 < \varepsilon < 1} \|u^\varepsilon\|_{C(\overline{\Omega}_\varepsilon)} < \infty$$

and

$$(1.6) \quad \sup_{0 < \varepsilon < 1} \|Du^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} < \infty.$$

Proposition 5. (*Cell problem*) Assume that $(\Omega 1)$ – $(\Omega 3)$, $(H 1)$ – $(H 4)$ and $(B 1)$ – $(B 4)$ hold. Then, for each $(p, u, x) \in \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N$, there exists a unique number $\lambda_2 \in \mathbf{R}$ such that the problem

$$(CPD) \quad \begin{cases} H(p + D_y v_2(y), u, x, y) \leq \lambda_2 & \text{in } \Omega, \\ H(p + D_y v_2(y), u, x, y) \geq \lambda_2 & \text{in } \overline{\Omega}, \end{cases}$$

has a bounded solution $v \in C^{0,1}(\overline{\Omega})$.

The problem (CPD) is of the *state-constraint* type. Problems of this type naturally arises in optimal control, where the dynamic programming equations have convex Hamiltonians H in the first variable p . Here, the interesting point is that the function H is not assumed to be convex in p .

We put $\lambda_2 = \overline{H}(p, u, x)$ and call $\overline{H}$ the effective Hamiltonian associated with the problem $(D)_\varepsilon$. The effective Hamiltonian $\overline{H}$ has the following properties.

Proposition 6. Assume that $(\Omega 1)$ – $(\Omega 3)$, $(H 1)$ – $(H 4)$ and $(B 1)$ – $(B 4)$ hold. Then:

(1) For each $R > 0$,

$$\overline{H} \in BUC(B(0, R) \times [-R, R] \times \mathbf{R}^N).$$

(2) For some $\gamma > 0$, the function $u \mapsto \overline{H}(p, u, x, y) - \gamma u$ is nondecreasing.

Theorem 2. *The Hamilton-Jacobi equation*

$$(1.7) \quad \max\{u(x) - \bar{b}(x), \bar{H}(Du(x), u(x), x)\} = 0 \quad \text{in } \mathbf{R}^N,$$

where $\bar{b}(x) = \min_{y \in \partial\Omega} b(x, y)$, has a unique solution $u \in BUC(\mathbf{R}^N)$ and

$$(1.8) \quad \lim_{\varepsilon \searrow 0} \sup_{x \in \Omega_\varepsilon} |u^\varepsilon(x) - u(x)| = 0.$$

2. Proof of main results.

We only sketch the proof in the case of the Dirichlet problem $(D)_\varepsilon$.

Proof of Proposition 4. Note that, by (H4), a bounded subsolution of $(D)_\varepsilon$ is Lipschitz continuous. Moreover, if (1.5) holds, then solutions satisfy (1.6). Noting that

$$u_1(x) = -A_1 \quad \text{and} \quad u_2(x) = A_1,$$

where $A_1 > 0$ is large enough are, respectively, a subsolution and a supersolution of $(D)_\varepsilon$. Then, using Perron's method and standard comparison arguments, we see that $(D)_\varepsilon$ has a unique bounded Lipschitz solution u^ε . Moreover, noting that the constant A_1 can be chosen independently of $\varepsilon > 0$, we conclude (1.5) and (1.6). ■

Outline of proof of Proposition 5. For $0 < \alpha < 1$, we consider the following approximate problem

$$(CP)_\alpha \quad \begin{cases} \alpha w^\alpha(y) + H(p + D_y w^\alpha(y), u, x, y) \leq 0 & \text{in } \Omega, \\ \alpha w^\alpha(y) + H(p + D_y w^\alpha(y), u, x, y) \geq 0 & \text{in } \bar{\Omega}. \end{cases}$$

Since

$$w_1(y) = -\frac{A_2}{\alpha} \quad \text{and} \quad w_2(y) = \frac{A_2}{\alpha}$$

are, respectively, a subsolution and a supersolution of $(CP)_\alpha$ if the constant A_2 is large enough, we get a unique Lipschitz solution w^α of $(CP)_\alpha$ by Perron's method for each $0 < \alpha < 1$. Moreover, by (H2), we have $w^\alpha(y + e_i) = w^\alpha(y)$ for all $1 \leq i \leq N$.

