

ASYMPTOTIC CYCLES ON TWO-DIMENSIONAL MANIFOLDS

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INTRODUCTION

In 1957, S.Schwartzman introduced the concept of asymptotic cycles. This concept represents how the trajectory of a flow rounds around the phase space in the homological meaning. Let us recall the definition.

Let M be a closed C^∞ Riemannian manifold and ψ_t a C^1 -flow on it. Choose p a point of M and consider a one cycle $\hat{C}_{T,p} = C_{T,p} + C'_{T,p}$, where $C_{T,p}$ denotes the trajectory from p to $\psi_T(p)$ and $C'_{T,p}$ a minimal geodesic from $\psi_T(p)$ to p .

DEFINITION The asymptotic cycle of p , denoted by A_p , is defined by

$$A_p = \lim_{T \rightarrow \infty} \frac{1}{T} [\hat{C}_{T,p}]$$

when the limit exists. (Here $[]$ denotes the homology class.)

It is easy to check that A_p is invariant under the flow ψ_t and is independent of the choice of Riemannian metrics.

Here we study the relations between asymptotic cycles and

the behaviour of trajectories on closed orientable two-manifolds. In Section 1 we give fundamental notations and statements of results, in Section 2 we describe the outline of the proof of Theorem 2 and Section 3 is a note for Theorem 3 and the remaining problem.

1. NOTATIONS AND STATEMENTS OF RESULTS

Throughout this paper, we suppose M is a closed orientable two-manifold and ϕ_t is a C^1 flow on M . And for simplicity, we assume that ϕ_t has only a finite number of equilibrium points.

If p is a point of M , $L_+(p)$ denotes the positive semi-trajectory departing at p and $\omega(p)$ the ω -limit set of p .

We call $L_+(p)$ exceptional and $\overline{L_+(p)}$ an exceptional domain if $L_+(p)$ is contained in $\omega(p)$, nowhere-dense, and neither an equilibrium point nor a periodic trajectory.

A subset C of M is called a circuit if C is a unicursal diagram consisting of equilibrium points and trajectories connecting them.

THEOREM 1 One and only one of the following eight cases occurs.

- (1) $L_+(p)$ is an equilibrium point.
- (2) $L_+(p)$ approaches one equilibrium point.
- (3) $L_+(p)$ winds around a circuit from one side.
- (4) $L_+(p)$ is a periodic trajectory.
- (5) $L_+(p)$ winds around a periodic trajectory from one side.

- (6) $L_+(p)$ is locally dense (namely, $\overline{L_+(p)}$ contains a non-empty open set).
- (7) $L_+(p)$ is exceptional.
- (8) $L_+(p)$ approaches one exceptional domain.

To state Theorem 2 and Theorem 3 , we need the following definition.

DEFINITION

- (i) $\alpha \in H_1(M; \mathbb{R})$ is rational if $\alpha \neq 0$ and there exist $k \in \mathbb{R}$ and $\alpha' \in H_1(M; \mathbb{Z})$ such that $\alpha = k\alpha'$.
- (ii) $\alpha \in H_1(M; \mathbb{R})$ is irrational if α is neither 0 nor rational.

THEOREM 2 Suppose A_p exists and is rational, then $L_+(p)$ is either of type (4) or of type (5).

THEOREM 3 If M is a two-dimensional torus, then A_p exists for all $p \in M$, and

- if $L_+(p)$ is of type (1), (2) or (3), then A_p is 0.
 - if $L_+(p)$ is of type (4) or (5), then A_p is rational or 0.
 - if $L_+(p)$ is of type (6), (7) or (8), then A_p is irrational or 0.
- Moreover $\omega(p_1) = \omega(p_2)$ implies $A_{p_1} = A_{p_2}$.

2. ASYMPTOTIC CYCLES OF SEMI-TRAJECTORIES OF TYPE (6)

It is immediate that the asymptotic cycle A_p is zero for a semi-trajectory $L_+(p)$ of type (1), (2) or (3). For $L_+(p)$ of

type (4) or (5), A_p is given by $A_p = \frac{1}{\tau}[C]$, hence is rational or zero. (Here $[C]$ denotes the homology class of the periodic trajectory and τ is its minimal period.)

The rest of this section is devoted to show that asymptotic cycles of semi-trajectories of type (6) are not rational. If $L_+(p)$ is of type (7) or (8), we can perform a similar computation and obtain that A_p is also never rational. These results and the previous observation imply Theorem 2.

Let p be a point of M with $L_+(p)$ locally dense, then by the orientability of M , we can construct a simple closed curve C which is transverse to the flow ψ_t and is contained in $\overline{L_+(p)}$. Consider the Poincaré map $\mathcal{P}$ of this flow with respect to C . $\mathcal{P}$ is defined on $L_+(p) \cap C$, a dense subset of C , hence $\mathcal{P}$ may not be defined at a point x only if x is of type (2), and the finiteness assumption for equilibrium points implies that the cardinal number of such points is at most finite.

