

Non-equilibrium dynamics in quantum many body systems and their gravity duals

:Eigenstate thermalization hypothesis and open quantum
systems

Taishi Kawamoto

January 13, 2026

Kyoto University

Yukawa Institute for Theoretical Physics

Center for Gravitational Physics and Quantum Information

Doctoral Thesis

Supervisor: Tadashi Takayanagi

Abstract

This thesis investigates important connection between two longstanding problems in theoretical physics: the information loss paradox in quantum gravity and the thermalization phenomena in quantum many-body systems. Although these problems originate from entirely different contexts, the AdS/CFT correspondence which is theoretical duality between chaotic quantum many body systems and gravity reveals that they are, in fact, two manifestations of the same underlying phenomenon.

Black holes exhibit thermodynamic behavior, and semiclassical gravity predicts information loss during Hawking evaporation, apparently contradicting unitarity. In the AdS/CFT framework, however, gravitational dynamics including black hole formation is mapped to the unitary time evolution of strongly coupled quantum many-body systems. This correspondence provides a precise reformulation of Maldacena's information loss problem: the late-time behavior of thermal correlation functions predicted by semiclassical gravity fails to match the exact unitary evolution of the dual quantum system.

A key ingredient in resolving this discrepancy is the eigenstate thermalization hypothesis (ETH), which characterizes thermalization in chaotic many-body systems. By studying the large-deviation technique for thermal correlation functions, we give a possible way of probing ETH for single non-integrable Hamiltonian with some non-trivial assumptions. In holographic systems, ETH captures the non-perturbative physics responsible for wormhole contributions in the gravitational path integral, thereby providing a quantitative and conceptual resolution of Maldacena's version of the information loss paradox.

Moreover the thesis further develops a holographic framework for open quantum systems, motivated by Hawking's original information loss problem in asymptotically flat spacetime. We are introducing non-Hermitian deformations such as imaginary Janus interfaces and brane-based constructions. We identify geometric counterparts of dissipative dynamics and measurement-induced like processes.

Publication List

This thesis is mainly based on the following papers

- [1] T. Kawamoto, *A strategy for proving the strong eigenstate thermalization hypothesis: chaotic systems and holography*, *JHEP* **01** (2025) 095 [2411.09746]
- [2] T. Kawamoto, R. Maeda, N. Nakamura and T. Takayanagi, *Holographic entanglement, pseudo-entropy and wormholes*, *International Journal of Modern Physics A* **0** (2025) 2548005 [<https://doi.org/10.1142/S0217751X25480057>]
- [3] T. Kawamoto, R. Maeda, N. Nakamura and T. Takayanagi, *Traversable AdS wormhole via non-local double trace or Janus deformation*, *JHEP* **04** (2025) 086 [2502.03531]
- [4] T. Kawamoto, S.-M. Ruan and T. Takayanagi, *Gluing AdS/CFT*, *JHEP* **07** (2023) 080 [2303.01247]
- [5] I. Akal, T. Kawamoto, S.-M. Ruan, T. Takayanagi and Z. Wei, *Page curve under final state projection*, *Phys. Rev. D* **105** (2022) 126026 [2112.08433]

and we also refer following papers

- [6] T. Kawamoto and Y.-k. Suzuki, *Entanglement entropy via double-cone regularization*, *Phys. Rev. D* **110** (2024) 046011 [2401.00219]
- [7] H. Kanda, T. Kawamoto, Y.-k. Suzuki, T. Takayanagi, K. Tasuki and Z. Wei, *Entanglement phase transition in holographic pseudo entropy*, *JHEP* **03** (2024) 060 [2311.13201]
- [8] T. Kawamoto, S.-M. Ruan, Y.-k. Suzuki and T. Takayanagi, *A half de Sitter holography*, *JHEP* **10** (2023) 137 [2306.07575]
- [9] P.-X. Hao, T. Kawamoto, S.-M. Ruan and T. Takayanagi, *Non-extremal island in de Sitter gravity*, *JHEP* **03** (2025) 004 [2407.21617]
- [10] Y. Fukusumi and T. Kawamoto, *Generalizing fusion rules by shuffle: Symmetry-based classifications of nonlocal systems constructed from similarity transformations*, 2512.02139
- [11] Y. Fukusumi and T. Kawamoto, *Generalizing quantum dimensions: Symmetry-based classification of local pseudo-Hermitian systems and the corresponding domain walls*, 2511.11059

Even though, we do not include in this thesis, we published following papers. These papers about non-equilibrium dynamics of boundary CFTs and their holographic duals

- [12] T. Kawamoto, T. Mori, Y.-k. Suzuki, T. Takayanagi and T. Ugajin, *Holographic local operator quenches in BCFTs*, *JHEP* **05** (2022) 060 [2203.03851]
- [13] I. Akal, T. Kawamoto, S.-M. Ruan, T. Takayanagi and Z. Wei, *Zoo of holographic moving mirrors*, *JHEP* **08** (2022) 296 [2205.02663]

Contents

1	Introduction	8
I	Blackhole, Information loss and holography	15
2	Black Holes and the Information Loss Problem	16
2.1	Black Hole Thermodynamics	16
2.2	Thermodynamic Entropy of Black Holes from Euclidean path integrals	21
2.3	Hawking Radiation	24
2.4	Holographic correspondence	26
2.5	Black-Hole Evaporation and the Information Loss Problem	27
3	AdS/CFT correspondence	28
3.1	AdS/CFT correspondence: Maldacena’s discussion	28
3.2	GKPW relation and holographic renormalization	32
3.3	Holographic entanglement entropy	39
3.4	SYK/JT correspondence	46
4	Deformations of AdS/CFT correspondence and wormholes	61
4.1	Double trace deformation	61
4.2	$T\bar{T}$ deformations and its generalization	65
4.3	AdS/BCFT and AdS/ICFT	80
4.4	Braneworld holography	86
II	Eigenstate Thermalization Hypothesis and Gravity	88
5	Thermalization and Equilibration in closed quantum many body systems	89
5.1	Typicality Theorems	90
5.2	Equilibration and Thermalization	96
5.3	Initial State Independent Upper Bound: Strong ETH	99
5.4	Initial State Dependent Upper Bound	101
5.5	MATE and MITE	107
5.6	ETH from Random Hamiltonian	125
5.7	Free probability theory and higher order generalization of ETH	128
5.8	Late time behavior of the spectral form factor and correlation functions	143

6	A Sufficient Condition for the Double Exponential Scaling	159
6.1	Concentration Inequality and Sub-Gaussian Properties	160
6.2	Almost Sub-Gaussianity of $\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$ and $\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}]$	168
7	Non-equilibrium evolution and information problems in AdS/CFT correspondence	177
7.1	Thermal correlators and information problems in AdS/CFT	177
7.2	Quantum quench in CFTs and its holographic duals	192
8	ETH in Holographic Theories	197
8.1	Bulk free theories	198
8.2	Adding $1/N$ Corrections	201
III	Open quantum systems and Gravity	209
9	Open Quantum Systems and Their Importance in Holography	210
9.1	Double Holography and Dissipation	210
9.2	Open Quantum Systems	214
10	Non-hermitian holography and wormholes	216
10.1	Traversable wormholes from Janus solutions	216
10.2	Computing geodesic length	216
10.3	Thermodynamic entropy and pseudo entropy	220
11	Gluing two AdS	222
11.1	Gluing AdS Spacetimes	223
11.2	Stress Tensor and $T\bar{T}$ Structure	224
11.3	Chiral Gluing Solutions	225
11.4	Non-Chiral Solutions (Perturbative)	226
11.5	Application: Half-dS Holography	227
12	Measurement and holographic BCFT and ICFT	228
12.1	Final state projection and its relation to Page curve	228
12.2	Conformal Defect and measurement like behavior	237
12.3	Measurement-induced phase transition-like behavior in holographic BCFT and its distinction from actual measurement	264
IV	Summary	267
13	Future problems and Conclusion	267

A	AdS spacetime, geodesic and its length	270
A.1	Geodesic in AdS spacetime via embedding method	270
A.2	Geodesic length of AdS spacetimes	273
B	Proof of Canonical typicality	275
C	Case without non-Resonant Condition	278
C.1	Proof of Theorem 17	279
D	Numerical evidence for the upper bound (6.32)	281
E	Numerical evidence of the inequality (8.23)	282

1 Introduction

There are two major and distinct problems in theoretical physics:

1. One concerns quantum gravity. We do not know the correct procedure for computing the quantum gravitational path integral. In particular, quantum effects of gravity are important inside black holes. We do not know the fate of particles that fall into a black hole, even though information should be preserved in physical dynamics.
2. The other problem concerns non-equilibrium physics. We expect that any sufficiently complicated initial state evolves toward thermal equilibrium in finite time. However, once we try to deduce this macroscopic behavior from the microscopic dynamics of (quantum) many-body systems, we immediately encounter a naive paradox [14]. The macroscopic world has a time arrow, whereas microscopic dynamics does not.

What we show in this thesis is that, under certain circumstances, these two problems are equivalent. The key notion is the AdS/CFT correspondence. We now explain this equivalence in more detail.

■ Black hole thermodynamics

Bekenstein and Hawking found that classical black holes obey the laws of thermodynamics [15, 16]. In particular, a static black hole represents a thermodynamic equilibrium state. A naive way to understand this is as follows. A black hole solution is characterized only by a few parameters such as mass, angular momentum, and electric charge. Suppose we throw some objects into a black hole; the black hole gains mass. The difference between the two black holes is then only the mass and is independent of the detailed information about the objects we threw in. Naively, this is because the objects cannot escape from the black hole, and we cannot retrieve their fine-grained information. This situation is completely analogous to thermodynamics: an equilibrium state is characterized by a few parameters such as energy and charge, and for a given equilibrium state, we cannot know how it was prepared [17].

■ Holographic correspondence

The surprising point is that this analogy is not merely superficial. Black hole thermodynamics is in fact the thermodynamics of some quantum many-body system. This observation stems from the entropy of the black hole. Bekenstein and Hawking found that the entropy of a black hole is proportional to the area of the horizon:

$$S_{\text{BH}} = \frac{\text{Area}[\text{horizon}]}{4G_N}. \quad (1.1)$$

AdS/CFT : Gravitational Collapse = Thermalization

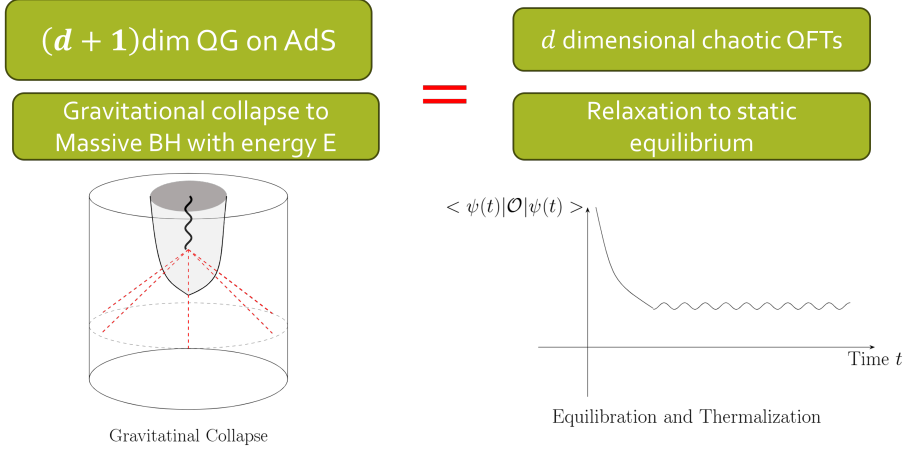


Figure 1: AdS/CFT relates gravitational collapse of a massive star into a black hole to the quantum many-body dynamics evolving from a non-equilibrium state to an equilibrium state, for example by a quantum quench.

In usual quantum many-body systems, entropy is an extensive quantity, i.e. it is proportional to the volume rather than the area of the boundary. Thus the black hole looks exotic. Susskind and 't Hooft argued that a black hole should be equivalent to a thermal system whose extensive entropy lives in one lower dimension [18, 19]. This idea is called the holographic correspondence.

■ AdS/CFT correspondence

Following this idea, Maldacena conjectured that this equivalence can be realized concretely [20]. He proposed that a $(d + 1)$ -dimensional quantum gravitational theory on AdS spacetime is equivalent to a d -dimensional strongly coupled quantum many-body system without gravity. This equivalence or duality is called the AdS/CFT correspondence. Although this duality is still a conjecture, there is abundant evidence [21]. By using this duality, we can understand the dynamics of black holes through the dynamics of quantum many-body systems, and vice versa.

In particular, black hole thermodynamics can be extended to non-equilibrium regimes. Gravitational dynamics is mapped to the non-equilibrium dynamics of quantum many-body systems. For example, gravitational collapse of a star corresponds to quantum thermalization from a non-equilibrium initial state [22]. See Figure 1.

■ Information loss problems

Using the AdS/CFT correspondence, we can tackle one of the most important problems in quantum gravity: the information loss problem. There are two versions of this problem:

1. **Hawking’s information loss problem [16]:** Suppose we have a massive star represented by some pure state; see Figure 2. Importantly, a black hole is not completely black: it emits thermal Hawking radiation and evaporates [23]. In asymptotically flat spacetime, this evaporation continues for a long time, and the black hole may eventually turn entirely into a collection of thermal radiation. The problem is that this thermal radiation is characterized only by a few parameters such as its temperature. The initial state, however, contains many parameters $\{c_n\}_n$,

$$|\Psi\rangle = \sum_n c_n |E_n\rangle, \quad (1.2)$$

and during the evaporation process we seemingly lose this information. However, any physical process in a closed system should be unitary and preserve information. In particular, holography tells us that gravity is equivalent to a usual quantum many-body system, where information is preserved. This naive discrepancy is called the information loss problem.

2. **Maldacena’s information loss problem [24]:** Within AdS/CFT, we can formulate the information problem more quantitatively. Consider throwing a light particle from the AdS boundary into an AdS black hole. Because the black hole emits thermal radiation, we can compare the original particle with the outgoing radiation. This is equivalent to computing the probability amplitude. According to the AdS/CFT dictionary [25, 26], the light particle corresponds to a low-energy operator $\mathcal{O}$ and the amplitude corresponds to the real-time evolution of a thermal correlator in a quantum many-body system,

$$\text{Tr}[\rho_{\text{can}}^{(\beta)} \mathcal{O}(t) \mathcal{O}(0)]. \quad (1.3)$$

By computing this correlator using semiclassical gravity, we find that it decays monotonically:

$$\text{Tr}[\rho_{\text{can}}^{(\beta)} \mathcal{O}(t) \mathcal{O}(0)] \sim e^{-t/t_{\text{dec}}} \quad (1.4)$$

where t_{dec} is some time scale. If we trust the semiclassical computation, this decay continues indefinitely and eventually approaches zero. However, from general principles of unitary quantum mechanics, the correlator cannot vanish, as shown in Figure 3. This discrepancy constitutes Maldacena’s information loss problem.

■ Eigenstate thermalization hypothesis and the resolution of Maldacena’s information loss problem

The natural questions are why gravity appears to lose information and where the information goes. The first question is easily answered. After the time scale

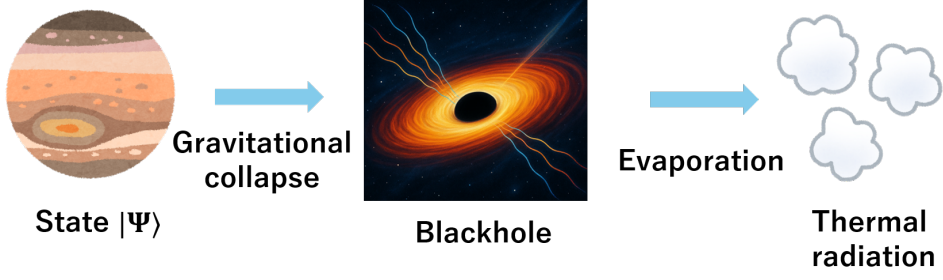


Figure 2: Hawking's information loss problem.

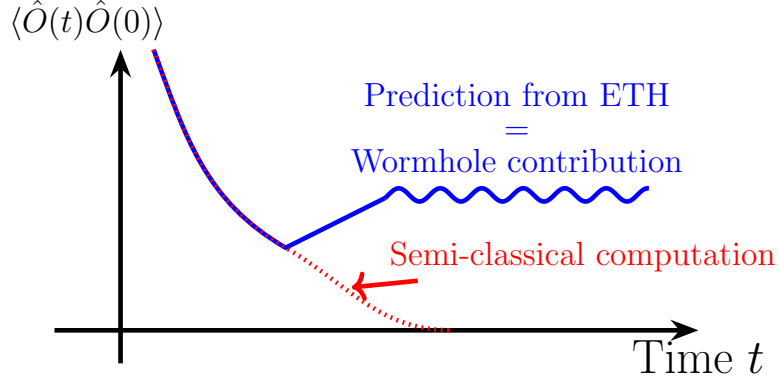


Figure 3: Time dependence of the two-point function $\text{Tr}(e^{-\beta H} \mathcal{O}(t) \mathcal{O}(0))$. The blue curve represents the contribution predicted by ETH and the associated wormhole saddle, while the red dotted curve shows the semiclassical computation. These two curves do not agree for $t \approx e^{S_{\text{BH}}}$. This is Maldacena's information loss problem.

when the correlation function becomes of order $e^{O(S_{\text{BH}})}$, semiclassical gravity loses its predictive power [24]. This is because $e^{-O(S_{\text{BH}})} = e^{-O(1/G_N)}$, so non-perturbative effects become important. The next question is how to understand these effects at late times.

Since gravitational dynamics is faithfully encoded in the non-equilibrium dynamics of the dual quantum many-body system, we should learn from the latter. The key point is that the relevant quantum many-body system is strongly coupled and therefore chaotic [27–29].

Chaotic systems are expected to thermalize. A possible mechanism for thermalization is the eigenstate thermalization hypothesis (ETH) [30–32]. This gives conditions for matrix elements of an operator $\mathcal{O}$ in the energy eigenbasis:

$$\langle E_n | \mathcal{O} | E_n \rangle - \text{Tr}[\rho_{\text{mc}} \mathcal{O}] \approx e^{-\frac{1}{2} S_{\text{mc}}}, \quad \langle E_n | \mathcal{O} | E_m \rangle \approx e^{-\frac{1}{2} S_{\text{mc}}}, \quad (1.5)$$

where S_{mc} is the microcanonical entropy. Once ETH is satisfied, one can show thermalization for any initial state. ETH also predicts the late-time behavior of correlation functions. Importantly, the small right-hand side of (1.5) contains the

factor $e^{-\mathcal{O}(S_{\text{mc}})}$, and via AdS/CFT we identify $S_{\text{mc}} = S_{\text{BH}}$. Thus ETH knows about non-perturbative effects, and by studying ETH in the dual quantum many-body system and translating its implications through the AdS/CFT dictionary, we can resolve Maldacena’s information paradox.