It follows from the construction of the solution that

$$\sup_{\alpha} \|\alpha w^{\alpha}\|_{C(\bar{\Omega})} < \infty.$$

By using this inequality, we obtain

$$\sup_{\alpha} \|Dw^{\alpha}\|_{L^{\infty}(\Omega)} < \infty.$$

We put $v^{\alpha}(y) = w^{\alpha}(y) - \min w^{\alpha}$. Then we have

$$\sup_{\alpha} \|v^{\alpha}\|_{C^{0,1}(\bar{\Omega})} < \infty.$$

Therefore,

$$v^{\alpha} \rightarrow v_2 \quad \text{and} \quad \alpha w^{\alpha} \rightarrow -\lambda_2 \quad \text{uniformly}$$

along a sequence as $\alpha \rightarrow 0$, for some $v \in C^{0,1}(\bar{\Omega})$ and $\lambda_2 \in \mathbf{R}$. This way we get a solution (v_2, λ_2) . We omit giving the proof of the uniqueness of λ (see [E2]). ■

We omit giving the proof of Proposition 6 (see [I4] or [HI]). Next, we will prove Theorem 2, where we use both the perturbed test function method (see [E1] and [E2]) and the test function used in the proof of comparison results (see [I2]).

Proof of Theorem 2. We put

$$\bar{u}(x) = \lim_{\varepsilon \rightarrow 0} \sup \{u^{\delta}(y) \mid |x - y| \leq \varepsilon, 0 < \delta < \varepsilon\}$$

and

$$\underline{u}(x) = \lim_{\varepsilon \rightarrow 0} \inf \{u^{\delta}(y) \mid |x - y| \leq \varepsilon, 0 < \delta < \varepsilon\}$$

for $x \in \mathbf{R}^N$. We will show that $\bar{u}$ and $\underline{u}$ are, respectively, a subsolution and a supersolution of (1.7).

Let $\varphi \in C^{1,1}(\mathbf{R}^N)$ and $\hat{x}$ be a maximum point of $\bar{u} - \varphi$. We may assume that $\lim_{|x| \rightarrow \infty} \varphi(x) = \infty$ and that $\bar{u} - \varphi$ attains a strict maximum at $\hat{x} \in \mathbf{R}^N$. Let $v \in C^{0,1}(\bar{\Omega})$ be a solution of Proposition 5 with $\lambda_2 = \overline{H}(D\varphi(\hat{x}), \bar{u}(\hat{x}), \hat{x})$. Then, we can find maximum

points $x^\varepsilon \in \bar{\Omega}_\varepsilon$ of $u^\varepsilon(x) - \varphi(x) - \varepsilon v\left(\frac{x}{\varepsilon}\right)$ satisfying $x^\varepsilon \rightarrow \hat{x}$ as $\varepsilon \rightarrow 0$. We are concerned with the case $x^\varepsilon \in \partial\Omega_\varepsilon$; the other case can be argued similarly and more easily.

By $(\Omega 2)$, there exist $\eta = \eta(x^\varepsilon) \in \mathbf{R}^N$ and $b > 0$ such that $B(x^\varepsilon + t\eta, tb) \subset \bar{\Omega}_\varepsilon$ for all $0 \leq t < b$. For $\alpha > 0$, we put

$$\Phi(x, y) = u^\varepsilon(x) - \varphi(x) - \varepsilon v\left(\frac{y}{\varepsilon}\right) - \left| \frac{x - y}{\alpha} - \eta \right|^2 - |y - x^\varepsilon|^2$$

on $\bar{\Omega}_\varepsilon \times \bar{\Omega}_\varepsilon$. Let $(x_\alpha^\varepsilon, y_\alpha^\varepsilon) \in \bar{\Omega}_\varepsilon \times \bar{\Omega}_\varepsilon$ be a maximum point of Φ . Then $x_\alpha^\varepsilon, y_\alpha^\varepsilon \rightarrow x^\varepsilon$ as $\alpha \rightarrow 0$. Since $\Phi(y_\alpha^\varepsilon + \alpha\eta, y_\alpha^\varepsilon) \leq \Phi(x_\alpha^\varepsilon, y_\alpha^\varepsilon)$, we have

$$\left| \frac{x_\alpha^\varepsilon - y_\alpha^\varepsilon}{\alpha} - \eta \right| \leq C\alpha$$

for some $C > 0$ independent of $\varepsilon > 0$. Moreover, we may assume that $x_\alpha^\varepsilon \in \Omega_\varepsilon$.

Since u^ε is a solution of $(D)_\varepsilon$, we obtain

$$H\left(D\varphi(x_\alpha^\varepsilon) + \frac{2}{\alpha}\left(\frac{x_\alpha^\varepsilon - y_\alpha^\varepsilon}{\alpha} - \eta\right), u^\varepsilon(x_\alpha^\varepsilon), x_\alpha^\varepsilon, \frac{x_\alpha^\varepsilon}{\varepsilon}\right) \leq 0$$

and

$$H\left(D\varphi(\hat{x}) + \frac{2}{\alpha}\left(\frac{x_\alpha^\varepsilon - y_\alpha^\varepsilon}{\alpha} - \eta\right) - 2(y_\alpha^\varepsilon - x^\varepsilon), \bar{u}(\hat{x}), \hat{x}, \frac{y_\alpha^\varepsilon}{\varepsilon}\right) \geq \bar{H}(D\varphi(\hat{x}), \bar{u}(\hat{x}), \hat{x}).$$