Now we define P-transformations.

DEFINITION $\mathcal{P}$ is called a P-transformation, if there exist distinct k points $p_1, \dots, p_k$ and distinct k points $q_1, \dots, q_k$ in S^1 , such that $\mathcal{P}$ is an orientation-preserving homeomorphism from $S^1 \setminus \{p_1, \dots, p_k\}$ to $S^1 \setminus \{q_1, \dots, q_k\}$.

For a P-transformation $\mathcal{P}$, $\mathcal{P}_R$ (resp. $\mathcal{P}_L$) denotes a right (resp. left) continuous extension of $\mathcal{P}$, and

$\bigcup_{n \in \mathbb{Z}} \mathcal{G}_R^n(\{p_1, \dots, p_k\})$ is denoted by $S(\mathcal{G})$. We call a point of $S(\mathcal{G})$ singular and a point of $S^1 \setminus S(\mathcal{G})$ regular.

LEMMA Let $\mathcal{G} : S^1 \rightarrow S^1$ be a P-transformation with a regular point x_0 satisfying $\overline{\{\mathcal{G}^n(x_0)\}_{n \geq 0}} = S^1$. Then the only closed invariant subsets under $\mathcal{G}_R$ or $\mathcal{G}_L$ are whole S^1 and the empty set.

PROPOSITION Let $\mathcal{G} : S^1 \rightarrow S^1$ be a P-transformation, then $\mathcal{G}_R$ or $\mathcal{G}_L$ has a non-trivial invariant measure on S^1 .

COROLLARY Let $\mathcal{G} : S^1 \rightarrow S^1$ be a P-transformation with a regular point x_0 satisfying $\overline{\{\mathcal{G}^n(x_0)\}_{n \geq 0}} = S^1$. Then every $\mathcal{G}_R$ (or $\mathcal{G}_L$) invariant measure μ satisfies the condition that $\text{supp. } \mu = S^1$ and $\mu(S(\mathcal{G})) = 0$. And for every regular point x , any cluster point of the sequence $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{\mathcal{G}^k(x)}$ gives a $\mathcal{G}_R$ (hence also $\mathcal{G}_L$ and $\mathcal{G}$) invariant measure. (Where δ denotes the Dirac measure.)

What we must prove is that if the asymptotic cycle exists for a semi-trajectory of type (6), then it is not rational.

For a semi-trajectory $L_+(p)$ of type (6), take a transeverse curve C_0 as before and the P-transformation $\mathcal{G}$ induced by the Poincaré map with respect to C_0 . Without loss of generality, we can assume p is contained in C_0 .

Let τ denote the first return time with respect to C_0 .

$$\tau(x) = \inf. \{ t > 0 : \varphi_t(x) \in C_0 \}$$

Then the n -th return time of p is given by

$$T(n) = \sum_{k=0}^{n-1} \tau(\varphi^k(p)) .$$

It is enough to show that every cluster point of the following sequence is irrational or zero :

$$\frac{1}{T(n)} [\hat{C}_{T(n),p}] .$$

Assume the contrary, then there exists a sequence n_i such that

$$\alpha = \lim_{i \rightarrow \infty} \frac{1}{T(n_i)} [\hat{C}_{T(n_i),p}]$$

is rational.

By the previous corollary, we can suppose that the following sequence converges to a φ -invariant measure, taking a subsequences if necessary.

$$\mu = \lim_{i \rightarrow \infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} \delta_{\varphi^k(p)}$$

Using this invariant measure, we will be led to a contradiction.