ETH was originally motivated by random matrix theory (RMT), a model of quantum chaotic systems. Srednicki and Deustch proposed that RMT arguments lead to the ETH ansatz

$$\langle E_n | \mathcal{O} | E_m \rangle = \mathcal{O}(E) \delta_{nm} + e^{-\frac{1}{2}S_{\text{mc}}(E)} F_{\mathcal{O}}(E, \omega) R_{nm}, \quad (1.6)$$

where R_{nm} is a random variable. In recent decades, the relation between random matrix theory and gravity has developed significantly. A concrete example is the JT/SYK correspondence [33, 34]: JT gravity is a two-dimensional dilaton gravity theory, and the SYK model is a Majorana quantum mechanics with random couplings. It has been shown that at late times, these two theories share random matrix universality [35–37]. In particular, Saad showed that by accurately incorporating non-perturbative wormhole contributions in JT gravity which obtains thermal correlators consistent with the ETH ansatz (1.6) [38].

This is promising, but not the whole story. JT/SYK correspondence is simple but quite different from higher-dimensional AdS/CFT because the SYK model involves ensemble averaging. To understand possible wormhole-like contributions in higher dimensions, we must seriously consider setups without ensemble averages.

Nevertheless, ETH should still hold. Although ETH was historically inspired by RMT, it is not fundamentally tied to it. ETH is a condition for thermalization that can be formulated without RMT and applies to any spacetime dimension with a single Hamiltonian. For example, there are numerically confirmed that the ETH holds in some non-integrable systems [39].

With this in mind, in Part II of this thesis we answer the following question:

Goal of Part II of this thesis

(Q) What is the precise and quantitative relation between ETH and the gravitational path integral with wormholes and black holes?

(A) By using large deviation techniques, we relate ETH to the long-time average of thermal correlators for any quantum many-body system. Applying this relation to AdS/CFT, we show some possible strategy for proving that holographic quantum many-body systems satisfy the (strong) ETH in any spacetime dimension, without performing exact diagonalization.

■ Open quantum systems and Hawking’s information loss problem

Hawking’s information loss problem involves evaporating black holes in asymptotically flat spacetime. To discuss holographic duals of flat spacetime, we need to generalize AdS/CFT. One interesting way is to couple AdS spacetime to a large external bath [40, 41]; see Figure 4. From the perspective of the boundary theory, this acts as a dissipative deformation. Thus we must consider a holographic correspondence for open quantum systems.

In typical open quantum systems [42], time evolution can be divided into two parts:

1. Time evolution with a non-Hermitian Hamiltonian,
2. Jump dynamics, such as measurements in quantum mechanics.

One goal of this thesis is to provide a concrete example of the holographic correspondence for these two effects. This is achieved by deforming the usual AdS/CFT framework.

Goal of Part III of this thesis

Construct the building blocks of the holographic correspondence for open quantum systems.

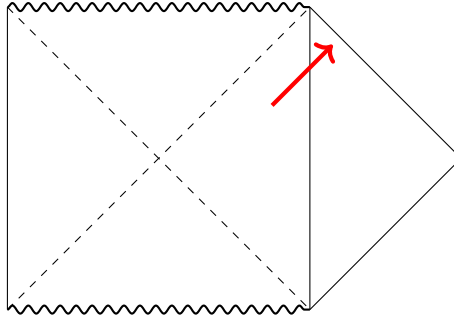


Figure 4: Setup of AdS coupled to a bath. Hawking radiation (red) can escape into the flat bath.

■ Organization of this thesis

This thesis is organized as follows. It is divided into three parts.

In Part I, we review the foundations of black hole thermodynamics and the information loss paradox (Section 2) and the AdS/CFT correspondence (Section 3). In Section 4, we discuss generalizations of AdS/CFT. Most of Part I is review, but some parts of Sections 4.2 and 4.4 include new results based on our papers [2, 3].

In Part II, we examine the relation between ETH and gravity in AdS/CFT. In Section 5, we review the foundations of statistical mechanics for closed quantum systems and ETH. In Section 6, we introduce a strategy for proving strong ETH

based on [1], which relates ETH to thermal correlation functions. In Section 7, we give examples of non-equilibrium dynamics in AdS/CFT and introduce Maldacena’s information loss problem. Finally, in Section 8, we explain why AdS/CFT satisfies the ETH criteria developed in Section 6.

In Part III, we discuss possible holographic duals of open quantum systems. In Section 9, we introduce the holographic setting of evaporating black holes and explain the motivation for studying open-system holography. In Section 10, we present a concrete realization of holography with non-Hermitian physics via imaginary Janus deformation, based on [2, 3]. In Section 11, we explain a setup of gluing two AdS/CFT systems as a possible model of open-system holography, based on [4, 8]. In Section 12, we see that boundaries and interfaces produce measurement-like behaviors, such as the decrease of entanglement entropy, based on [5, 7, 10, 11].

In Part IV, we give a summary and conclusion.

Part I

Blackhole, Information loss and holography

2 Black Holes and the Information Loss Problem

In this section, we review the basics of blackhole thermodynamics, Hawking radiation, holographic principle and information loss paradox.

2.1 Black Hole Thermodynamics

A black hole, which is a solution to Einstein's equations in general relativity, is a configuration that obeys thermodynamics. First, we show that a classical black-hole configuration satisfies the first law of thermodynamics. Below we review the method based on Noether's theorem due to Wald [43, 44]. Boldface symbols denote differential forms. Consider a field theory in $d + 1$ dimensions that includes gravity. Let ϕ denote all fields, including the gravitational field. The variation of the Lagrangian is

$$\delta\mathbb{L} = \mathbb{E} \delta\phi + d\mathbb{S}[\phi, \delta\phi] \quad (2.1)$$

where

$$\mathbb{E} = \frac{\delta\mathbb{L}}{\delta\phi}, \quad (2.2)$$

$$\mathbb{S} = (d^{d-1}x)_A \left(\frac{\partial\mathbb{L}}{\partial\phi_{,A}} \delta\phi + \frac{\partial\mathbb{L}}{\partial\phi_{,A,B}} \delta\phi_{,B} + \cdots \right). \quad (2.3)$$

The object $\mathbb{S}$ is called the symplectic potential. From this we obtain the equations of motion

$$\mathbb{E} \triangleq 0, \quad (2.4)$$

where $\triangleq$ denotes equality upon using the equations of motion.

On the other hand, if δ_s is a variation corresponding to a symmetry of the action, then there exists a d -form $\mathbb{X}$ such that

$$\delta_s \mathbb{L} = d\mathbb{X}. \quad (2.5)$$

In particular, when the symmetry is a diffeomorphism generated by a vector field ξ ,

$$\delta_\xi \mathbb{L} = \mathcal{L}_\xi \mathbb{L} \quad (2.6)$$

$$= d(i_\xi \mathbb{L}), \quad (2.7)$$

so that $\mathbb{X} = i_\xi \mathbb{L}$. Using the identity valid for any differential form,

$$\mathcal{L}_\xi = d i_\xi + i_\xi d, \quad (2.8)$$

and the fact that $\mathbb{L}$ is a $(d+1)$ -form, we obtain

$$\delta_\xi \mathbb{L} = d(i_\xi \mathbb{L}) \quad (2.9)$$

$$\Leftrightarrow d(\mathbb{S}[\phi, \delta_\xi \phi] - i_\xi \mathbb{L}) = 0. \quad (2.10)$$

Hence, the Noether current associated with ξ is the d -form

$$\mathbb{J} = \mathbb{S}[\phi, \delta_\xi \phi] - i_\xi \mathbb{L}, \quad d\mathbb{J} \triangleq 0. \quad (2.11)$$

Using this current, the Noether charge $H[\xi]$ defined on a Cauchy surface Σ is

$$H[\xi] = \int_\Sigma \mathbb{J}. \quad (2.12)$$

That this charge is conserved follows, for two Cauchy surfaces Σ_t, Σ_{t+dt} , from the on-shell relation

$$dt \frac{dH[\xi]}{dt} = \left[\int_{\Sigma_{t+dt}} - \int_{\Sigma_t} \right] \mathbb{J} = \int_V d\mathbb{J} \triangleq 0 \quad (2.13)$$

where V is volume surrounded by Σ_{t+dt} and Σ_t . Since the Noether current is closed on-shell, by Poincaré's lemma there exists a $(d-1)$ -form $\mathbb{Q}$ such that

$$\mathbb{J} \triangleq d\mathbb{Q}. \quad (2.14)$$

This $(d-1)$ -form $\mathbb{Q}$ is called the Wald charge. Consequently,

$$H[\xi] = \int_\Sigma \mathbb{J} \triangleq \int_{\partial\Sigma} \mathbb{Q}. \quad (2.15)$$

To derive the first law of black-hole thermodynamics, consider a perturbation $\delta\phi$ around a solution ϕ to the equations of motion. For this perturbation,

$$\delta\mathbb{J} = \delta\mathbb{S}[\phi, \delta_\xi \phi] - i_\xi \delta\mathbb{L}, \quad (2.16)$$

where we assume $\delta\xi = 0$. Furthermore,

$$i_\xi \delta\mathbb{L} = i_\xi (\mathbb{E} \delta\phi + d\mathbb{S}[\phi, \delta\phi]) \quad (2.17)$$

$$\triangleq \mathcal{L}_\xi \mathbb{S}[\phi, \delta\phi] - d(i_\xi \mathbb{S}[\phi, \delta\phi]), \quad (2.18)$$

so altogether

$$\delta\mathbb{J} = \Omega[\phi, \delta\phi, \delta_\xi \phi] + d(i_\xi \mathbb{S}[\phi, \delta\phi]), \quad (2.19)$$

where we defined

$$\Omega[\phi, \delta\phi, \delta_\xi\phi] := \delta\mathbb{S}[\phi, \delta_\xi\phi] - \delta_\xi\mathbb{S}[\phi, \delta\phi]. \quad (2.20)$$

Using the $(d-1)$ -form

$$\mathbb{K}_\xi[\delta\phi, \phi] = -\delta\mathbb{Q}_\xi + i_\xi\mathbb{S}[\phi, \delta\phi], \quad (2.21)$$

the variation of the Noether charge is defined by

$$\Delta H[\xi] = - \int_{\partial\Sigma} \mathbb{K}_\xi[\delta\phi, \phi] = \int_{\partial\Sigma} \delta\mathbb{Q}_\xi - \int_{\partial\Sigma} i_\xi\mathbb{S}[\phi, \delta\phi]. \quad (2.22)$$

Note that this differs from the earlier definition by a surface integral of the interior product of the symplectic potential. Variations of thermodynamic quantities—such as the mass variation in the Schwarzschild black hole—are of this type. In general, $\Delta H[\xi]$ need not be integrable with respect to δ . Now we consider blackhole solutions with bifurcation surface B . We take Cauchy slice Σ such that $\partial\Sigma = B \cup \partial\Sigma_\infty$ where $\partial\Sigma_\infty$ is subset of asymptotic infinity. See figure 5 for intuition.

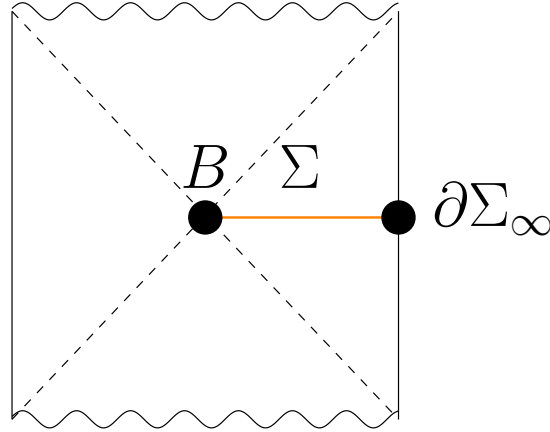


Figure 5: Example of blackholes solution.

Suppose there are Killing vectors corresponding to time translations t and rotations m . Then the variations of energy and angular momentum are defined by

$$\Delta\mathcal{E} = \delta H[t] = \int_{\partial\Sigma_\infty} [\delta\mathbb{Q}_t - i_t\mathbb{S}[\phi, \delta\phi]], \quad (2.23)$$

$$\Delta\mathcal{J} = - \int_{\partial\Sigma_\infty} [\delta\mathbb{Q}_m - i_m\mathbb{S}[\phi, \delta\phi]]. \quad (2.24)$$

This energy corresponds to the ADM mass with possible long-range-correlation cor-

rections. If there exists a $(d-1)$ -form $\mathbb{B}$ such that

$$\delta \int_{\partial \Sigma_\infty} i_t \mathbb{B} = \int_{\partial \Sigma_\infty} i_t \mathbb{S}[\phi, \delta \phi], \quad (2.25)$$

then we can integrate over δ and write

$$\mathcal{E} = \int_N (\mathbb{Q}_t - i_t \mathbb{B}). \quad (2.26)$$

Let B denote the bifurcation surface, and consider a Killing vector that can be written as

$$\xi = t + \Omega_H m. \quad (2.27)$$

Since $\mathcal{L}_\xi \phi = 0$, we have $\Omega[\phi, \delta \phi, \delta_\xi \phi] = 0$, and hence

$$\delta \mathbb{J}[\xi] = d i_\xi \mathbb{S}[\phi, \delta \phi]. \quad (2.28)$$

On the other hand, by definition $\delta \mathbb{J} \triangleq \delta d\mathbb{Q}$, so

$$\delta \mathbb{Q} - i_\xi \mathbb{S}[\phi, \delta \phi] = 0. \quad (2.29)$$

Integrating this over Σ , we obtain

$$0 = \int_\Sigma d(\mathbb{Q} - i_\xi \mathbb{S}[\phi, \delta \phi]) = \int_{\partial \Sigma} (\mathbb{Q} - i_\xi \mathbb{S}[\phi, \delta \phi]) = \Delta H[\xi]. \quad (2.30)$$

Since $\partial \Sigma = B \cup \partial \Sigma_\infty$, splitting the surface terms yields

$$\int_B (\delta \mathbb{Q}_\xi - i_\xi \mathbb{S}[\phi, \delta \phi]) = \int_{\partial \Sigma_\infty} (\delta \mathbb{Q}_\xi - i_\xi \mathbb{S}[\phi, \delta \phi]) \quad (2.31)$$

$$= \Delta \mathcal{E} - \Omega_H \Delta \mathcal{J}, \quad (2.32)$$

where in the last equality we used linearity of the charge variation in ξ . On the bifurcation surface B , $\xi = 0$, so $i_\xi \mathbb{S}[\phi, \delta \phi] = 0$. Therefore,

$$\int_B \delta \mathbb{Q}_\xi = \Delta \mathcal{E} - \Omega_H \Delta \mathcal{J}. \quad (2.33)$$

We can eliminate the explicit appearance of ξ from $\mathbb{Q}$ through the following scheme:

1. Second and higher derivatives of a Killing vector can be expressed everywhere

in spacetime using the identity

$$\nabla_A \nabla_B \xi^C = R_{ABD}^C \xi^D, \quad (2.34)$$

so $\mathbb{Q}$ can be written solely in terms of ξ and $\nabla\xi$.

2. On the bifurcation surface B ,

$$\xi = 0, \quad (2.35)$$

$$\nabla_A \xi_B = \kappa \epsilon_{AB}, \quad (2.36)$$

where the second relation follows from Frobenius' theorem and the definition of the surface gravity κ .

In this way, we obtain a Wald charge $\tilde{\mathbb{Q}}$ that is independent of the explicit form of ξ . The $\tilde{\mathbb{Q}}$ depends only on the dynamical fields of the theory and their derivatives, is invariant under diffeomorphisms on B , and is also the Wald charge associated with the Killing vector $\tilde{\xi} = \xi/\kappa$ normalized to unit surface gravity. For an infinitesimal perturbation with the Killing vector held fixed, we then have

$$\delta\mathbb{Q} = \kappa \delta\tilde{\mathbb{Q}}. \quad (2.37)$$

Defining the Wald entropy by

$$S_{\text{Wald}} = 2\pi \int_B \tilde{\mathbb{Q}}, \quad (2.38)$$

we obtain

$$\kappa \Delta S_{\text{Wald}} = \Delta\mathcal{E} - \Omega_H \Delta\mathcal{J}. \quad (2.39)$$

For simple gravitational theories which is mild deviation of the Einstein gravity we see [45]

$$S_{\text{Wald}} = \frac{\text{Area}[B]}{4G_N} \quad (2.40)$$

and κ can be regarded as a temperature. This is the first law of black-hole thermodynamics.

2.2 Thermodynamic Entropy of Black Holes from Euclidean path integrals

Next, we show that classical black-hole solutions have finite thermodynamic entropy from Euclidean path integral [46]. Consider the action of a gravitational theory

$$I_{\text{tot}} = I_{\text{EH}}[G] + I_{\text{GH}}[\gamma] + I_{\text{mat}}[X; G], \quad (2.41)$$

$$I_{\text{EH}} = \frac{1}{16\pi G_N} \int_M d^{d+1}x \sqrt{-G} (R[G] - 2\Lambda), \quad (2.42)$$

$$I_{\text{GH}} = \frac{1}{8\pi G_N} \int_{\partial M} d^d x \sqrt{-\gamma} K, \quad (2.43)$$

where the boundary term I_{GH} is required to make the variational problem well-defined. $I_{\text{mat}}[X; G]$ is the matter action. For simplicity, we will set the matter fields to zero hereafter. The equations of motion are Einstein's equations. We compute the thermodynamic entropy of a static black-hole solution [47]. For simplicity, we proceed in the spherically symmetric case (2.48) with no matter fields. The Helmholtz free energy of the gravitational theory in equilibrium is

$$F(\beta) = -\frac{1}{\beta} \log Z(\beta), \quad (2.44)$$

$$Z(\beta) = \text{Tr}[e^{-\beta H}] = \int_{\text{KMS}} \mathcal{D}G e^{-I_{\text{grav}}}, \quad (2.45)$$

with $I_{\text{grav}} = I_{\text{EH}} + I_{\text{GH}}$. The thermodynamic entropy is

$$S(\beta) = \left(1 - \beta \frac{\partial}{\partial \beta}\right) \log Z(\beta). \quad (2.46)$$

Here the subscript KMS on the measure indicates that the Euclidean time defined by the timelike Killing vector satisfies the periodic boundary condition with inverse temperature β . When Newton's constant G_N is much smaller than other dimensionful scales, the saddle-point approximation gives

$$Z(\beta) \sim \sum_{G \in \text{saddle}} e^{-I_{\text{grav}}[G]} \sim e^{-I_{\text{grav}}[G_{\text{min}}]} + \dots, \quad (2.47)$$

where the sum runs over solutions of Einstein's equations satisfying the periodic boundary condition, and G_{min} denotes the configuration with minimal on-shell action. The saddle that minimizes the action depends on β . For example, with compact asymptotic AdS boundary, thermal AdS dominates at low temperature while the AdS black hole dominates at high temperature [48]. In what follows we simply write G_{min} as G and consider cases where a black-hole solution provides the

dominant saddle. For simplicity, consider a static, spherically symmetric black-hole solution in $d + 1$ dimensions with the background metric

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \quad (2.48)$$

where $f(r_H) = 0$ and $f'(r_H) > 0$ at the event horizon $r = r_H$. Here $d\Omega_{d-1}^2$ is the metric on the unit $(d - 1)$ -sphere S^{d-1} .