Sending $\alpha \rightarrow 0$ first and $\varepsilon \rightarrow 0$, we get

$$\bar{H}(D\varphi(\hat{x}), \bar{u}(\hat{x}), \hat{x}) \leq 0.$$

Now, we show that $\bar{u}(x) \leq \bar{b}(x)$. If there exists $\tilde{x} \in \mathbf{R}^N$ such that $\bar{u}(\tilde{x}) > \bar{b}(\tilde{x})$, then we can show that there exist $\varepsilon > 0$ and $\tilde{x}_\varepsilon \in \partial\Omega_\varepsilon$ such that $u^\varepsilon(\tilde{x}_\varepsilon) > b(\tilde{x}_\varepsilon, \frac{\tilde{x}_\varepsilon}{\varepsilon})$. Let $r > 0$, $A > 0$ and x_A be a maximum point of $u^\varepsilon(x) - A|x - \tilde{x}_\varepsilon - rn(\tilde{x}_\varepsilon)|$. Since $x_A \rightarrow \tilde{x}_\varepsilon$ as $r = \frac{1}{A}$ and $A \rightarrow \infty$ and

$$H\left(A \frac{x_A - \tilde{x}_\varepsilon - rn(\tilde{x}_\varepsilon)}{|x_A - \tilde{x}_\varepsilon - rn(\tilde{x}_\varepsilon)|}, u^\varepsilon(x_A), x_A, \frac{x_A}{\varepsilon}\right) > 0$$

for $A > 0$ large enough by (H4), we have $x_A \in \partial\Omega_\varepsilon$ and $u^\varepsilon(x_A) \leq b(x_A, \frac{x_A}{\varepsilon})$. Therefore, sending $A \rightarrow \infty$, we get a contradiction. Thus we have proved that $\bar{u}$ is a subsolution of (1.7).

Similarly, we can prove that $\underline{u}$ is a supersolution of (1.7). By comparison arguments, we have $\bar{u} = \underline{u}$ and conclude the proof. ■

References

- [A] H. Attouch, *Variational Convergence for Functions and Operators* (New York: Pitman, 1984).
- [BL] G. Barles and P.-L. Lions, Fully nonlinear Neumann type boundary conditions for first-order Hamilton-Jacobi equations, *Nonlinear Anal. Theory Methods Appl.* **16** (1991), 143–153.
- [BP] G. Barles and B. Perthame, Discontinuous solutions of deterministic optimal stopping- time problems, *Modél. Math. Anal. Num.* **21** (1987), 557–579.
- [BLP] A. Bensoussan, J. L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures* (Amsterdam: Noth Holland, 1978).
- [C1] M. C. Concordel, Periodic homogenization of Hamilton-Jacobi equations: 1. Additive eigenvalues and variational formula, preprint.
- [C2] M. C. Concordel, Periodic homogenization of Hamilton-Jacobi equations: 2. Eikonal equations, preprint.
- [CL] M. G. Crandall, and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.*, **277** (1983), 1–42.
- [CIL] M. G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.* **27** (1992), 1–67.
- [DI] P. Dupuis and H. Ishii, On oblique derivative problems for fully nonlinear second order elliptic equations on nonsmooth domains, *Nonlinear Anal. Theory Methods Appl.* **12** (1991), 1123–1138.

- [E1] L. C. Evans, The perturbed test function technique for viscosity solutions of partial differential equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **111** (1989), 359–375.
- [E2] L. C. Evans, Periodic homogenisation of fully nonlinear partial differential equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **120** (1992), 245–265.
- [HI] K. Horie and H. Ishii, Homogenization of Hamilton-Jacobi equations on domains with small scale periodic structure, preprint.
- [I1] H. Ishii, Perron's method for Hamilton-Jacobi equations, *Duke Math. J.*, **55** (1987), 369–384.
- [I2] H. Ishii, A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations, *Ann. Scuola Norm. Sup. Pisa* **16** (1989), 105–135.
- [I3] H. Ishii, Fully nonlinear oblique derivative problems for nonlinear second-order elliptic PDE's, *Duke Math. J.*, **62** (1991), 633–661.
- [I4] 石井仁司, 粘性解とその応用, *数学*, **46** (1995), 97–110.
- [L] P.-L. Lions, Neumann type boundary conditions for Hamilton-Jacobi equations, *Duke Math. J.*, **52** (1985), 793–820.
- [LVP] P.-L. Lions, G. Papanicolaou and S. R. S. Varadhan, Homogenization of Hamilton-Jacobi equations, Preprint.
- [S] H. M. Soner, Optimal control with state-space constraint I, *SIAM J. Control Optim.* **24** (1986), 552–562.