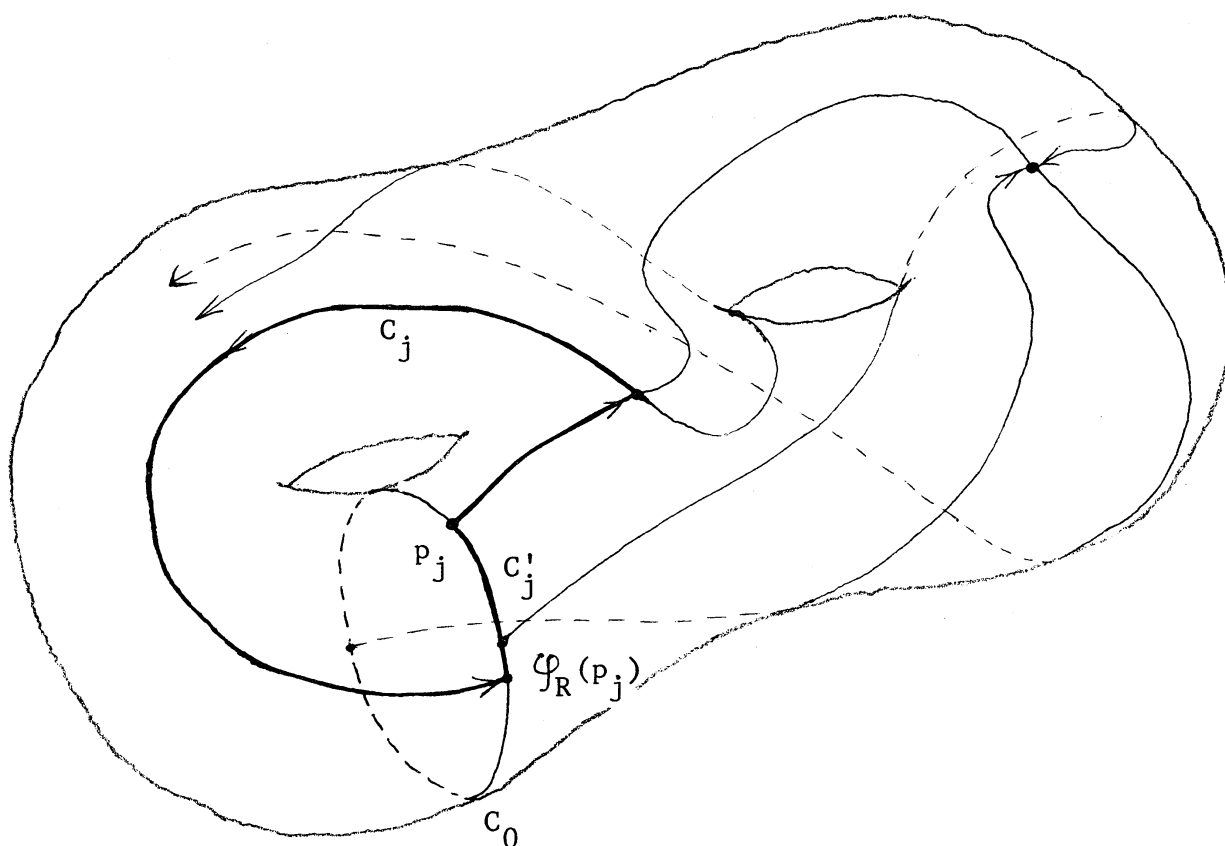
Let γ_0 be the homology class represented by C_0 , then the intersection number of α and γ_0 is given as follows:

$$\begin{aligned} \alpha \circ \gamma_0 &= \lim_{i \rightarrow \infty} \frac{1}{T(n_i)} [\hat{C}_{T(n_i),p}] \circ [C_0] \\ &= \lim_{i \rightarrow \infty} \frac{1}{T(n_i)} n_i \end{aligned}$$

$$= \left(\int_{C_0} \tau(x) d\mu(x) \right)^{-1}$$

As in the definition of P-transformations, let $p_1, \dots, p_k$ be points in C_0 where φ is not defined and the ordering is compatible to the orientation of C_0 .

Consider the integral homology class $\gamma_j = [\hat{C}_j] = [C_j + C'_j]$, where C_j denotes the 'trajectory' from p_j to $\varphi_R(p_j)$ (more precisely, C_j is the limit of the segment of the trajectory from x to $\varphi(x)$ as x approaches p_j from the right side) and C'_j is the segment from $\varphi_R(p_j)$ to p_j in C_0 .



The intersection number of α and γ_j is given by

$$\alpha \circ \gamma_j = \lim_{i \rightarrow \infty} \frac{1}{T(n_i)} \sum_{k=0}^{n_i-1} \chi_{(\varphi_R(p_j), p_j)}(\varphi^k(p))$$

where $\chi_{(\varphi_R(p_j), p_j)}$ denotes the characteristic function of the open interval $(\varphi_R(p_j), p_j)$.

So, we obtain the following equation.

$$\alpha \circ \gamma_j = \left(\int_{C_0} \tau d\mu \right)^{-1} \cdot \mu((\varphi_R(p_j), p_j))$$

If we put $a_j = \mu((\varphi_R(p_j), p_j))$, then this equation becomes

$$a_j = \frac{\alpha \circ \gamma_j}{\alpha \circ \gamma_0}.$$

Hence, by the assumption that α is rational, a_j is a rational number for all j .

Let us introduce the coordinate in C_0 by measure μ . Since $\varphi|_{(p_j, p_{j+1})}$ is continuous and preserves μ , it follows that $\varphi|_{(p_j, p_{j+1})}(x) = x - a_j$. But all a_j 's are rational numbers, this contradicts the fact that φ has a regular point with a dense orbit. Thus we obtain the desired result.

3. CONVERGENCES OF ASYMPTOTIC CYCLES

In the case of a two-dimensional torus, every P-transformation induced by a semi-trajectory of type (6) has a continuous extension on S^1 . Then Theorem 3 is obtained from the fact that every

homeomorphism of S^1 with a dense orbit is uniquely ergodic.

We expect that the result of Theorem 3 holds also for a surface of higher genus. This is essentially reduced to the next problem.

PROBLEM Is every P-transformation with a dense positive-orbit uniquely ergodic ?

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