In asymptotically flat and asymptotically AdS spacetimes, the on-shell action diverges because of the infinite volume. We therefore consider the regulated gravitational action

$$I[G] = I_{\text{grav}'}[G, \gamma], \quad (2.49)$$

$$I_{\text{grav}'}[G] = I_{\text{reg}}[G, \gamma] + I_{\text{ct}}[\gamma], \quad (2.50)$$

$$I_{\text{reg}}[G, \gamma] := \frac{1}{16\pi G_N} \int_{r_H + \epsilon \leq r \leq r_\infty} d^{d+1}x \sqrt{G} (R[G] - 2\Lambda) \quad (2.51)$$

$$+ \frac{1}{8\pi G_N} \int_{r_\infty} d^d x \sqrt{\gamma} K[\gamma], \quad (2.52)$$

where γ_{ab} is the induced metric at the IR cutoff surface r_∞ , and h_{ab} is the induced metric at $r = r_H + \epsilon$. The counterterm I_{ct} depends on the boundary condition and the spacetime dimension. We will discuss the asymptotically AdS case in the next chapter. For the asymptotically flat case, one may use [49]

$$I_{\text{ct}} = \frac{1}{8\pi G_N} \int d^d x \sqrt{\gamma} K_0, \quad (2.53)$$

where K_0 is the mean curvature of the surface $r = r_H$ embedded in the reference metric

$$ds_0^2 = f(r_H) dt_E^2 + \frac{dr^2}{f(r_H)} + r^2 d\Omega_{d-1}^2. \quad (2.54)$$

As long as the total action is finite, the precise choice of counterterms is arbitrary; as we will see, the thermodynamic entropy is independent of this choice.

We take $\epsilon \ll r_H \ll r_\infty$ and send $\epsilon \rightarrow 0$ and $r_\infty \rightarrow \infty$ after the computation. The thermodynamic entropy is

$$S(\beta) = - \left(1 - \beta \frac{\partial}{\partial \beta} \right) I[G]. \quad (2.55)$$

For a static black hole, the on-shell Lagrangian is time-independent, so

$$I_{\text{grav}}[G] = \beta L(\beta), \quad (2.56)$$

and therefore

$$S = \beta^2 \frac{\partial L}{\partial \beta}. \quad (2.57)$$

A direct computation yields

$$\begin{aligned} \partial_\beta L = & \frac{1}{16\pi G_N} \int d^d x \left[-\sqrt{G} \mathbb{E}^{AB} \partial_\beta G_{AB} + \partial_C \left(\sqrt{G} (G^{CA} \nabla^B \partial_\beta G_{AB} - G^{AB} \nabla^C \partial_\beta G_{AB}) \right) \right] \\ & + \int d^{d-1} x \partial_\beta (\sqrt{\gamma} K[\gamma] - L_{\text{ct}}), \end{aligned} \quad (2.58)$$

where $\mathbb{E}^{AB}$ is the left-hand side of Einstein's equations and hence vanishes on-shell. Upon integrating by parts, boundary terms arise at $r = r_H$ and $r = r_H + \epsilon$. The contribution at $r = r_H$ combines with the Gibbons–Hawking term and vanishes due to the boundary conditions imposed when solving the equations of motion. Thus only the horizon contributes to the entropy:

$$\partial_\beta L = \frac{1}{16\pi G_N} \int_{r_H+\epsilon} d^{d-1} x \sqrt{G} n_A (G^{CA} \nabla^A \partial_\beta G_{AB} - G^{AB} \nabla^C \partial_\beta G_{AB}). \quad (2.59)$$

Approximating the near-horizon geometry by the Rindler metric (2.62) and evaluating the integrand explicitly, we obtain

$$S(\beta) = \frac{r_H^{d-1} \text{Vol}[S^{d-1}]}{4G_N} = \frac{\text{Area}}{4G_N}, \quad (2.60)$$

where Area is the area of the horizon. The same formula holds more generally for static spacetimes [46, 50]. This is the Bekenstein–Hawking entropy.

Also the inverse temperature is determined from the horizon singularity. Near the horizon the metric becomes

$$ds^2 \approx -f'(r_H)(r - r_H) dt^2 + \frac{dr^2}{f'(r_H)(r - r_H)} + r_H^2 d\Omega_{d-1}^2. \quad (2.61)$$

In what follows, we treat $f(r)$ as a smooth function with a simple zero at $r = r_H$.

Define a new radial coordinate ρ by $\rho = \frac{\sqrt{r-r_H}}{2f'(r_H)}$. Then

$$ds^2 \approx -4f'(r_H)^2 \rho^2 dt^2 + d\rho^2 + r_H^2 d\Omega_{d-1}^2. \quad (2.62)$$

Upon Wick-rotating $t_E = -it$, the first two terms become

$$ds^2 \approx 4f'(r_H)^2 \rho^2 dt_E^2 + d\rho^2. \quad (2.63)$$

Unless the Euclidean time t_E is periodic with period $\beta_H = \frac{\pi}{f'(r_H)}$, the resulting space has a conical singularity at the horizon; in particular, the Ricci scalar acquires a

delta-function singularity. If the metric solves Einstein's equations and the matter fields do not have such delta-function singularities in their stress tensor, this conical singularity is not allowed. Hence, t_E must be identified with period $\beta_H = \frac{\pi}{f'(r_H)}$. Due to this periodicity, all correlation functions of a field theory on this background satisfy the KMS condition. That is, the field theory is thermal with inverse temperature $\beta = \beta_H$.

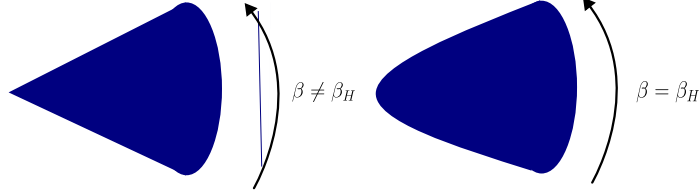


Figure 6: Left) Spacetime with $\beta \neq \beta_H$: a conical singularity appears at the horizon $r = r_H$. Right) $\beta = \beta_H$.

2.3 Hawking Radiation

Next, we consider a QFT on blackhole [23,47,51,52]. In Hawking's original work [23], one considers quantum fields on a black hole formed by gravitational collapse and shows thermal radiation by comparing Bogoliubov coefficients between past and future null infinity $\mathcal{I}^-$ and $\mathcal{I}^+$. This thermal radiation is called Hawking radiation. The essence common to both derivations is the presence of two distinct notions of time in the theory or state. For instance, in the static black-hole example above, there is the time t of an observer at infinity and the time generated by the Killing vector associated with the horizon. In Hawking's setup in asymptotically flat space-time, one has the Minkowski time at infinity and the proper time of an observer falling into the black hole. Each time coordinate comes with its own Hamiltonian and Fock space. In finite systems, Fock spaces generated by different Hamiltonians are related by a unitary transformation, so a state in one Fock space can be expressed as a linear combination of states in the other. However, in infinite systems such as quantum field theory, the transformation between the two Fock spaces involves divergent integrals unless regularized, and becomes a singular unitary transformation; a state in one Fock space cannot be written as a regular linear combination of states in the other [53,54]. For example, consider infinitely many harmonic oscillators α_k and their Bogoliubov transforms $\alpha_k(\theta) = \alpha_k + \theta_k$. Each oscillator defines a vacuum and a Fock space. If $|\theta\rangle$ denotes the vacuum of $\alpha_k(\theta)$ and $|0\rangle$ the vacuum of α_k , then

$$|\theta\rangle = \exp\left\{-\frac{1}{2} \int d^d k |\theta_k|^2 - \int d^d k \theta_k \alpha_k^\dagger\right\} |0\rangle, \quad (2.64)$$

and $\langle 0|\theta\rangle = 0$. Such a phenomenon also appears in discussions of phase transitions, where distinct Fock spaces correspond to different phases. In the case of Hawking radiation, the two Fock spaces correspond to the two notions of time and are related in this way [55]. The vacuum with respect to one time appears as a condensate of particle pairs with respect to the other. Consider an observer near the horizon measuring with respect to the Killing time who looks at the Minkowski vacuum at infinity: this state exhibits pair creation and annihilation with respect to the Killing-time particles. Because of the presence of the horizon, one particle of each pair falls into the black hole and never returns, while the other escapes to $r \rightarrow \infty$ and is observed as thermal radiation.

One important remark is that we need to distinct the entropy of Hawking radiation from the entropy of the blackhole. The former one is purely matter contribution similar to the Unruh effect. On the other hand, the latter is purely contribution from the classical gravity. Sometimes we can think of that the entropy of the Hawking radiation as the blackhole entropy with "fine tuning" UV cut-off of the QFT on the blackhole. To see this, let us consider t'Hooft's brickwall model [56]: we set a hard wall on the stretch horizon $r = r_H + \epsilon$. Since the wall hide the horzion, there is no classical gravity contribution to the blackhole entropy. Also the wall play a role of UV regularization. For example, the QFT in Rinder wadge have the natural energy which is nothing but a boost operator. This boost operator has continuous spectrum and shows the UV divergence. In the recent terminology of the operator algebra the continuous spectrum of the boost operator is refereed as the type III_1 von Neumann algebra [57, 58]. Once we set the stretch horizon, the spectrum is discretized. Then the leading contribution is coming from the matter thermal entropy. For free field, by using the heat kernel method, we can concretely compute the matter thermal entropy is proportional to the area of black hole divide by the UV cut-off

$$S_{\text{mat}} = \# \cdot \frac{\text{Area}}{\epsilon^{d-1}} + \dots \quad (2.65)$$

where $\#$ is some numerical constant depends on the theory we consider [59]. By fine-tuning the $\epsilon = O(l_{\text{pl}})$ we can obtain the Bekenstein-Hawking formula (2.60).

Actually we have the another interpretation for matter contribution. Following to [60], the brickwall type regularization can be considered as the regularization of the entangling surface. Thus, the matter entropy (2.65) can be considered as the entanglement entropy between the inside and out side of the blackhole horizon. Now we consider a generalized entropy

$$S_{\text{gen}} := S_{\text{BH}} + S_{\text{mat}} \quad (2.66)$$

which is total thermal entropy in the quantum gravity path integral. Susskind and Uglum considered that the entanglement entropy of the string theory is finite since there is minimum length scale l_s and corresponding this S_{gen} is finite [61]. At the level of the small but finite G_N by doing the renormalization of Newton constant $\frac{1}{4G_N^{\text{ren}}} = \frac{1}{4G_N} + \frac{\#}{\epsilon^d}$, we obtain the finite S_{gen} . In the recent terminology the finiteness of the generalized entropy is referred as the type II von Neumann algebra [62].

2.4 Holographic correspondence

In the previous section we see the following facts:

- The blackholes which are the solution of classical equation of motion of the gravitational theories possess non-trivial entropy.
- The blackhole entropy is proportional to the area of blackhole and inverse of gravitational coupling.

In usual (quantum) many body systems, the thermodynamic entropy is extensive for the system size [17]. In this sense, the blackhole entropy is exotic. We may explain this exotic scaling of blackhole entropy by thinking that area law of blackhole entropy in $d+1$ dimensional gravitational theory is considered as the usual extensive thermodynamic entropy in d dimensional quantum many body systems. By proceeding this consideration in more deeply, t'Hooft and Susskind suggest that this exotic behavior of blackhole entropy in gravitational theory is explained by holographic principle: $d+1$ dimensional gravitational theory is equivalent to d dimensional quantum many body systems [18, 19].

In usual statistical physics, the thermodynamic entropy is emergent concept raised from the thermodynamic limit. For example, when we write down the thermal partition function in terms of spectral function $\rho(E) := \sum_n \delta(E - E_n)$, we see

$$Z(\beta) = \text{Tr}[e^{-\beta H}] = \sum_n e^{-\beta E_n} = \int dE \rho(E) e^{-\beta E}. \quad (2.67)$$

When we assume $\rho(E) = e^{Vs(E/V)+o(V)}$ with convex function $s(u)$, we write

$$Z(\beta) = \int dE e^{V(s(E/V)-\beta E/V)+o(V)} \approx e^{S(E_*)-\beta E_*} \quad (2.68)$$

with thermodynamic relation $\partial S/\partial E|_{E_*} = \beta$ in the saddle point approximation under thermodynamic limit $V \rightarrow \infty$ [63]. Similarly in the gravitational theory gravitational entropy is also emergent in some thermodynamic limit. Indeed the semi classical limit $G_N \rightarrow 0$ play a role of thermodynamic limit. We consider a

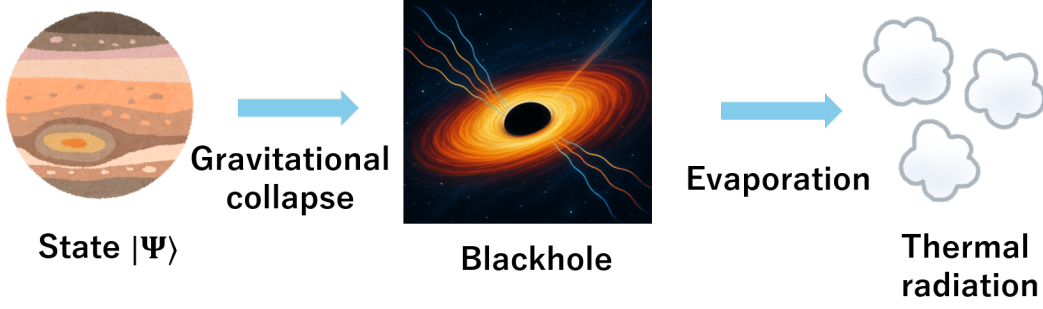


Figure 7: Process of gravitational collapse and evaporation

quantum gravity partition function

$$Z_{\text{grav}}(\beta) = \int \mathcal{D}G \mathcal{D}X e^{-I_{\text{grav}}[G] - I_{\text{matter}}[X;G]} \approx e^{S_{\text{BH}} - \beta E_*}. \quad (2.69)$$

and by taking the "naive" thermodynamic limit $G_N \rightarrow 0$ we obtain the Bekenstein-Hawking entropic formula (2.60). In the holographic principle $1/G_N$ corresponds to the number of local degree of freedom. The AdS/CFT correspondence is beautiful example which realize these discussion in concretely.

2.5 Black-Hole Evaporation and the Information Loss Problem

So far we have treated a black-hole solution as a fixed background, but now we consider the back reaction of Hawking radiation. From the viewpoint of an observer outside the horizon, the in-falling partner of a created pair carries negative energy, so the black hole loses mass as it emits Hawking radiation. In an asymptotically flat spacetime without boundaries, the Hawking radiation does not return to the black hole, and the mass is expected to eventually go to zero; that is, the black hole ultimately evaporates.

Now consider a pure state $|\Psi\rangle$ with large energy. According to general relativity, such a quantum state undergoes gravitational collapse to form a black hole and begins to emit Hawking radiation. Eventually, the black hole evaporates, leaving a thermal state consisting of Hawking quanta. See figure 7 for illustration. From the semiclassical analysis above, this final thermal state is a mixed canonical ensemble. Naively, the process of formation and evaporation of a black hole thus appears to evolve a pure state into a mixed state, seemingly violating unitarity [64]. On the other hand, superstring theory, a candidate for quantum gravity, is a unitary theory; hence the final state after evaporation ought to be pure. The apparent non-unitarity of the evaporation process is called the information loss problem. Later we will

see more concrete realization of information paradox in the context of AdS/CFT correspondence.

3 AdS/CFT correspondence

The AdS/CFT correspondence is a concrete realization of the holographic correspondence. They are duality between quantum gravity with asymptotic AdS structure and some conformal field theories(CFT). Originally, it is conjectured by Maldacena by using the super string theory [20]. The duality relation is still conjecture but there are many justification [21]. Moreover, the AdS/CFT correspondence is nowadays considered as a duality relation which can be general more than the string theory. For example we can see some matching of symmetry, correlation functions, entanglement structure for examples for some simple setting which may not have string theory interpretation [65].

In this section, we review the basic of AdS/CFT correspondence and their dictionary.

3.1 AdS/CFT correspondence: Maldacena's discussion

Maldacena pointed out that some QFTs obtained from the excitations on certain branes match the string theory on AdS times some compact manifolds in certain limits. Especially, he showed that the large N limit of certain CFTs in various dimensions are equivalent to the supergravity on the product of AdS, spheres and other compact manifolds. The insight comes from the two equivalent views for the brane configuration.

3.1.1 World volume theory point of view

Let us consider a brane configuration where N parallel D3-branes are sitting. There are naively two kinds of string excitations.

- open strings from D3-branes,
- closed strings from ten-dimensional empty space.

In this situation, we take a low-energy limit in the sense that the energy scale E we take into account satisfies $E \ll 1/l_s$. Then only the massless modes survive.

- open strings: vector multiplet on the D3-brane, $\mathcal{N} = 4$ $U(N)$ super Yang Mills theory (SYM) ,
- closed strings: ten-dimensional gravity supermultiplet, type II B super gravity(SUGRA).

The low-energy action is expected to take the form:

$$I_{\text{tot}} = I_{\text{bulk}} + I_{\text{brane}} + I_{\text{int}}. \quad (3.1)$$

Taking the weak-gravity limit $\kappa \rightarrow 0$, the interaction between branes and closed strings vanishes. So the world volume theory on the D3-branes and ten-dimensional SUGRA decouple. More precisely, the decoupling limit is the following: keeping the energy scale E fixed and taking $l_s \rightarrow 0$, that is $El_s \rightarrow 0$, while keeping all the dimensionless parameters g_s, N fixed. Then the Newton constant $\kappa \sim g_s \alpha'^2 \rightarrow 0$.

3.1.2 SUGRA point of view

The SUGRA solution that carries D3-brane charges is

$$ds^2 = f^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2), \quad (3.2)$$

$$F_5 = (1 + *) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge df^{-1}, \quad (3.3)$$

$$e^\phi = g_s, \quad f = 1 + \frac{R^4}{r^4}, \quad R^4 = 4\pi g_s \alpha'^2 N. \quad (3.4)$$

The proper energy E_p at r and the energy for an observer at infinity satisfy

$$d\tau_r E_p = d\tau_\infty E, \quad E = f^{-1/4} E_p. \quad (3.5)$$

Thus the redshift means that the same object brought closer and closer to $r = 0$ appears to have lower and lower energy for the observer at infinity. Next, we consider the low-energy limit. To this end, we consider the throat around the brane with string-scale radius. We want to consider the $\alpha' \rightarrow 0$ limit, but now we want to consider the duality between the world volume theory and the bulk, so we should keep fixed the energies of the objects in the throat (the near-horizon region). Then we can consider arbitrarily excited string states. More precisely, we fix $l_s E_p$. In this limit, we also keep the energy from infinity

$$E_\infty \sim E_p \frac{r}{l_s} \sim E_p \frac{l_s r}{\alpha'}, \quad (3.6)$$

which is the energy of the field theory on the brane. With this setup, the low-energy excitations from the point of view of an observer at infinity are

- the long-wavelength massless particles in the bulk region: ten-dimensional SUGRA,
- any kind of string excitation in the near-horizon region.

These two excitations decouple because of the following reasons:

- the long-wavelength low-energy massless particles in the bulk do not get absorbed into the throat since their absorption rate is $\sigma \sim \omega^3 R^8$. Naively speaking, in this low-energy limit, the wavelength of the bulk particles becomes much bigger than the typical gravitational size of the brane $\sim R$;
- similarly, the excitations that live very close to $r = 0$ find it harder and harder to climb up the gravitational potential and escape to the asymptotic region [66].

To consider the near-horizon region in the low-energy limit, it is convenient to introduce the fixed coordinate $U = r/\sqrt{\alpha'}$. Then, taking $\alpha' \rightarrow 0$, $r \rightarrow 0$ with fixed U, g_s, N , we obtain

$$ds^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \sqrt{4\pi g_s N} \frac{dU^2}{U^2} + \sqrt{4\pi g_s N} d\Omega_5^2 \right]. \quad (3.7)$$

This is the metric of AdS_5 times S^5 . The important point is that the AdS radius and the radius of the S^5 match,

$$L_{\text{AdS}}^2 = R_{S^5}^2 = \alpha' \sqrt{4\pi g_s N}. \quad (3.8)$$

3.1.3 Duality

Finally we have two different points of view for the same configuration in the appropriate low-energy limit.

- World volume theory: four-dimensional $\mathcal{N} = 4$ $U(N)$ SYM + ten-dimensional SUGRA in the flat background.
- SUGRA: type II B string on $\text{AdS}_5 \times S^5$ (near horizon) + ten-dimensional SUGRA in the flat background.

Here + means the two theories decouple. So it is natural to conjecture that : The four-dimensional $\mathcal{N} = 4$ $U(N)$ SYM is equivalent to type II B string on $\text{AdS}_5 \times S^5$.

There are some comments for $\text{AdS}_5/\text{CFT}_4$ which will be important in later discussions.

- In the above argument, the validity of the SUGRA solution requires that the curvature of the solution is much smaller than the string scale,

$$\frac{R^4}{l_s^4} \sim \frac{L_{\text{AdS}}^2}{l_s^2} \sim g_s N \sim g_{\text{YM}}^2 N = \lambda \gg 1. \quad (3.9)$$

Also, the D-brane description above is valid when the D-brane is a heavy object, that is

$$g_s \ll 1. \quad (3.10)$$

So we need large N in the above argument. This can also be seen from the suppression of quantum gravity effects:

$$\frac{R^4}{l_p^4} \sim N \gg 1. \quad (3.11)$$

- We can trust perturbation theory in SYM when

$$\lambda = g_{\text{YM}}^2 N = g_s N \sim \frac{R^4}{l_s^4} \ll 1. \quad (3.12)$$

So the AdS/CFT correspondence is a strong/weak duality in the sense that the region where the theory can be easily treated is different and the hard part on one side can be computed from the easy part of the other side. Another important point is that the $1/N$ expansion in SYM corresponds to the expansion of the quantum fluctuations in quantum gravity. Especially, Hawking radiation is a $1/N$ effect.

- The isometry of $\text{AdS}_5 \times S^5$ corresponds to the global symmetry of the SYM. The isometry of AdS_5 is $SO(4, 2)$ and this is exactly the conformal symmetry of the four-dimensional SYM. The $SO(6)$ isometry of S^5 is the R-symmetry of the $\mathcal{N} = 4$ supersymmetry in SYM.
- The SYM coupling is given by the string VEV

$$\tau = \frac{4\pi i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi} = \frac{i}{g_s} + \frac{\chi}{2\pi}, \quad (3.13)$$

and g_s, χ are the VEVs of the dilaton and the RR 0-form in string theory. The $SL(2, \mathbb{Z})$ duality in SYM corresponds to the $SL(2, \mathbb{Z})$ symmetry of type II B string on flat space. The $\text{AdS}_5 \times S^5$ background is also invariant under the $SL(2, \mathbb{Z})$ duality. We also note that N is given by the number of D3-branes or equivalently the flux of the RR 5-form,

$$N = \int_{S_\infty^5} F_5. \quad (3.14)$$

- When the D3-branes do not coincide, there appear strings stretched between branes. They are W bosons in the world volume theory. The mass of the W

boson is the tension times the length of the string, thus $M_W \sim r/\alpha' = U$, which is kept fixed in the low-energy limit.

- By considering other types brane configuration we obtain AdS/CFT correspondence with other dimensions.

Moreover we can discuss AdS/CFT without string theory. As we will see from right now, by considering some gravitational theory on AdS and moderate dictionary we can see matching of symmetry, correlation functions and so on.

3.2 GKPW relation and holographic renormalization

3.2.1 GKPW relation

From the above discussion, the AdS/CFT correspondence is essentially a relation between the closed string scattering process in the gravitational theory and open string propagation on the brane. From the gauge theory perspective, the open string propagation is the same as the correlation functions of gauge-invariant operators. Gubser-Klebanov-Polyakov and Witten suggested that this relation is formulated by the equivalence of the quantum gravity partition function with certain boundary conditions and the sourced partition functions of CFTs [25, 26]:

$$Z_{\text{grav}}[X \sim X_0] = \left\langle \exp \left(- \int_{\partial M} X_0 \mathcal{O}_X \right) \right\rangle_{\text{CFT}} =: e^{W_{\text{CFT}}[X_0]} \quad (3.15)$$

Here $X \sim X_0$ means that we do the gravitational path integral with the Dirichlet boundary condition X_0 times some conformal factor. Often we consider the saddle-point approximation for the gravitational theory, that is

$$\underline{I}_{\text{grav}}[X_0] = -W_{\text{CFT}}[X_0], \quad (3.16)$$

where the underline denotes the on-shell action. With this relation, the correlation functions are, in order,

$$\langle \mathcal{O}_X(x) \rangle = \left. \frac{\delta \underline{I}_{\text{grav}}}{\delta \phi_{(0)}(x)} \right|_{\phi_{(0)}=0}, \quad (3.17)$$

$$\langle \mathcal{O}_X(x) \mathcal{O}_X(y) \rangle = - \left. \frac{\delta \underline{I}_{\text{grav}}}{\delta \phi_{(0)}(x) \delta \phi_{(0)}(y)} \right|_{\phi_{(0)}=0} \quad (3.18)$$

$$= \left. \frac{\delta \langle \mathcal{O}_X(x) \rangle_{\phi_{(0)}}}{\delta \phi_{(0)}(y)} \right|_{\phi_{(0)}=0}. \quad (3.19)$$

Now we try to do some concrete computations of correlation functions via the

dictionary. The two-point function can be computed as follows.

3.2.2 Holographic renormalization

■ Scale/radius duality

In QFT, we have UV divergences. Also, the gravitational theory on AdS space-time has divergences which come from the infinite volume of asymptotic AdS space-time. Thus these divergences are IR divergences from the bulk point of view. On the other hand, thermodynamics is IR physics of the boundary QFTs. This thermodynamics is related to black hole thermodynamics in the bulk. The black hole thermodynamics is governed by the black hole horizon and thus the physics in the deep interior of the bulk. From these discussions, we may think there is a duality relation between the energy scale of the boundary QFTs and the bulk distance from the boundary. More clearly, in the asymptotic AdS spacetime, there is a canonical ADM coordinate system called the Fefferman-Graham coordinate:

$$\begin{aligned} ds^2 &= dr^2 + e^{2r} \gamma_{ab}(r, x^a) dx^a dx^b, \\ \gamma_{ab}(r, x^a) &= g_{(0)ab}(x^a) + e^{-2r} g_{(2)ab}(x^a) + e^{-4r} g_{(4)ab}(x^a) + \cdots \end{aligned} \quad (3.20)$$

The limit $r \rightarrow \infty$ corresponds to the conformal boundary. The statement of the scale/radius duality is that we can identify the energy scale Λ_{QFT} of the boundary QFTs with the radial coordinate: $\Lambda_{\text{QFT}} \sim r$. See Figure 8.

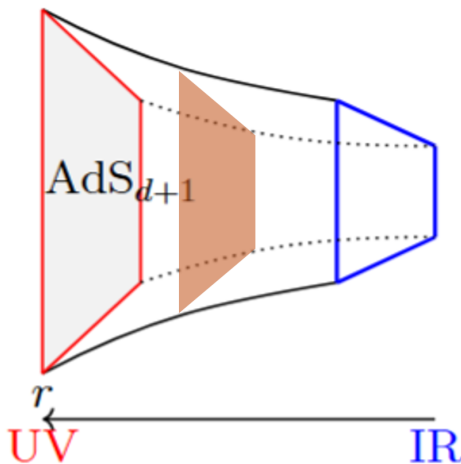


Figure 8: Sketch of scale radius duality in AdS spacetime. Large r corresponds to UV degree of freedom of boundary QFTs and small r corresponds to IR part.

■ Holographic renormalization

The holographic renormalization is the scheme for regulating the IR divergences of the gravitational theory on AdS by putting counterterms, similar to the computa-

tion of the black hole entropy [67–69]. In AdS/CFT correspondence, the holographic renormalization plays the role of the renormalization of the effective action of the boundary CFTs [70].

There is a systematic method for holographic renormalization that relies only on the induced fields on the cut-off surface [71–73]. In this method, we use the canonical formalism of the AdS gravitational theory with the radial coordinate playing the role of time. We consider the ADM decomposition

$$ds^2 = dr^2 + h_{ab}(r, x^a) dx^a dx^b. \quad (3.21)$$

Here we set the lapse to be one and the shift vectors to be zero. We denote the constant r surface by Σ_r . The induced metric on Σ_r , h_{ab} , is related to the Fefferman-Graham coordinate with $h_{ab} = e^{2r} \gamma_{ab}$. For the asymptotic AdS geometry it is useful to consider the following dilatation operator,

$$\delta_D := \int_{\Sigma_r} d^2x \, 2h_{ab} \frac{\delta}{\delta h_{ab}}. \quad (3.22)$$

This operator is basically equivalent to the r derivative for large r . In this note, we consider the pure gravity case. The task we do is solving the Hamilton-Jacobi equation. To this end let us consider the Hamilton’s principal function $\mathcal{S}$ in the form

$$\mathcal{S}[h] = \int_{\Sigma_r} d^2x \sqrt{h} \mathcal{L}. \quad (3.23)$$

We consider the spectral decomposition for the dilatation operator,

$$\mathcal{S} = \sum_{k=0} \mathcal{S}_{2-2k}, \quad \delta_D \mathcal{S}_{2k} = 2k \mathcal{S}_{2k}. \quad (3.24)$$

We also have the spectral decomposition for the Lagrangian density

$$\mathcal{L} = \sum_{k=0} \mathcal{L}_{2k}, \quad (3.25)$$

where each $\mathcal{L}_{(2k)}$ is defined up to a total derivative. From the general discussion in the Hamilton-Jacobi formalism, we have the identity

$$\pi^{ab} \delta_D h_{ab} = 2\pi = \frac{1}{\sqrt{h}} \delta_D (\sqrt{h} \mathcal{L}) + (\text{total derivative}). \quad (3.26)$$

The total derivative term can be absorbed into the Lagrangian density and we do

not care about it. We also have, for each eigenvalue sector,

$$\pi_{(2k)}^{ab} \delta_D h_{ab} = \frac{1}{\sqrt{h}} \delta_D (\sqrt{h} \mathcal{L}_{(2k)}). \quad (3.27)$$

In the pure gravity case, it is known that we have the following decomposition

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{(0)} + \tilde{\mathcal{L}}_{(2)} \log e^{-2r} + \mathcal{L}_{(2)} + \mathcal{L}_{(4)} + \cdots, \\ \delta_D \mathcal{L}_{(n)} &= -n \mathcal{L}_{(n)}, \quad n \neq 2, \quad \delta_D \tilde{\mathcal{L}}_{(2)} = -2 \tilde{\mathcal{L}}_{(2)}, \\ \delta_D \mathcal{L}_{(2)} &= -2 \mathcal{L}_{(2)} - 2 \tilde{\mathcal{L}}_{(2)}. \end{aligned} \quad (3.28)$$

This is because we have the formula $\delta_D \gamma_{ab} = 2\gamma_{ab}$. The transformation law of $\mathcal{L}_{(2)}$ shows that $\mathcal{L}_{(2)}$ cannot be a polynomial of the induced fields and is thus a non-local functional. As mentioned, the dilatation operator δ_D is roughly the r derivative, thus we can read off the scaling $\mathcal{L}_{(n)} \sim e^{-nr}$, $\sqrt{h} \sim e^{2r}$. Thus we define the counterterm

$$S_{\text{ct}} = - \int_{\Sigma_r} d^2x \sqrt{h} \left(\mathcal{L}_{(0)} + \tilde{\mathcal{L}}_{(2)} \log e^{-2r} \right). \quad (3.29)$$

We also have the renormalized on-shell action

$$S = S_{\text{reg}} + S_{\text{ct}} = \int_{\Sigma_r} d^2x \sqrt{h} \mathcal{L}_{(2)}. \quad (3.30)$$

Recursive relation for $\mathcal{L}_{(n)}, \tilde{\mathcal{L}}_{(2)}$

Let us introduce the method to compute the $\mathcal{L}_{(n)}$ ($n < 2$) and $\tilde{\mathcal{L}}_{(2)}$. To this end we consider the expansion for the canonical momenta,

$$\pi^{ab} := \frac{1}{\sqrt{h}} \frac{\delta}{\delta h_{ab}} \int_{\Sigma_r} d^2x \sqrt{h} \mathcal{L} = \pi_{(0)}^{ab} + \tilde{\pi}_{(2)}^{ab} \log e^{-2r} + \pi_{(2)}^{ab} + \pi_{(4)}^{ab} + \cdots. \quad (3.31)$$

Note that $\delta_D \pi^a_{b(n)} = -n \pi^a_{b(n)}$. By using this relation, the identity (3.26), and the Hamiltonian constraint we can obtain a recursive algorithm for $\mathcal{L}_{(n)}$ and $\tilde{\mathcal{L}}_{(2)}$.

1. We know that from the Einstein action the canonical momentum is defined as¹

$$\pi^{ab} = -\frac{1}{16\pi G_N} (K h^{ab} - K^{ab}). \quad (3.32)$$

In the Fefferman-Graham coordinates

$$K_{ab} = \frac{1}{2} \frac{\partial}{\partial r} \gamma_{ab}. \quad (3.33)$$

¹There is a difference of the volume factor $\sqrt{h}$ from the usual definition.

Also we know that γ_{ab} behaves as

$$h_{ab} \sim e^{2r} g_{(0)ab}. \quad (3.34)$$

This shows the leading term

$$\pi_{(0)}^{ab} = -\frac{1}{16\pi G_N} h^{ab}. \quad (3.35)$$

2. By using the identity (3.26), we have

$$\begin{aligned} & 2(\pi_{(0)} + \pi_{(2)} + \tilde{\pi}_{(2)} \log e^{-2r} + \pi_{(4)} + \dots) \\ &= 2\mathcal{L}_{(0)} + 0 \cdot \tilde{\mathcal{L}}_{(2)} \log e^{-2r} - 2\tilde{\mathcal{L}}_{(2)} + 0 \cdot \mathcal{L}_{(2)} - 2\mathcal{L}_{(4)} + \dots \end{aligned} \quad (3.36)$$

By picking up the equal-weight terms we have the equations

$$\begin{aligned} \mathcal{L}_{(0)} &= \pi_{(0)} = -\frac{1}{8\pi G_N}, \quad \tilde{\pi}_{(2)} = 0, \\ \tilde{\mathcal{L}}_{(2)} &= -\pi_{(2)}, \\ \mathcal{L}_{(4)} &= -\pi_{(4)}, \quad \dots \end{aligned} \quad (3.37)$$

Note that there is no equation to determine $\mathcal{L}_{(2)}$.

3. To determine $\pi_{(n)}$ from $\pi_{(0)}^{ab}$ we should have equations between the $\pi_{(n)}$. These are obtained from the Hamiltonian constraint,

$$0 = \mathcal{H} = 16\pi G_N (\pi_a^b \pi_b^a - \pi^2) + \frac{1}{16\pi G_N} R[h] + \frac{1}{8\pi G_N}. \quad (3.38)$$

By expanding this equation in the dilatation operator, we have

$$16\pi G_N (\pi_{(0)a}^b \pi_{(0)b}^a - \pi_{(0)}^2) + \frac{1}{8\pi G_N} = 0, \quad (3.39)$$

$$\frac{1}{16\pi G_N} R[h] + 32\pi G_N (\pi_{(0)}^{ab} \pi_{(2)ab} - \pi_{(0)} \pi_{(2)}) = 0, \quad (3.40)$$

$$2\pi_{(4)a}^b \pi_{(0)b}^a + \pi_{(2)a}^b \pi_{(2)b}^a - 2\pi_{(4)} \pi_{(0)} - \pi_{(2)}^2 = 0. \quad (3.41)$$

4. From $\pi_{(n)}$ we can read off the Lagrangian density $\mathcal{L}_{(n)}$. And from these $\mathcal{L}_{(n)}$, we can in turn determine $\pi_{(n)}^{ab}$.

From the second equation of the Hamiltonian constraint we obtain

$$\pi_{(2)} = -\frac{1}{32\pi G_N} R[h], \quad \tilde{\mathcal{L}}_{(2)} = \frac{1}{32\pi G_N} R[h]. \quad (3.42)$$

Thus we have a counterterm

$$S_{\text{ct}} = -\frac{1}{8\pi G_N} \int_{\Sigma_r} d^2x \sqrt{h} \left(-1 + \frac{1}{4} R[h] \log e^{-2r} \right). \quad (3.43)$$

We also have

$$\tilde{\pi}_{(2)}^{ab} = \frac{1}{64\pi G_N} \left(R[h]^{ab} - \frac{1}{2} R[h] h^{ab} \right) = 0. \quad (3.44)$$

This is consistent with the condition.

The renormalized sourced one-point function is defined as

$$\langle T^{ab} \rangle_{\text{ren}} := -\frac{2}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{CFT}}}{\delta g_{(0)ab}} = -2 \lim_{r_c \rightarrow \infty} e^{2r_c} \pi^{(2)ab}, \quad (3.45)$$

From the momentum constraint at order d , we have

$$D_b \pi_{(d)a}^b = 0 \quad \Leftrightarrow \quad D_b \langle T^{ab} \rangle_{\text{ren}} = 0. \quad (3.46)$$

Next, consider the diffeomorphism $r \rightarrow r + \delta\sigma(x)$, which corresponds to a Weyl transformation of the boundary theory. Then we find

$$\delta_\sigma S_{\text{ren}} = \int d^d x \delta\sigma \delta_D(\sqrt{g_{(0)}} \mathcal{L}_{(d)}) = \int d^d x \sqrt{g_{(0)}} \delta\sigma \left(-2 \tilde{\mathcal{L}}_{(d)} \right). \quad (3.47)$$

It is therefore natural to define the conformal anomaly $\mathcal{A}$ as

$$\mathcal{A} = 2 \tilde{\mathcal{L}}_{(d)}. \quad (3.48)$$

From (3.37), we obtain

$$\langle T_a^a \rangle_{\text{ren}} = \mathcal{A}. \quad (3.49)$$

For example,

$$\begin{aligned} d=2: \quad \mathcal{A} &= \frac{1}{16\pi G_N} R[g_{(0)}], \\ d=4: \quad \mathcal{A} &= \frac{1}{320G_N} \left(-\frac{1}{3} R[g_{(0)}]^2 + R_{ab}[g_{(0)}] R^{ab}[g_{(0)}] \right) \propto E_{(4)} + I_{(4)}, \\ E_{(4)} &= R_{abcd}[g_{(0)}] R^{abcd}[g_{(0)}] - 4R_{ab}[g_{(0)}] R^{ab}[g_{(0)}] + R[g_{(0)}]^2, \\ I_{(4)} &= -R_{abcd}[g_{(0)}] R^{abcd}[g_{(0)}] + 4R_{ab}[g_{(0)}] R^{ab}[g_{(0)}] - \frac{1}{3} R[g_{(0)}]^2. \end{aligned} \quad (3.50)$$

For $d = 2$, we can relate the central charge and Newton constant by

$$c = \frac{3L_{\text{AdS}}}{2G_N^{(3)}}, \quad (3.51)$$

known as the Brown-Henneaux formula [74].

3.2.3 Large N factorization and QFT on AdS

In a large N gauge theory or large c CFT, it is typically expected or proved that the single-trace light operators show large N or large c factorization.² This is the statement that the OPE (operator product expansion) is given by a generalized free theory in the large N limit. For large but finite N we expect that

$$\begin{aligned} \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_{2n}(x_{2n}) \rangle &= \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle \langle \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle \cdots \langle \mathcal{O}_{2n-1}(x_{2n-1}) \mathcal{O}_{2n}(x_{2n}) \rangle \\ &+ (\text{permutations}) + O\left(\frac{1}{N^\#}\right). \end{aligned} \quad (3.52)$$

This operator algebraic structure for light operators can be obtained by the weakly coupled QFT on AdS spacetime. For example, for a scalar operator we consider

$$I_{\text{scalar}}[\Phi] = \int_M d^{d+1}x \sqrt{G} [\nabla_A \Phi \nabla^A \Phi + m^2 \Phi^2] + \frac{\lambda_\Phi}{N^\#} \int d^{d+1}x \sqrt{G} \Phi^3 + \cdots. \quad (3.53)$$

Now we try to evaluate the leading part of the scalar correlation functions. We consider a free massive scalar field on vacuum AdS. The on-shell action is evaluated by GKPW dictionary (3.15) as

$$\begin{aligned} I_{\text{scalar}} &= \int d^d x [z^{-(d-1)} \Phi(z, x) \partial_z \Phi(z, x)]_{z=\epsilon} \\ &= \int d^d k \Phi(k) \Phi(-k) \frac{z K_\nu(|k|z) \partial_z K_\nu(|k|z)|_{z=\epsilon}}{\epsilon^d K_\nu(|k|\epsilon)^2} \\ &\equiv \int d^d k \Phi(k) \Phi(-k) G(k), \end{aligned} \quad (3.54)$$

where the Green function is computed as (in the $\epsilon \rightarrow 0$ limit)

$$G(k) = - \left(\frac{\epsilon |k|}{2} \right)^{2\nu} \cdot 2\nu \cdot \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}. \quad (3.55)$$

²Here the terminology “light” refers to operators whose insertions correspond to perturbative excitations of the bulk gravity theory.

It is useful to note that the scalar field solution behaves near the boundary $z \rightarrow 0$ as

$$\Phi(z, x) \simeq \frac{\pi}{2 \sin(\nu\pi)} e^{ikx} \left(\frac{1}{\Gamma(1-\nu)} \left(\frac{\epsilon|k|}{2} \right)^{-\nu} z^{d-\Delta_+} - \frac{1}{\Gamma(1+\nu)} \left(\frac{\epsilon|k|}{2} \right)^{\nu} z^{\Delta_+} \right) \quad (3.56)$$

We write this as

$$\Phi(z, x) \simeq \alpha(x) z^{d-\Delta_+} + \beta(x) z^{\Delta_+}. \quad (3.57)$$

α and β are interpreted as the source (external field) and the expectation value $\langle \mathcal{O}(x) \rangle$ in the dual CFT, respectively. We also find the relation

$$\beta(k) = -G(k) \alpha(k). \quad (3.58)$$

This relation will be directly derived from

$$G(k) = \frac{\delta \langle \mathcal{O}(x) \rangle}{\delta J(x)}, \quad \beta(k) = \langle \mathcal{O}(k) \rangle, \quad \alpha(k) = J(k). \quad (3.59)$$

3.3 Holographic entanglement entropy

There is a famous RT/HRT prescription for computing the entanglement entropy holographically [75, 76]. The RT formula is as follows:

$$S_A = \min_{\Gamma_A} \frac{\text{Area}[\Gamma_A]}{4G_N}. \quad (3.60)$$

Here, the minimum is taken over co-dimension two surfaces Γ_A such that $\partial \Gamma_A = \partial A$, and Γ_A is homologous to A . The Γ_A that realizes the minimum is called the Ryu-Takayanagi (RT) surface.

■ Simple example: vacuum entanglement

Now we consider a simple case. We consider a vacuum AdS_3 spacetime, for example with Poincare coordinates.

$$ds_{\text{Poincare}}^2 = \frac{dz^2 + dt_E^2 + dx^2}{z^2}, \quad z = e^{-r}. \quad (3.61)$$

For three dimensional spacetime, the codimension 2 extreme surface is just a geodesics. Then by using the embedding method discussed in the Appendix A, we can easily compute the geodesics length connecting two points $P = (z_P = \epsilon, t_P = t_b, x_P = \frac{l}{2})$

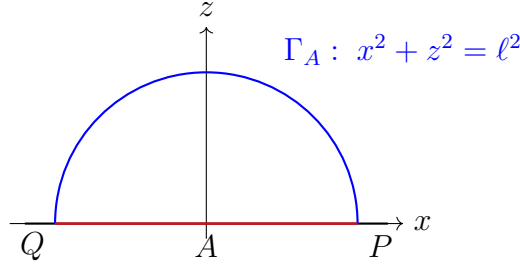


Figure 9: The sketch of the RT surface for the Poincare AdS₃.

and $Q = (z_Q = \epsilon, t_Q = t, x_Q = -\frac{l}{2})$,

$$\text{Dist}(P : Q) = L_{\text{AdS}} \cosh^{-1} \left(\frac{2\epsilon^2 + l^2}{2\epsilon^2} \right). \quad (3.62)$$

Then by using the relation (3.51), we obtain the entanglement entropy for subsystem A such that $\partial A = \{P, Q\}$, by

$$S_A = \frac{\text{Dist}(P : Q)}{4G_N} = \frac{c}{3} \log \left(\frac{l}{\epsilon} \right) + O(\epsilon^0) \quad (3.63)$$

with relation $\epsilon \ll l$. This logarithmic behavior is universal in gapless 1 + 1 dimensional systems and we can explicitly show this for two dimensional CFTs including holographic CFTs [77, 78]. In this case, the RT surface is given by

$$x^2 + z^2 = l^2 \quad (3.64)$$

as in Figure 9.

■ Heuristic proof via gravitational replica

There is a heuristic proof of the RT formula [79–81]. I am going to give a quick review of the proof. The key points are as follows:

- In AdS/CFT, the planar limit $c \rightarrow \infty$ corresponds to the weak-gravity limit, and we only consider the saddle point of the gravitational path integral with prescribed boundary conditions.
- In the QFT replica computation, we consider the replica manifold B_q and compute the partition function on the replica manifold $Z[B_q]$. To this end, we compute the on-shell action of the bulk manifold M_q with boundary conditions $\partial M_q = B_q$.

The general strategy for the proof is following:

1. Consider solutions of the Einstein equation with boundary conditions $\partial M_q = B_q$.

2. Evaluate the on-shell action for these solutions.
3. Compute the Rényi entropy given by

$$S_A^{(q)} = \frac{1}{1-q} \log \left(\frac{\text{Tr } \rho_A^q}{(\text{Tr } \rho_A)^q} \right) \quad (3.65)$$

$$= \frac{1}{1-q} \log \left(\frac{Z[B_q]}{Z[B]^q} \right) \quad (3.66)$$

$$= \frac{1}{1-q} \log \left(\frac{Z_{\text{grav}}[M_q]}{Z_{\text{grav}}[M]^q} \right). \quad (3.67)$$

Since we are now focusing on the semi-classical limit, we will have

$$\begin{aligned} Z_{\text{grav}}[M] &= e^{-I_{\text{grav}}[M]}, \\ I_{\text{grav}}[M] &= I_{\text{EH}}[M] + I_{\text{GH}}[\partial M]. \end{aligned} \quad (3.68)$$

Here we use the notation that $I_{\text{grav}}[M]$ contains the boundary terms for all the boundaries that M has. The computation of the partition function $Z_{\text{grav}}[M_q]$ is a bit tricky and we will see it later.

4. Analytically continue $q \in \mathbb{Z}$ to $q \in \mathbb{R}$ and take the $q \rightarrow 1$ limit.

The argument in [81] can be divided into two parts:

1. Kinematical part: Providing the essence of how to implement analytic continuation.
2. Dynamical part: Ensuring that the contribution of the saddle point configuration gives the modular entropy.

Now, we go into the details.

Kinematical part

There are a few key assumptions:

- Assumption 1: We can extend the replica symmetry $\mathbb{Z}_q$ on the boundary into the bulk.
- Assumption 2: The fixed points of the $\mathbb{Z}_q$ action in the bulk are co-dimension 2 loci.

Inspired by Assumption 1, we can consider the orbifold by $\mathbb{Z}_q$:

$$\hat{M}_q := M_q / \mathbb{Z}_q. \quad (3.69)$$

The $\hat{M}_q$ is almost topologically equivalent to the original manifold M , but $\hat{M}_q$ has a singular locus e_q , which is the set of fixed points of the $\mathbb{Z}_q$ action. Recalling the

fact that the entangling surface ∂A is the $\mathbb{Z}_q$ fixed locus on the boundary replicated manifold and is a co-dimension 2 surface on the boundary, the singular locus e_q is some extension of the entangling surface into the bulk. This intuition is locally valid in the neighborhood of the conformal boundary but may not be true globally. So a natural question to ask is when the assumptions are true. The answer is as follows: as long as one has a family of replica symmetries parametrized by a positive integer q and any result is smooth for integer q , we can extend the local statement about the singular locus to a global one.

We now move to the gravitational setup. From now on, we focus on $\hat{M}_q$ rather than M_q . As mentioned above, $\hat{M}_q$ is almost the same as M , except for the co-dimension 2 singular locus e_q . We can think of the manifold $\hat{M}_q$ as the manifold M with a conical defect e_q , which causes backreaction and deforms M into $\hat{M}_q$. The tension of the defect e_q is given by:

$$T_q = \frac{1}{4G_N} \frac{q-1}{q}. \quad (3.70)$$

This is because when the manifold has a co-dimension two conical defect, the Ricci curvature behaves as:

$$R = \alpha \delta(y). \quad (3.71)$$

Here, α denotes the deficit angle, and y denotes coordinates transverse to the defect. Now the orbifold possesses the deficit angle $\alpha = 2\pi - \frac{2\pi}{q}$. Suppose the defect can be described by the action:

$$I_{\text{brane}} = T_q \int_{e_q} \sqrt{h}, \quad (3.72)$$

where h_{ab} describes the induced metric on the locus; then from Einstein's equation, we finally obtain the tension.

Let us summarize the discussion above: we can compute the geometry of $\hat{M}_q$ and also M_q by starting from the original manifold M with a co-dimension two defect e_q with tension T_q . This claim is important because the effect of the replica index q is converted into the tension of the defect, and this ensures that we can safely analytically continue the integer q to real values.

Consider the manifold $M_q(\epsilon)$, which is M_q with a hollowed-out tubular neighborhood of radius ϵ around the $\mathbb{Z}_q$ fixed line. Recall that M_q is a smooth manifold. Then the on-shell action of M_q is

$$I_{\text{grav}}[M_q] = \lim_{\epsilon \rightarrow 0} I_{\text{grav}}[M_q(\epsilon)] = q \lim_{\epsilon \rightarrow 0} I_{\text{grav}}[\hat{M}_q(\epsilon)] =: q I[\hat{M}_q]_{\text{exploit brane}}. \quad (3.73)$$

Here we would like to stress that $I[\hat{M}_q]_{\text{exploit brane}}$ does not possess the boundary

term for the co-dimension one surface $e_q(\epsilon)$.

Finally, we conclude the kinematical part with a few comments about the advantages of the approach:

- As mentioned above, when we consider the analytic continuation of the integer q to real values, in the gravitational perspective, q is just a parameter for the tension of the cosmic brane e_q , and the analytic continuation should be safe.
- We can write down the metric of the original manifold M around the brane:

$$ds^2 \sim dx^2 + dt^2 + (\gamma_{ij} + 2K_{ij}^x x + 2K_{ij}^t t) dy^i dy^j, \quad (3.74)$$

where K_{ij}^a , $a = x, t$, are the extrinsic curvatures with respect to the normal vectors $a = x, t$.

Dynamical part

Here we want to figure out which configuration dominates the partition function and compute the on-shell action in practice. The metric of M_q around the brane e_q in polar coordinates $x \pm it_E = e^{\pm i\tau}$, y^i , should satisfy the following conditions:

- $\mathbb{Z}_q$ symmetry, i.e. invariance under $\tau \rightarrow \tau + 2\pi$,
- periodicity $\tau \sim \tau + 2\pi$,
- smoothness of M_q ,
- in the $q \rightarrow 1$ limit, it coincides with the original metric (3.74).

An ansatz for such a metric is the following:

$$ds^2 = (q^2 dr^2 + d\tau^2) + (\gamma_{ij}(y) + 2K_{ij}^x r^q \cos \tau + 2K_{ij}^t r^q \sin \tau) dy^i dy^j + \dots \quad (3.75)$$

This is natural from the conditions, but let us add some comments on why r^q appears in the second term. To this end, let us consider a massless real scalar field on the background $ds_{(0)}^2 = q^2 dr^2 + d\tau^2 + \gamma_{ij}(y) dy^i dy^j$. The equation of motion for the mode $\phi = f(r) e^{i\omega\tau} Y_\omega(y)$ is:

$$\frac{1}{q^2} r \frac{d}{dr} \left(r \frac{d}{dr} f \right) - \omega^2 f = 0. \quad (3.76)$$

Consider solutions of this equation around the brane $r \sim 0$ with an ansatz $f \sim r^s$. We obtain $s = \pm q\omega$. Then around the brane:

$$\phi \sim (r^q e^{-i\tau})^\omega Y_\omega(y). \quad (3.77)$$

So the on-shell mode arises from the combination $r^q e^{-i\tau}$ rather than $r e^{i\tau}$. Note that we have the replica symmetry which requires 2π periodicity in τ , so we learn that $\omega \in \mathbb{Z}$. Also the r -expansion is in powers of r^q . This guarantees that when q is an integer the field is smooth everywhere.

The computation is the same for the perturbed metric $G_{AB} = G_{AB}^{(0)} + \delta G_{AB}$.

The quotient operation does not change the functional form of the metric. Thus, we expect the metric of $\hat{M}_q$ around the defect to be again:

$$ds^2 = (q^2 dr^2 + d\tau^2) + (\gamma_{ab}(y) + 2K_{ab}^x r^q \cos \tau + 2K_{ab}^t r^q \sin \tau) dy^i dy^j + \dots \quad (3.78)$$

Importantly, the range of the angle τ is different from that in M_q . It is $\tau \sim \tau + 2\pi$ by construction. This corresponds to the conical singularity at the edge with the deficit angle $2\pi(1 - \frac{1}{q})$. Once we have an ansatz, we can directly compute the field equations. Then we obtain

$$\text{curvature} \sim \frac{q-1}{r} K^a + \dots \quad (3.79)$$

Actually, none of the higher terms can cancel this since they are just regular polynomials. So the only way the equations of motion are satisfied is simply $K^a = 0$ for both $a = x, t$. However, because we are now considering a static spacetime and it should have reflection symmetry $t \rightarrow -t$, K^t automatically vanishes. Thus the only condition we should impose is $K^x = 0$. This leads us to the conclusion that, to satisfy the equations of motion, e_q should be a minimal surface in the $q \rightarrow 1$ limit.

Finally we move to the evaluation of the on-shell action and the entropy. To evaluate the on-shell action we would in principle need the global information about the bulk manifold. But there is a trick. To compute the entanglement entropy we use, rather than the Rényi entropy, the so-called modular entropy defined by

$$\tilde{S}_A^{(q)} := \frac{1}{q^2} \partial_q \left(\frac{q-1}{q} S_A^{(q)} \right). \quad (3.80)$$

It is easy to see that this matches the entanglement entropy in the $q \rightarrow 1$ limit. The Rényi entropy is expressed in terms of the gravitational action as

$$S_A^{(q)} = \frac{1}{1-q} (\log Z_{\text{grav}}[M_q] - q \log Z_{\text{grav}}[M]) = \frac{q}{q-1} \left(I[\hat{M}_q]_{\text{exploit locus}} - I_{\text{grav}}[M] \right), \quad (3.81)$$

and the modular entropy now becomes

$$\tilde{S}_A^{(q)} = \frac{1}{q^2} \partial_q I[\hat{M}_q]_{\text{exploit locus}}. \quad (3.82)$$

The on-shell metric depends on q and we can regard ∂_q as a deformation of the on-shell metric.

To evaluate the q -derivative in an elegant manner, let us recall the variational problem in general relativity. In general, any variation of the Einstein–Hilbert action for some manifold M with boundary ∂M is

$$\delta I_{\text{EH}}[M] = \int_M [\mathbb{E}^{AB} \delta G_{AB} + d\Theta(G_{AB}, \delta G_{AB})] \triangleq \int_{\partial M} \Theta(G_{AB}, \delta G_{AB}), \quad (3.83)$$

where $\mathbb{E}^{AB}$ is a $(d+1)$ -form expressing the equations of motion (the Einstein equations), and $\Theta(G, \delta G)$ is a presymplectic form. The total action is $I_{\text{grav}}[M] = I_{\text{EH}}[M] + I_{\text{GH}}[\partial M]$. The Gibbons–Hawking term $I_{\text{GH}}[\partial M]$ is added so that we have a well-defined variational problem; that is,

$$\begin{aligned} \delta I_{\text{grav}}[M] &= \int_M \mathbb{E}^{AB} \delta G_{AB}, \\ \int_{\partial M} \Theta(G_{AB}, \delta G_{AB}) + \delta I_{\text{GH}}[\partial M] &= 0. \end{aligned} \quad (3.84)$$

Let us go back to the problem of evaluating (3.82). One key point is that our gravitational action $I[\hat{M}_q]_{\text{exploit locus}}$ has the Gibbons–Hawking term only for the conformal boundary B_q and not for $e_q(\epsilon)$. Thus

$$\delta I[\hat{M}_q]_{\text{exploit locus}} = \lim_{\epsilon \rightarrow 0} \delta (I_{\text{grav}}[\hat{M}_q(\epsilon)] - I_{\text{GH}}[e_q(\epsilon)]) = - \lim_{\epsilon \rightarrow 0} \delta I_{\text{GH}}[e_q(\epsilon)] \quad (3.85)$$

for any variation δ . In the last line, we use the fact that $\hat{M}_q(\epsilon)$ satisfies the Einstein equation since there is no singularity, and M is also an on-shell geometry. So

$$\partial_q I[\hat{M}_q]_{\text{exploit locus}} = - \lim_{\epsilon \rightarrow 0} \partial_q I_{\text{GH}}[e_q(\epsilon)] \quad (3.86)$$

and

$$\lim_{\epsilon \rightarrow 0} \delta I_{\text{GH}}[e_q(\epsilon)] = \frac{1}{8\pi G_N} \int_{e_q(\epsilon)} \sqrt{h} K. \quad (3.87)$$

With the above metric (3.78) and using the fact $K^a = 0$, we can evaluate the extrinsic curvature K as

$$K = \frac{1}{q\epsilon}, \quad (3.88)$$

and $\partial_q h = 0$. Finally,

$$- \lim_{\epsilon \rightarrow 0} \partial_q I_{\text{GH}}[e_q(\epsilon)] = \frac{\text{Area}[e_q]}{4q^2 G_N}. \quad (3.89)$$

So taking the limit and we obtain

$$S_A = \frac{\text{Area}[\Gamma_A]}{4G_N}. \quad (3.90)$$

For the time-dependent non-equilibrium situation, by repeating the above argument in the Schwinger-Keldysh path integral, we obtain the HRT formula [76]

$$S_A = \min_{\Gamma_A^{\text{Ext}}} \text{Ext}_{\Gamma_A} \left[\frac{\text{Area}[\Gamma_A]}{4G_N} \right]. \quad (3.91)$$

with the appropriate extremal surface in the Lorentzian geometry rather than the minimal surface. Minimal surface does not meaning since we can always take surface to be null.

By considering the perturbation of the replica gravitational path integral, we can discuss the quantum corrections to the RT/HRT formula. Especially we need to consider a bulk QFT entanglement between regions separated by the RT surface. In that case we may think there is a bulk UV divergence from matter entanglement. However, similar to the generalized entropy, we can avoid this divergence by renormalizing the Newton constant. Especially we can easily verify this point in AdS/CFT by using the scale/radius duality. Suppose we consider asymptotic AdS spacetime whose boundary is given by a cylinder with finite spatial volume. In this case there is no IR-type divergence in the boundary. From scale/radius duality, we can relate the bulk UV structure to the IR structure in the boundary QFT. So there should not be any bulk UV divergence and they should be renormalized.

3.4 SYK/JT correspondence

In this subsection, we give a quick review of the SYK model and JT gravity, which is a nice example of low-dimensional holography [33, 34, 82]. Here we refer to “low dimension” as holographic correspondence between two-dimensional gravity and one-dimensional quantum mechanics. Since in two-dimensional spacetime the Einstein–Hilbert action is topological,

$$\int_M d^2x \sqrt{G} (R[G] + 2) + 2 \int_{\partial M} dy \sqrt{\gamma} K[\gamma] = 4\pi \chi[M], \quad (3.92)$$

where $\chi[M]$ is the Euler characteristic. To get non-trivial gravitational dynamics we need to add new degrees of freedom which are not topological. Also, on the boundary side, one-dimensional conformal theories have vanishing Hamiltonian because of the traceless condition from conformal symmetry,

$$H = 0. \quad (3.93)$$

Thus there is no non-trivial time evolution. To get interesting dynamics we need to deviate from this conformal fixed point. In conclusion, to get non-trivial holography for low-dimensional systems we need to consider nearly AdS / nearly CFT correspondence. The SYK/JT correspondence is a particular realization of nearly AdS / nearly CFT correspondence.

3.4.1 SYK models

The SYK models are defined by theories of Majorana fermion quantum mechanics with q -body interactions. Here we consider the Schwinger–Keldysh path integral for SYK models for later convenience. The Schwinger–Keldysh path integral of the SYK model is given by

$$\begin{aligned}
Z_{\text{SK}} &= \int \mathcal{D}\psi_{\pm} \exp\{iS_{\text{tot}}[\psi_+, \psi_-]\}, \\
iS_{\text{tot}}[\psi_+, \psi_-] &= iS_{\text{SYK}}[\psi_+] - iS_{\text{SYK}}[\psi_-] \\
&= -\frac{1}{2} \int_I dt \sum_{a=\pm} \epsilon_a \psi_a^i(t) \partial_t \psi_a^i(t) \\
&\quad - i^{q/2+1} \int_I dt \sum_{1 \leq i_1 < \dots < i_q \leq N} j_{i_1 i_2 \dots i_q} f_q(t) \sum_a \epsilon_a \psi_a^{i_1 i_2 \dots i_q}(t),
\end{aligned} \tag{3.94}$$

where

$$\epsilon_+ = -\epsilon_- = 1, \tag{3.95}$$

and $f_q(t)$ is some function of time.

J -integral

We now perform the $j_{i_1 i_2 \dots i_q}$ integral. The probabilistic measure is given by the Gaussian ensemble,

$$\begin{aligned}
[dj_{i_1 i_2 \dots i_q}] &= \prod_{1 \leq i_1 < \dots < i_q \leq N} dj_{i_1 i_2 \dots i_q} P_q(j_{i_1 i_2 \dots i_q}), \\
P_q(j_{i_1 i_2 \dots i_q}) &= A \exp \left\{ -\frac{1}{2\sigma_q^2} \sum_{1 \leq i_1 < \dots < i_q \leq N} j_{i_1 i_2 \dots i_q}^2 \right\}, \\
\sigma_q^2 &= \frac{J_q^2 (q-1)!}{N^{q-1}}, \quad A = \sqrt{\frac{1}{2\pi\sigma_q^2}}.
\end{aligned} \tag{3.96}$$

The relevant integral for our action is

$$\int [dj_{i_1 i_2 \dots i_q}] \exp \left\{ - \sum_{1 \leq i_1 < \dots < i_q \leq N} j_{i_1 i_2 \dots i_q} B^{i_1 i_2 \dots i_q} \right\} = \exp \left\{ \frac{\sigma_q^2}{2} \sum_{i_1 i_2 \dots i_q} (B^{i_1 i_2 \dots i_q})^2 \right\}. \quad (3.97)$$

Here we take

$$B^{i_1 i_2 \dots i_q} = i^{\frac{q}{2}+1} \int dt f_q(t) \sum_a \epsilon_a \psi_a^{i_1 i_2 \dots i_q}(t). \quad (3.98)$$

After simplification, we obtain

$$\begin{aligned} & \exp \left\{ \frac{\sigma_q^2}{2} \sum_{i_1 i_2 \dots i_q} (B^{i_1 i_2 \dots i_q})^2 \right\} \\ &= \exp \left\{ - \frac{N i^q J_q^2}{2q} \sum_{ab} s_{ab} \int dt_1 \int dt_2 f_q(t_1) f_q(t_2) \left(\frac{i}{N} \sum_i \psi_a^i(t_1) \psi_b^i(t_2) \right)^q \right\}, \end{aligned} \quad (3.99)$$

where

$$s_{ab} = \epsilon_a \epsilon_b, \quad s_{++} = s_{--} = 1, \quad s_{+-} = s_{-+} = -1. \quad (3.100)$$

Introducing bi-local fields G, Σ

Importantly, we have a nice mean-field expression in terms of Keldysh Green functions

$$G_{ab}(t_1, t_2) := -\frac{i}{N} \sum_i \langle \psi_a^i(t_1) \psi_b^i(t_2) \rangle, \quad (3.101)$$

and the self-energy $\Sigma_{ab}(t_1, t_2)$. To see this, we insert the identity in the path integral:

$$\begin{aligned} 1 &= \int \mathcal{D}G_{ab} \delta \left[G_{ab}(t_1, t_2) + \frac{i}{N} \sum_i \psi_a^i(t_1) \psi_b^i(t_2) \right] \\ &= \int \mathcal{D}G_{ab} \mathcal{D}\Sigma_{ab} \exp \left\{ i \sum_{\alpha, \beta} \int dt_1 \int dt_2 \Sigma_{ab}(t_1, t_2) G_{ab}(t_1, t_2) + \frac{i}{N} \sum_i \psi_a^i(t_1) \psi_b^i(t_2) \right\} \\ &= \int \mathcal{D}G_{ab} \mathcal{D}\Sigma_{ab} \exp \left\{ -N \sum_{\alpha, \beta} \int dt_1 \int dt_2 s_{ab} \Sigma_{ab}(t_1, t_2) G_{ab}(t_1, t_2) \right\} \\ &\quad \times \exp \left\{ \frac{i}{N} \sum_i \int dt_1 \int dt_2 \sum_{\alpha, \beta} \psi_a^i(t_1) \Sigma_{ab}(t_1, t_2) \psi_b^i(t_2) \right\}, \end{aligned} \quad (3.102)$$

where in the last equality we changed variables $\Sigma_{ab} \rightarrow \frac{iN}{2} s_{ab} \Sigma_{ab}$. The reason for introducing the s_{ab} factor is to match expressions with complex time contours. Finally,

substituting the above expression, we obtain

$$\begin{aligned}
Z_{\text{SK}} = & \int \mathcal{D}G_{ab} \mathcal{D}\Sigma_{ab} \mathcal{D}\psi_a \\
& \times \exp \left[-\frac{1}{2} \int dt \sum_i \sum_a \epsilon_a \psi_a^i(t) \partial_t \psi_a^i(t) \right. \\
& \quad \left. - \frac{i}{2} \int dt_1 dt_2 \sum_{ab} s_{ab} \sum_i \psi_a^i(t_1) \Sigma_{ab}(t_1, t_2) \psi_b^i(t_2) \right] \\
& \times \exp \left[-\frac{N(i)^q J_q^2}{2q} \int dt_1 dt_2 \sum_{ab} s_{ab} f_q(t_1) f_q(t_2) (G_{ab}(t_1, t_2))^q \right. \\
& \quad \left. + \int dt_1 dt_2 \sum_{ab} s_{ab} \Sigma_{ab}(t_1, t_2) G_{ab}(t_1, t_2) \right].
\end{aligned} \tag{3.103}$$

Fermion path integrals

Finally, by completing the fermion path integral we obtain the effective action in terms of the bi-local fields

$$\begin{aligned}
Z_{\text{SK}} = & \int \mathcal{D}G_{ab} \mathcal{D}\Sigma_{ab} \exp\{N I_{\text{eff}}[G, \Sigma]\}, \\
I_{\text{eff}}[G, \Sigma] = & \frac{1}{2} \text{Tr} \log [-i(G_0^{-1} - \Sigma')] - \frac{i^q J_q^2}{2q} \int dt_1 \int dt_2 \sum_{ab} s_{ab} f_q(t_1) f_q(t_2) (G_{ab}(t_1, t_2))^q \\
& - \frac{1}{2} \int dt_1 \int dt_2 \sum_{ab} s_{ab} \Sigma_{ab}(t_1, t_2) G_{ab}(t_1, t_2),
\end{aligned} \tag{3.104}$$

where

$$\begin{aligned}
(G_0^{-1})_{ab}(t_1, t_2) &= \epsilon_a \delta_{ab} i \partial_{t_1} \delta(t_1 - t_2), \\
\Sigma'_{ab}(t_1, t_2) &= s_{ab} \Sigma_{ab}(t_1, t_2).
\end{aligned} \tag{3.105}$$

■ Saddle point equation

In the large N limit, we can consider the saddle point equation. We obtain

$$\begin{aligned}
\frac{\delta I_{\text{eff}}}{\delta \Sigma'_{ab}} = 0 & \Leftrightarrow G_0^{-1} - G^{-1} = \Sigma', \\
\frac{\delta I_{\text{eff}}}{\delta G_{ab}} = 0 & \Leftrightarrow \Sigma_{ab} = -i^q J_q^2 f_q(t_1) f_q(t_2) (G_{ab}(t_1, t_2))^{q-1}.
\end{aligned} \tag{3.106}$$

■ Kadanoff-Baym equation

Finally, from the saddle-point-equation we derive the Kadanoff-Baym equations.

Firstly, we multiply G from the left and right for (3.106) and we obtain

$$\begin{aligned}\epsilon_a i \partial_{t_1} G_{ab}(t_1, t_2) &= \delta_{ab} \delta(t_1 - t_2) + \int_I dt_3 \sum_c s_{ac} \Sigma_{ac}(t_1, t_3) G_{cb}(t_3, t_2), \\ -\epsilon_a i \partial_{t_2} G_{ab}(t_1, t_2) &= \delta_{ab} \delta(t_1 - t_2) + \int_I dt_3 \sum_c G_{ac}(t_1, t_3) s_{cb} \Sigma_{cb}(t_3, t_2).\end{aligned}\tag{3.107}$$

Next, we consider the $ab = -+$ component of these equalities. To derive the Kadanoff-Baym equations, we need to recall the basis of the non-equilibrium Green functions:

The correlators on closed time contours are related to the non-equilibrium Green functions as follows:

$$\begin{aligned}G_{++}(t_1, t_2) &= G^F(t_1, t_2), \quad G_{--}(t_1, t_2) = G^{\bar{F}}(t_1, t_2), \\ G_{+-}(t_1, t_2) &= G^>(t_1, t_2), \quad G_{-+}(t_1, t_2) = G^<(t_1, t_2).\end{aligned}\tag{3.108}$$

The Feynman and retarded and advanced Green functions are defined from the Wightman functions as

$$\begin{aligned}G^F(t_1, t_2) &= \Theta(t_1 - t_2) G^>(t_1, t_2) + \Theta(t_2 - t_1) G^<(t_1, t_2), \\ G^{\bar{F}}(t_1, t_2) &= \Theta(t_2 - t_1) G^>(t_1, t_2) + \Theta(t_1 - t_2) G^<(t_1, t_2), \\ G^R(t_1, t_2) &= \Theta(t_1 - t_2) (G^>(t_1, t_2) - G^<(t_1, t_2)), \\ G^A(t_1, t_2) &= \Theta(t_2 - t_1) (G^<(t_1, t_2) - G^>(t_1, t_2)).\end{aligned}\tag{3.109}$$

We can find that the non-equilibrium Green functions are not independent. Explicitly, we can find following relations,

$$\begin{aligned}G^F(t_1, t_2) + G^{\bar{F}}(t_1, t_2) &= G^>(t_1, t_2) + G^<(t_1, t_2) \\ G^{\bar{F}}(t_1, t_2) &= -G^R(t_1, t_2) + G^>(t_1, t_2) \\ G^F(t_1, t_2) &= G^A(t_1, t_2) + G^>(t_1, t_2).\end{aligned}\tag{3.110}$$

We then rewrite (3.107) and obtain the Kadanoff-Baym equations:

$$\begin{aligned}i \partial_{t_1} G^>(t_1, t_2) &= \int_I dt_3 [\Sigma^R(t_1, t_3) G^>(t_3, t_2) + \Sigma^>(t_1, t_3) G^A(t_3, t_2)], \\ -i \partial_{t_2} G^>(t_1, t_2) &= \int_I dt_3 [G^R(t_1, t_3) \Sigma^>(t_3, t_2) + G^>(t_1, t_3) \Sigma^A(t_3, t_2)], \\ \Sigma^>(t_1, t_2) &= -i^q J_q^2 f_q(t_1) f_q(t_2) (G^>(t_1, t_2))^{q-1}, \\ \Sigma^<(t_1, t_2) &= -i^q J_q^2 f_q(t_1) f_q(t_2) (G^<(t_1, t_2))^{q-1}.\end{aligned}\tag{3.111}$$

Note that for Majorana fermion models, we have the relation

$$G^<(t_1, t_2) = -G^>(t_1, t_2) \quad (3.112)$$

and the Kadanoff-Baym equation is the equation to determine $G^>(t_1, t_2)$.

Finally let us substitute the expression of Σ into the Kadanoff-Baym equation (3.111). Then we obtain

$$\begin{aligned} & i\partial_{t_1} G^>(t_1, t_2) \\ &= -i^q \int_{-\infty}^{t_1} dt_3 J_q(t_1) J_q(t_3) [(G^>(t_1, t_3))^{q-1} - (G^<(t_1, t_3))^{q-1}] G^>(t_3, t_2) \\ &+ i^q \int_{-\infty}^{t_2} dt_3 J_q(t_1) J_q(t_3) (G^>(t_1, t_3))^{q-1} [G^>(t_3, t_2) - G^<(t_3, t_2)] \end{aligned} \quad (3.113)$$

and

$$\begin{aligned} & -i\partial_{t_2} G^>(t_1, t_2) \\ &= -i^q \int_{-\infty}^{t_1} dt_3 J_q(t_3) J_q(t_2) [G^>(t_3, t_2) - G^<(t_3, t_2)] (G^>(t_3, t_2))^{q-1} \\ &+ i^q \int_{-\infty}^{t_2} dt_3 J_q(t_3) J_q(t_2) G^>(t_1, t_3) [(G^>(t_3, t_2))^{q-1} - (G^<(t_3, t_2))^{q-1}]. \end{aligned} \quad (3.114)$$

The task of SYK model is solve this integral equations.

■ Euclidean correlator

We can also consider the Euclidean propagator

$$G_{ab}^E(\tau) := \frac{1}{N} \sum_{i=1}^N \langle \mathcal{T}_E \psi_a^i(\tau) \psi_b^i(0) \rangle, \quad (3.115)$$

$$:= \frac{1}{N} \sum_{i=1}^N [\langle \psi_a^i(\tau) \psi_b^i(0) \rangle \theta(\tau) - \langle \psi_b^i(0) \psi_a^i(\tau) \rangle \theta(-\tau)], \quad (3.116)$$

$$G_{ab}^{E,\beta}(\tau) := \frac{1}{N} \sum_{i=1}^N \langle \mathcal{T}_E \psi_a^i(\tau) \psi_b^i(0) \rangle_\beta. \quad (3.117)$$

To get some feelings, we consider the free Majorana fermion theory at $J = 0$. In this case, the Hamiltonian is zero and

$$\langle \psi_a^i(\tau) \psi_b^i(0) \rangle = \langle \psi_a^i(0) \psi_b^i(0) \rangle = \frac{1}{2} \delta_{ab}, \quad (3.118)$$

from the anti-commutation relations. This means

$$G_{0,ab}^E(\tau) = \frac{1}{2}\delta_{ab} \operatorname{sgn}(\tau), \quad (3.119)$$

$$G_{0,ab}^{E,\beta}(\tau) = \frac{1}{2}\delta_{ab} \operatorname{sgn}\left(\sin\left(\frac{\pi\tau}{\beta}\right)\right). \quad (3.120)$$

Note that the Green function is antisymmetric,

$$G_{ab}^E(-\tau) = -G_{ba}^E(\tau), \quad G_{ab}^{E,\beta}(-\tau) = -G_{ba}^{E,\beta}(\tau), \quad (3.121)$$

which also follows from the anti-commutation relations.

In Euclidean signature we have the effective action

$$\langle Z \rangle_J = \int \mathcal{D}G_{ab}^E \mathcal{D}\Sigma_{ab}^E e^{-NI_{\text{eff}}[G^E, \Sigma^E]}, \quad (3.122)$$

$$I_{\text{eff}}[G^E, \Sigma^E] := \quad (3.123)$$

$$- \frac{1}{2} \log \det \left[(G_0^E)^{-1}(\tau_1, \tau_2) - \Sigma_{ab}^E(\tau_1, \tau_2) \right] \quad (3.124)$$

$$+ \frac{1}{2} \int d\tau_1 d\tau_2 \sum_{a,b} \left(\Sigma_{ab}^E(\tau_1, \tau_2) G_{ab}^E(\tau_1, \tau_2) - \frac{(i)^{\frac{q}{2}}}{q} (G_{ab}^E(\tau_1, \tau_2))^q \right), \quad (3.125)$$

$$(G_0^E)^{-1}(\tau_1, \tau_2) := -\delta_{ab} \delta(\tau_1 - \tau_2) \partial_{\tau_1}. \quad (3.126)$$

At large N , we can use the saddle point approximation for the path integral. The saddle point equation is exactly the Schwinger–Dyson equation,

$$G_{ab}^E(\tau_1, \tau_2) = G_{0,ab}^E(\tau_1, \tau_2) + \int d\tau_3 d\tau_4 G_{0,ac}^E(\tau_1, \tau_3) \Sigma_{cd}^E(\tau_3, \tau_4) G_{db}^E(\tau_4, \tau_2), \quad (3.127)$$

$$\Sigma_{ab}^E(\tau_1, \tau_2) = J^2 (G_{ab}^E(\tau_1, \tau_2))^{q-1}, \quad (3.128)$$

where we define $G_{ab}^E(\tau_1, \tau_2) = G_{ab}^E(\tau_1 - \tau_2)$ and self-energy $\Sigma_{ab}^E(\tau_1, \tau_2) = \Sigma_{ab}^E(\tau_1 - \tau_2)$. The first equation can be simplified in Fourier space,

$$G_{ab}^E(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} G_{ab}^E(\tau). \quad (3.129)$$

Note that by using the formula for the Fourier transform of the sign function, we see

$$G_{0,ab}^E(\omega) = \frac{i}{\omega + i0^+} \Rightarrow (G_0^E)^{-1}(\omega) = -i\delta_{ab}\omega, \quad (3.130)$$

and the first equation is rewritten as

$$(G^E)_{ab}^{-1}(\omega) = -i\delta_{ab}\omega - \Sigma_{ab}^E(\omega). \quad (3.131)$$

The Schwinger–Dyson equations can be solved

- numerically for all q ,
- exactly when $q = 2$,
- analytically in a $q \rightarrow \infty$ expansion,
- in the IR, $\omega \sim \tau^{-1} \ll J$ and strong coupling $\beta J \gg 1$.

■ Conformal symmetry in the IR

Let us consider the IR regime where $\omega \sim \tau^{-1} \ll J$ and strong coupling $\beta J \gg 1$ holds. In this case, as an approximation we can drop the G_0^E terms,

$$(G^E)_{ab}^{-1}(\omega) \approx -\Sigma_{ab}^E(\omega) \Leftrightarrow \int d\tau_2 G_{ab}^E(\tau_1, \tau_2) \Sigma_{bc}^E(\tau_2, \tau_3) \approx -\delta(\tau_1 - \tau_3). \quad (3.132)$$

Then, there is no dimensionful parameter in the Schwinger–Dyson equations and we find that the equations possess the reparametrization symmetry,

$$G_{ab}^E(\tau_1, \tau_2) \rightarrow G_{f,ab}^E(\tau_1, \tau_2) := G_{ab}^E(f(\tau_1), f(\tau_2))(f'(\tau_1)f'(\tau_2))^\Delta, \quad (3.133)$$

$$\Sigma_{ab}^E(\tau_1, \tau_2) \rightarrow \Sigma_{E,ab}^E(\tau_1, \tau_2) := \Sigma_{ab}^E(f(\tau_1), f(\tau_2))(f'(\tau_1)f'(\tau_2))^{(q-1)\Delta}, \quad (3.134)$$

with $\Delta = \frac{1}{q}$. As one particular solution, we obtain

$$G_{\text{conf},ab}^E(\tau_1, \tau_2) = B \frac{\text{sgn}(\tau_1 - \tau_2)}{|J(\tau_1 - \tau_2)|^{2\Delta}} \delta_{ab}. \quad (3.135)$$

For the finite-temperature case,

$$G_{\text{conf},ab}^{E,\beta}(\tau_1, \tau_2) = B_\beta \frac{\text{sgn}\left(\sin\left(\frac{\pi(\tau_1 - \tau_2)}{\beta}\right)\right)}{\left|\sin\left(\frac{\pi(\tau_1 - \tau_2)}{\beta}\right)\right|^{2\Delta}} \delta_{ab}. \quad (3.136)$$

In Lorentzian signature

$$G_{\text{conf}}(t_1 - t_2) \propto e^{-\frac{\pi\Delta}{\beta}(t_1 - t_2)} \quad (3.137)$$

and we see exponential decays.

■ Thermodynamics

By using the bi-local field expression, we can compute the thermodynamic quantities. We have two ways of computing when we have disorder average

1. First way is we compute the averaged Euclidean partition function $\langle Z \rangle_J$ and define the free energy

$$F = -\frac{1}{\beta} \log Z_J. \quad (3.138)$$

2. In the context of spin-glass theory, we have another way of performing the average, called the replica trick:

$$\beta \langle F \rangle_J := -\langle \log Z \rangle_J = -\lim_{n \rightarrow 0} \partial_n \langle Z^n \rangle_J. \quad (3.139)$$

It is known that the two definitions give the same result in the large N limit since the replica off-diagonal components are suppressed [83]. Indeed we can find the off-diagonal solution which is sub-dominant and

$$\langle Z^n \rangle_J = (\langle Z \rangle_J)^n + o\left(\frac{1}{N}\right). \quad (3.140)$$

By integrating out the couplings and the fermions, we obtain the effective action in terms of the bi-local fields $G_{ab}^E(\tau_1, \tau_2)$ and $\Sigma_{ab}^E(\tau_1, \tau_2)$. By substituting the solution of the saddle point equation (3.127), we can discuss the thermodynamics at large N . It is known that in the IR regime, i.e. at low temperature,

$$\beta F = \beta E_0 + N \left[-S_0 - \frac{2\pi^2 C}{\beta J} + O\left(\frac{1}{(\beta J)^2}\right) \right] + \frac{3}{2} \log \beta J + \text{const} + O\left(\frac{1}{N}\right). \quad (3.141)$$

Here, we denote S_0 as the ground state entropy ($S_0 \approx 0.232$) and E_0 as the ground state energy.

■ Schwarzian action

As we saw above, in the IR regime $\beta J \sim \tau J \gg 1$, we drop the kinetic term $(G_0^E)_{ab}^{-1}$ and find the reparametrization symmetry of the equations of motion. In terms of the effective action, this reparametrization symmetry is a symmetry of the following action,

$$I_{\text{repa.}}[G^E, \Sigma^E] := -\frac{1}{2} \log \det [-\Sigma_{ab}^E(\tau_1, \tau_2)] \quad (3.142)$$

$$+ \frac{1}{2} \int d\tau_1 d\tau_2 \sum_{a,b} \left(\Sigma_{ab}^E(\tau_1, \tau_2) G_{ab}^E(\tau_1, \tau_2) - \frac{(i)^{\frac{q}{2}}}{q} (G_{ab}^E(\tau_1, \tau_2))^q \right), \quad (3.143)$$

where we dropped the derivative ∂_τ in the determinant. Since the Schwinger–Dyson equation at low energy is invariant under the reparametrization transformation, the

action also has the symmetry,

$$I_{\text{repa.}}[G_{\text{conf.},f}^E, \Sigma_{\text{conf.},f}^E] = I_{\text{repa.}}[G_{\text{conf.}}^E, \Sigma_{\text{conf.}}^E]. \quad (3.144)$$

Also, all of the solutions of the Schwinger–Dyson equations are obtained from the reparametrization $f(\tau) : \mathbb{S}_1 \rightarrow \mathbb{S}_1 \in \text{Diff}(\mathbb{S}_1)$. That is, the solution spontaneously breaks the reparametrization symmetry. Since the conformal solution (3.135) is invariant under the $\text{SL}(2, \mathbb{R}) \subset \text{Diff}(\mathbb{S}_1)$ action,

$$f \rightarrow \frac{af + b}{cf + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1, \quad (3.145)$$

the spontaneous breaking occurs from $\text{Diff}(\mathbb{S}_1)$ to $\text{SL}(2, \mathbb{R})$. Then, we expect that at large N and at low energy

$$\begin{aligned} \langle Z \rangle_J &\approx \sum_{(G^E, \Sigma^E) \in \{\text{solutions of SD eq.}\}} e^{-NI_{\text{repa.}}[G^E, \Sigma^E]} \\ &= \int_{f \in \text{Diff}(\mathbb{S}_1)/\text{SL}(2, \mathbb{R})} \mathcal{D}f e^{-NI_{\text{repa.}}[G_{\text{conf.},f}^E, \Sigma_{\text{conf.},f}^E]}. \end{aligned} \quad (3.146)$$

In the original path integral of G^E and Σ^E , we do not have any action for f and also no Fadeev–Popov ghosts coming from the inverse of the volume factor. Thus, we see the divergence

$$\langle Z \rangle_J \approx \text{Vol}(\text{Diff}(\mathbb{S}_1)/\text{SL}(2, \mathbb{R})) \times e^{-NI_{\text{repa.}}[G_{\text{conf.}}^E, \Sigma_{\text{conf.}}^E]}. \quad (3.147)$$

To solve this divergence we recall that the reparametrization symmetry is just an emergent symmetry at the exact IR limit $\beta J \rightarrow \infty$ and is explicitly broken in the effective action $I_{\text{eff}}[G^E, \Sigma^E]$ for finite βJ . So for large but finite βJ , we have the $1/\beta J$ correction to the low-energy action $I_{\text{repa.}}$ and f is actually a pseudo Nambu–Goldstone mode. This correction should satisfy the following properties:

1. it explicitly breaks the reparametrization symmetry;
2. it does not break the $\text{SL}(2, \mathbb{R})$ symmetry;
3. it vanishes in the $\beta J \rightarrow \infty$ limit;
4. it scales as $O(N)$.

The natural action satisfying these properties is the Schwarzian action³

$$I_{\text{Sch}}[f] = -\frac{N\alpha_S}{J} \int d\tau \text{Sch}(f, \tau), \quad \text{Sch}(f, \tau) = \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2} \left(\frac{f''(\tau)}{f'(\tau)} \right)^2. \quad (3.149)$$

and the path integral is given by

$$\langle Z \rangle_J \approx \int \frac{\mathcal{D}f}{\text{Vol}(\text{SL}(2, \mathbb{R}))} \exp \left[-N (I_{\text{repa.}}[G_{\text{conf.},f}^E, \Sigma_{\text{conf.},f}^E] + I_{\text{Sch}}[f]) \right]. \quad (3.150)$$

For the finite-temperature case, one usually uses

$$f(\tau) = \tan \frac{\pi\varphi(\tau)}{\beta}, \quad (3.151)$$

and

$$I_{\text{Sch}}[\varphi] = -\frac{N\alpha_S}{J} \int_{-\beta/2}^{\beta/2} d\tau \text{Sch} \left(\tan \frac{\pi\varphi(\tau)}{\beta}, \tau \right). \quad (3.152)$$

The Schwarzian action is also derived from the original effective action I_{eff} by considering the fluctuation around the solution of the Schwinger-Dyson equation.

■ Large q SYK model

Now we try to consider large q limit of SYK models without taking conformal limit [34, 84]. This is one solvable limit to get some analytical result of SYK models. Now we consider the large q expansion,

$$G^>(t_1, t_2) = -\frac{i}{2} e^{\frac{1}{q}g(t_1, t_2)} = -\frac{i}{2} \left(1 + \frac{1}{q}g(t_1, t_2) \right). \quad (3.153)$$

By using the relations

$$G^<(t_1, t_2) = \frac{i}{2} e^{\frac{1}{q}g(t_2, t_1)} = \frac{i}{2} \left(1 + \frac{1}{q}g(t_2, t_1) \right). \quad (3.154)$$

Then, $O(1/q)$ order of (3.113) and (3.114) are given by

$$\begin{aligned} \frac{\partial}{\partial t_1} g(t_1, t_2) &= 2 \int_{-\infty}^{t_2} dt_3 \mathcal{J}(t_1) \mathcal{J}(t_3) e^{g(t_1, t_3)} - \int_{-\infty}^{t_1} dt_3 \mathcal{J}(t_1) \mathcal{J}(t_3) [e^{g(t_1, t_3)} + e^{g(t_3, t_1)}], \\ \frac{\partial}{\partial t_2} g(t_1, t_2) &= 2 \int_{-\infty}^{t_1} dt_3 \mathcal{J}(t_3) \mathcal{J}(t_2) e^{g(t_3, t_2)} - \int_{-\infty}^{t_2} dt_3 \mathcal{J}(t_3) \mathcal{J}(t_2) [e^{g(t_3, t_2)} + e^{g(t_2, t_3)}] \end{aligned} \quad (3.155)$$

³There is another well-known expression,

$$\text{Sch}(f, \tau) = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2. \quad (3.148)$$

where we define canonical choice of coupling $\mathcal{J}(t)$ in large q SYK models by

$$\mathcal{J}(t)^2 = \frac{qJ^2(t)}{2^{q-1}}. \quad (3.156)$$

By combining the two equations (3.155), we obtain

$$\frac{\partial^2}{\partial t_1 \partial t_2} g(t_1, t_2) = 2\mathcal{J}(t_1)\mathcal{J}(t_2)e^{g(t_1, t_2)}. \quad (3.157)$$

For time-independent coupling $\mathcal{J}(t) = \mathcal{J}$, this is just a equation of motion of Liouville theory and we know general solutions,

$$g(t_1, t_2) = \log \left[\frac{-h'_1(t_1)h'_2(t_2)}{\mathcal{J}^2(h_1(t_1) - h_2(t_2))^2} \right] \quad (3.158)$$

with arbitrary function $h_1(t_1)$ and $h_2(t_2)$. Importantly, there is $SL(2, \mathbb{C})$ gauge symmetry

$$h_\alpha(t) \rightarrow \frac{ah_\alpha(t) + b}{ch_\alpha(t) + d}, \quad a, b, c, d \in \mathbb{C}, ad - bc = 1. \quad (3.159)$$

One typical way of fixing this gauge symmetry in thermal physics is

$$h_1(t) = \frac{ae^{i\theta}e^{\sigma t} + b}{ce^{i\theta}e^{\sigma t} + d}, h_2(t) = \frac{ae^{-i\theta}e^{\sigma t} + b}{ce^{-i\theta}e^{\sigma t} + d} \quad (3.160)$$

with complex parameters a, b, c, d and real parameters σ and θ . Then, independent of choice of a, b, c, d we obtain

$$g(t_1, t_2) = \log \left[\frac{-\sigma^2}{4\mathcal{J}^2 \sinh^2 \left(\frac{\sigma(t_1 - t_2)}{2} + i\theta \right)} \right]. \quad (3.161)$$

This is seemingly surprising. By just gauge fixing, we obtain the static solution *i.e.*, the two time solution $g(t_1, t_2)$ is a function of $t_1 - t_2$. Then some people may wonder that the large q SYK cannot have non-equilibrium dynamics at least in the leading part of large q expansion. However, this is not true. In non-equilibrium situation we prepare some initial states and we use the gauge fixing to represent that states. When we have external driving, for example, we obtain non-trivial two time solutions⁴. Now we try to compute the solution in thermal equilibrium. First

⁴However, in some quantum quench protocol, it is known that the relaxation time is very rapid $O(1/q^2)$ and we obtain equilibrium solution for $t_1 > 0, t_2 > 0$ [84].

we need to take care about the algebra of Majorana fermion. We see

$$G^>(t, t) = -\frac{i}{N} \sum_i \langle \psi^i(t) \psi^i(t) \rangle = -\frac{i}{2}. \quad (3.162)$$

This leads

$$g(t, t) = 0. \quad (3.163)$$

And we obtain

$$\sigma = 2\mathcal{J} \sin \theta. \quad (3.164)$$

Also we consider the KMS condition for thermal equilibrium situation

$$G^>(t_1 + i\beta, t_2) = -G^<(t_1, t_2) \Rightarrow g(t_1 + i\beta, t_2) = g(t_1, t_2). \quad (3.165)$$

This determines

$$\beta = \frac{2(\pi - 2\theta)}{\sigma}. \quad (3.166)$$

In [34], the other parameter v satisfying

$$\beta\mathcal{J} = \frac{\pi v}{\cos\left(\frac{\pi v}{2}\right)} \quad (3.167)$$

is used. We have simple relation

$$v = 1 - \frac{2\theta}{\pi} \Leftrightarrow \sin \theta = \cos \frac{\pi v}{2}. \quad (3.168)$$

the conformal limit $v \rightarrow 1$ is thus $\theta \rightarrow 0$. With this choice of σ and θ we obtain

$$e^{g(t_1, t_2)} = \frac{\cos^2\left(\frac{\pi v}{2}\right)}{\cosh^2\left(\pi v \left(\frac{t}{\beta} - \frac{i}{2}\right)\right)}. \quad (3.169)$$

We see that the correlation function exponentially decay

$$G^>(t_1, t_2) = -\frac{i}{2} e^{\frac{1}{q}g(t_1, t_2)} \propto e^{-\frac{2\pi v}{q\beta}|t_1 - t_2|}, \quad |t_1 - t_2| \gg \beta/v. \quad (3.170)$$

Interestingly, it is known that the large q expansion matches to numerical solution even in $q = 4$ if we take plot the form of $e^{\frac{1}{q}g(t_1, t_2)}$ [85]. However, if we plot with the form of $G = -\frac{i}{2}(1 + \frac{1}{q}g(t_1, t_2))$, we see the matching with more large q . Note that from the expression, we can estimate the relaxation time

$$t_{\text{rel}} = \frac{q\beta}{2\pi v} \quad (3.171)$$

which is shortest in conformal limit $v \rightarrow 1$. This means that conformal limit makes

the large q SYK more chaotic. Indeed, we can exactly compute the out-of-time-order correlator (OTOC) exactly and read off quantum Lyapunov exponents [34, 86, 87],

$$\lambda_L = \frac{2\pi v}{\beta}. \quad (3.172)$$

And conformal limit saturates the Lyapunov bound $\lambda_L = \frac{2\pi}{\beta}$ in [29].

3.4.2 JT gravity

The JT gravity configurations consist of cutting out different shapes from AdS_2 . We will see this in JT gravity on Poincaré AdS. In Euclidean signature, AdS_2 is just a hyperbolic disk:

$$ds^2 = \frac{dt^2 + dz^2}{z^2}. \quad (3.173)$$

The Euclidean JT action is the following

$$\begin{aligned} I_{\text{JT}}[G, \phi] = & \frac{\phi_0}{16\pi G_N} \left[\int_M d^2x \sqrt{G} R[G] + 2 \int_{\partial M} du \sqrt{h} K[h] \right] \\ & - \frac{1}{16\pi G_N} \left[\int_M d^2x \sqrt{G} \phi (R[G] + 2) + 2 \int_{\partial M} du \sqrt{h} \phi_b K[h] \right], \end{aligned} \quad (3.174)$$

where K is the extrinsic curvature of the boundary. The variation (or the path integral) with respect to the bulk dilaton requires that $R = -2$, that is, the bulk metric is locally AdS. The first term is topological and its action value is given by the Euler number of the manifold we consider. So the dynamical degrees of freedom are only on the boundary. This degree of freedom comes from different ways to cut out a chunk of the hyperbolic geometry, see Figure 10.

Now we discuss the independent ways of such cuttings. As we said, the first term is just topological, and thus whenever the manifold M is a simply connected chunk of the hyperbolic disk the actions have the same value. In other words, there is an enormous amount of symmetry from continuous deformations. But the second term breaks this degeneracy.

Let us denote the boundary trajectory as $(t(u), z(u))$. The parameter u is the time coordinate of the boundary. To make the boundary metric uniform, we demand

$$G|_{\text{bdry}} = \frac{1}{\epsilon^2} = \frac{t'(u)^2 + z'(u)^2}{z(u)^2}. \quad (3.175)$$

Suppose the cutoff ϵ is small compared to every other length scale we discuss. Then

$$z(u) = \epsilon t'(u) + O(\epsilon^3). \quad (3.176)$$

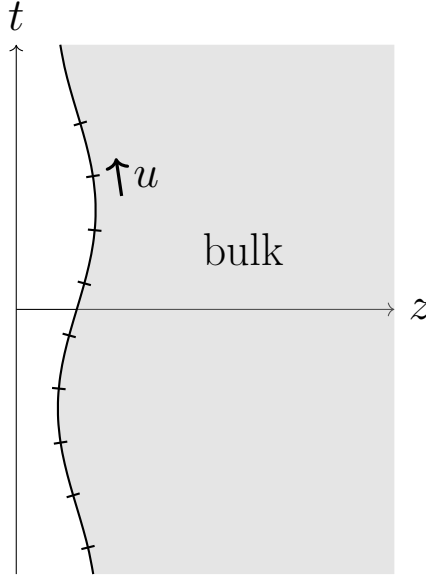


Figure 10: The sketch of boundary curve.

We treat the way of cutting out as the dynamical degree of freedom, so the function $t(u)$ is the dynamical variable in this model. This is called the boundary particle. The Einstein–Hilbert term is topological and has the same value for all $t(u)$. This is the symmetry of the action called the reparametrization of the boundary, $u \rightarrow f(u)$. Notice that most of these reparametrizations, i.e. transformations of $t(u)$, map a given cut-out of the hyperbolic disk to a different one. However, the translations and rotations map it to the same one. These generate an $SL(2, \mathbb{R})$ subgroup of the reparametrization symmetry:

$$t(u) \rightarrow \frac{at(u) + b}{ct(u) + d}, \quad ad - bc = 1. \quad (3.177)$$

Because of the non-trivial dilaton ϕ , the second term is not invariant under the reparametrization and gives a nontrivial finite action for $t(u)$. One can notice this fact from the following viewpoints:

- In the context of a (nearly) extremal black hole, this translates to: the very near-horizon region enjoys the reparametrization symmetry, but as we move to the asymptotic boundary this symmetry is broken.
- In terms of the holographic RG, the reparametrization is just the conformal symmetry of the CFT_1 . This is expressed by the vanishing Hamiltonian. But once we move a little to the UV along an irrelevant deformation which corresponds to a blown-up boundary condition of the dilaton, the symmetry is explicitly broken.

Next we want to argue the action for $t(u)$. The boundary dilaton term is the only candidate. Suppose the value of the boundary dilaton is constant:

$$\phi_b = \frac{\bar{\phi}_r}{\epsilon}. \quad (3.178)$$

Then

$$I_{\text{Sch}}[t(u)] = -\frac{\bar{\phi}_r}{8\pi G_N} \int du \text{Sch}(t(u), u). \quad (3.179)$$

We can obtain this expression from the direct computation of the extrinsic curvature, but the fact that the Schwarzian derivative is the unique $SL(2, \mathbb{R})$ -invariant functional constructed out of a few derivatives leads us to this expression without such a calculation. We now have the same Schwarzian action from the small deviation from the conformal limit of SYK models and from the boundary curve fluctuation of JT gravity. So, at least in this energy regime, we have the holographic duality between SYK and JT gravity. Later we will see that this duality also holds in the very late-time regime where the quantum systems show random matrix universality.

4 Deformations of AdS/CFT correspondence and wormholes

In this section we discuss the deformation of CFTs or SYK models and their effect in gravitational systems. We discuss these models:

- double trace deformation by light operators,
- the analogue of double trace deformation in SYK/JT models,
- $T\bar{T}$ deformation and its generalisation.
- AdS/BCFT correspondence and AdS/ICFT correspondence
- Braneworld holography

4.1 Double trace deformation

Now we consider some primary operator of CFTs $\mathcal{O}$. For later use we focus on a case where $\mathcal{O}$ is light. We consider the following deformation of boundary QFTs:

$$S_{[\mu+\Delta\mu]} = S_{[\mu]}^{(0)} + \int d^d x J_{[\mu]} \mathcal{O}(x) + \frac{\Delta\mu}{2} \int d^d x \mathcal{O}(x) F(-\square) \mathcal{O}(x). \quad (4.1)$$

Here $F(x)$ is a function such that it decays as $x \rightarrow \infty$ and $J_{[\mu]}(x)$ is a source of the operator $\mathcal{O}(x)$ in the theory at μ . This equation defines families of QFTs labelled by

a parameter μ . To discuss the flow equation of the source $J_{[\mu]}(x)$ and the expectation value $\mathcal{O}_{[\mu]}$ let us consider the Hubbard–Stratonovich transformation,

$$\begin{aligned}
& \int \mathcal{D}X e^{-S_{[\mu+\Delta\mu]}} \\
&= \int \mathcal{D}X e^{-S_{[\mu]}^{(0)} - \int d^d x J_{[\mu]} \mathcal{O}(x) - \frac{\Delta\mu}{2} \int d^d x \mathcal{O}(x) F(-\square) \mathcal{O}(x)} \\
&= \int \mathcal{D}X \mathcal{D}\psi e^{-S_{[\mu]}^{(0)} - \int d^d x J_{[\mu]}(x) \mathcal{O}(x) - \int d^d x \psi(x) \mathcal{O}(x) + \frac{\Delta\mu}{2} \int d^d x \psi(x) F(-\square) \psi(x)}.
\end{aligned} \tag{4.2}$$

The saddle point equation in $\Delta\mu \rightarrow 0$ is given by

$$\begin{aligned}
\frac{1}{\Delta\mu} F(-\square) \psi^*(x) - F(-\square) \mathcal{O}(x) &= 0, \\
\psi^*(x) &= \Delta\mu \mathcal{O}(x).
\end{aligned} \tag{4.3}$$

Then the flow equation is given by

$$\begin{aligned}
J_{[\mu+\Delta\mu]}(x) - J_{[\mu]}(x) &= \psi^*(x), \\
\partial_\mu J_{[\mu]}(x) &= F(-\square) \mathcal{O}_{[\mu]}(x).
\end{aligned} \tag{4.4}$$

And the flow equation of the effective action is read off as

$$\partial_\mu W_{[\mu]} = \frac{1}{2} \int d^d x \mathcal{O}_{[\mu]}(x) F(-\square) \mathcal{O}_{[\mu]}(x). \tag{4.5}$$

Then consider the variation of the source which keeps the flow equation of the source (4.4),

$$\partial_\mu \delta J_{[\mu]} = F(-\square) \delta \mathcal{O}_{[\mu]} = 0. \tag{4.6}$$

Now we see that

$$\partial_\mu \delta W_{[\mu]} = \int \delta J_{[\mu]} \partial_\mu \mathcal{O}_{[\mu]} = \delta \frac{1}{2} \int d^d x \mathcal{O}_{[\mu]}(x) F(-\square) \mathcal{O}_{[\mu]}(x) = 0. \tag{4.7}$$

Thus,

$$\partial_\mu \mathcal{O}_{[\mu]} = 0. \tag{4.8}$$

Then the flow equation is solved and we find

$$\begin{aligned}
J_{[\mu]}(x) &= J_{[0]}(x) + \mu F(-\square) \mathcal{O}_{[0]}(x), \\
W_{[\mu]} &= W_{[0]} + \frac{\mu}{2} \int d^d x \mathcal{O}_{[0]}(x) F(-\square) \mathcal{O}_{[0]}(x).
\end{aligned} \tag{4.9}$$

In the GKPW dictionary, we can identify the non-normalizable mode $\alpha(x)$ with $J_{[0]}(x)$ and the normalizable mode $\beta(x)$ with $\mathcal{O}_{[0]}(x)$. Therefore for the bulk field

$\Phi(x)$ in the bulk dual theory we assign the modified boundary conditions

$$J_{[\mu]}(x) = \mu F(-\square)\beta(x) + \alpha(x) \quad (4.10)$$

for a given source $J_{[\mu]}(x)$ [88].

■ Double trace deformation: two sides

As a variant of the double trace deformation we consider deformations of two CFTs [89]. We consider two CFTs $\text{CFT}_{(1)}$ and $\text{CFT}_{(2)}$ and two operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(x)$ from each theory. Now we consider a deformation

$$\begin{aligned} S_{[\mu+\Delta\mu]}^{\text{tot}}[X_1, X_2] &= S_{[\mu]}^{\text{tot}}[X_1, X_2] + \int d^d x J_{[\mu]}^{(1)}(x) \mathcal{O}_1(x) + \int d^d x J_{[\mu]}^{(2)}(x) \mathcal{O}_2(x) \\ &\quad + \Delta\mu \int d^d x \mathcal{O}_1(x) \mathcal{O}_2(x), \end{aligned} \quad (4.11)$$

$$S_{[0]}^{\text{tot}}[X_1, X_2] = S^{\text{CFT}_{(1)}}[X_1] + S^{\text{CFT}_{(2)}}[X_2].$$

Similar to the single-side case, we consider the Hubbard–Stratonovich transformation,

$$\begin{aligned} &\int \mathcal{D}X_1 \mathcal{D}X_2 e^{-S_{[\mu+\Delta\mu]}^{\text{tot}}[X_1, X_2]} \\ &= \int \mathcal{D}X_1 \mathcal{D}X_2 e^{-S_{[\mu]}^{\text{tot}}[X_1, X_2] - \int d^d x J_{[\mu]}^{(1)}(x) \mathcal{O}_1(x) - \int d^d x J_{[\mu]}^{(2)}(x) \mathcal{O}_2(x)} e^{-\Delta\mu \int d^d x \mathcal{O}_1(x) \mathcal{O}_2(x)} \\ &= \int \mathcal{D}\psi_{1,2} \mathcal{D}X_{1,2} e^{\frac{1}{\Delta\mu} \int d^d x \psi_1(x) \psi_2(x) - S_{[\mu]}^{\text{tot}}[X_1, X_2]} \\ &\quad \times e^{-\int d^d x (J_{[\mu]}^{(1)}(x) + \psi_1(x)) \mathcal{O}_1(x) - \int d^d x (J_{[\mu]}^{(2)}(x) + \psi_2(x)) \mathcal{O}_2(x)} \end{aligned} \quad (4.12)$$

In the saddle point for the integral of $\psi_{1,2}(x)$ in $\Delta\mu \rightarrow 0$, we see

$$\psi_1^*(x) = \Delta\mu \mathcal{O}_2(x), \quad \psi_2^*(x) = \Delta\mu \mathcal{O}_1(x). \quad (4.13)$$

Then we identify

$$J_{[\mu+\Delta\mu]}^{(1)}(x) = J_{[\mu]}^{(1)}(x) + \psi_1^*(x) = J_{[\mu]}^{(1)}(x) + \Delta\mu \mathcal{O}_2(x), \quad (4.14)$$

$$J_{[\mu+\Delta\mu]}^{(2)}(x) = J_{[\mu]}^{(2)}(x) + \psi_2^*(x) = J_{[\mu]}^{(2)}(x) + \Delta\mu \mathcal{O}_1(x). \quad (4.15)$$

Then we obtain

$$\partial_\mu J_{[\mu]}^{(1)}(x) = \mathcal{O}_{[\mu]}^{(2)}(x), \quad \partial_\mu J_{[\mu]}^{(2)}(x) = \mathcal{O}_{[\mu]}^{(1)}(x). \quad (4.16)$$

By substituting the saddle point we obtain the flow equation of the effective action

$$\partial_\mu W_{[\mu]} = \int d^d x \mathcal{O}_{[\mu]}^{(1)}(x) \mathcal{O}_{[\mu]}^{(2)}(x). \quad (4.17)$$

Now we consider a variation of sources while keeping the flow equations

$$\partial_\mu \delta J_{[\mu]}^{(1)}(x) = \delta \mathcal{O}_{[\mu]}^{(2)}(x), \quad \partial_\mu \delta J_{[\mu]}^{(2)}(x) = \delta \mathcal{O}_{[\mu]}^{(1)}(x). \quad (4.18)$$

Then we have two expressions for the variation of the effective action

$$\begin{aligned} \delta \partial_\mu W_{[\mu]} &= \partial_\mu \int d^d x \left(\mathcal{O}_{[\mu]}^{(1)} \delta J_{[\mu]}^{(1)}(x) + \mathcal{O}_{[\mu]}^{(2)} \delta J_{[\mu]}^{(2)}(x) \right) \\ &= \partial_\mu \int d^d x \left(\mathcal{O}_{[\mu]}^{(1)} \delta \mathcal{O}_{[\mu]}^{(2)} + \delta \mathcal{O}_{[\mu]}^{(1)} \mathcal{O}_{[\mu]}^{(2)} \right) = 0. \end{aligned} \quad (4.19)$$

From these we see

$$\partial_\mu \mathcal{O}_{[\mu]}^{(1)}(x) = 0, \quad \partial_\mu \mathcal{O}_{[\mu]}^{(2)}(x) = 0. \quad (4.20)$$

Then we see the solution of the flow equation of sources,

$$J_{[\mu]}^{(1)}(x) = J_{[0]}^{(1)}(x) + \mu \mathcal{O}_{[0]}^{(2)}(x), \quad J_{[\mu]}^{(2)}(x) = J_{[0]}^{(2)}(x) + \mu \mathcal{O}_{[0]}^{(1)}(x). \quad (4.21)$$

Now we consider a case where the deformation $\int \mathcal{O}_1(x) \mathcal{O}_2(x)$ is a relevant deformation. In this case we expect that the effect in the bulk appears deep inside the bulk and does not change the asymptotic structure of AdS spacetime. Now we think that two spacetimes before deformation are connected in some sense and we have a single scalar Φ . The field has asymptotics

$$\begin{aligned} \Phi &\approx z^{d-\Delta} \alpha^{(1)}(x) + z^\Delta \beta^{(1)}(x) \quad \text{for asymptotics in boundary 1 } (z \rightarrow 0), \\ &\approx (z')^{d-\Delta} \alpha^{(2)}(x) + (z')^\Delta \beta^{(2)}(x) \quad \text{for asymptotics in boundary 2 } (z' \rightarrow 0). \end{aligned} \quad (4.22)$$

Then, based on the GKPW relation, we identify the non-normalizable mode for each source as $\alpha^{(a)}(x) = J_{[0]}^{(a)}(x)$, $a = 1, 2$ and the normalizable mode with the expectation value $\beta^{(a)}(x) = \mathcal{O}_{[0]}^{(a)}(x)$. Then, for given sources $J_{[\mu]}^{(a)}(x)$, we have the deformed boundary conditions for the bulk theory,

$$\begin{aligned} J_{[\mu]}^{(1)}(x) &= \alpha^{(1)}(x) + \mu \beta^{(2)}(x), \\ J_{[\mu]}^{(2)}(x) &= \alpha^{(2)}(x) + \mu \beta^{(1)}(x). \end{aligned} \quad (4.23)$$

Now using the relation $\alpha^{(a)}(k) = -G(k) \beta^{(a)}(k)$ in (3.58), we obtain the correlation

functions between the two CFT operators,

$$G_{[\mu]}^{12}(k) = \frac{\langle \mathcal{O}_{[\mu]}^{(2)} \rangle}{J_{[\mu]}^{(1)}} \Big|_{J_{[\mu]}=0} = \frac{\beta^{(2)}}{\alpha^{(1)}} \Big|_{J_{[\mu]}=0} = \frac{\mu G(k)^2}{1 - \mu^2 G(k)^2}. \quad (4.24)$$

That is, there is a non-trivial correlation between the two holographic CFTs. In the bulk, this correlation is highly related to the existence of a wormhole connecting two asymptotic boundaries. Indeed, in [89] they compute the gravitational perturbation corresponding to the deformations via the matter stress tensor and find traversable wormholes. Especially, the deformed boundary conditions (4.23) induce a Casimir effect in the bulk and allow us to violate the energy conditions which protect the traversable wormhole. In these computations, the deformation is relevant deformation and the wormhole appear in the bulk inside. Similar things happened in SYK/JT [90]

We can also consider a marginal or irrelevant deformations. In these, we expect that the effect of deformation appears around the conformal boundary. Especially, we expect that the two conformal boundaries are glued at least in gravitational perpetuation. In [91, 92], this set up is discussed in detail. In the bulk, from the deformed boundary conditions, we have a interaction between fields in distinct AdSs. It is known that because of this interaction the self energy of graviton is non zero and thus graviton acquire mass. From the boundary side, this is easily understood. Before deformation, we have two conformal CFTs. In total we have two conformal symmetry. This corresponds to the two massless gravitons. However, after deformation, we have one conformal symmetry and thus there is only one massless graviton. The other gyration should be massive. Similar massive graviton also appears in Karch-Randall type brane world gravity [93].

4.2 $T\bar{T}$ deformations and its generalization

The $T\bar{T}$ deformation is an irrelevant deformation of two dimensional QFTs defined by flow equation [94–96]

$$\partial_\mu S_{[\mu]} = -\frac{1}{2} \int d^2x (\sqrt{\gamma} (T^{ab} T_{ab} - T^2))_{[\mu]} =: -\frac{1}{2} \int d^2x \sqrt{\gamma_{[\mu]}} (\mathcal{O}_{T\bar{T}})_{[\mu]}. \quad (4.25)$$

This flow equations define a family of QFTs labeled by μ . Interestingly, this flow equations are exactly solved as we will see. Also the energy spectrum and some correlation functions of deformed theories are exactly evaluated. We also have a nice holographic interpretation called "finite cut-off proposal". The proposal suggests that the $T\bar{T}$ deformed theories are holographically dual to the gravitational theories with finite cut-off surface [4, 97, 98].

In our paper [2, 3], we explore generalization of $T\bar{T}$ deformation including many derivative or non-local kernel and discuss their gravity dual. Interestingly, the dual gravitational theories no longer than the Einstein gravity and indeed non-local. By using the Coleman's result we may relate the non-locality to some microscopic wormhole.

4.2.1 Usual $T\bar{T}$ deformation and finite cut-off holography

■ QFT computation

In this section, we concentrate on the $T\bar{T}$ deformation purely from the perspective of quantum field theory.

Consider a two-dimensional QFT with action $S_{[0]}$, defined on a background metric $\gamma_{[0]}$. We introduce a deformation by means of the $T\bar{T}$ flow equation,

$$\frac{\partial}{\partial\mu} S_{[\mu]} = -\frac{1}{2} \int d^2x (\sqrt{\gamma} (T^{ab} T_{ab} - T^2))_{[\mu]} =: -\frac{1}{2} \int d^2x \sqrt{\gamma_{[\mu]}} (\mathcal{O}_{T\bar{T}})_{[\mu]}. \quad (4.26)$$

We denote the metric associated with the theory $S_{[\mu]}$ by $\gamma_{[\mu]}$, and define the stress tensor as

$$T_{[\mu]ab} := \frac{2}{\sqrt{\gamma_{[\mu]}}} \frac{\delta S_{[\mu]}}{\delta \gamma_{[\mu]}^{ab}}. \quad (4.27)$$

We adopt a formulation in which the background metric itself evolves along the flow [98, 99]. This viewpoint will be particularly convenient when we later discuss holographic dual descriptions. In analogy with double-trace deformations of scalar operators, we employ a Hubbard–Stratonovich transformation.

We now present an alternative derivation of the $T\bar{T}$ flow equation. Cardy showed that the $T\bar{T}$ deformation can be understood in terms of a random-geometry path integral [100]. The key idea is to apply a Hubbard–Stratonovich transformation to the $T\bar{T}$ interaction,

$$\begin{aligned} S_{T\bar{T}} &= \frac{1}{2} \int d^2x \sqrt{\gamma} \mathcal{O}_{T\bar{T}} \\ &= -\frac{1}{2} \int d^2x \sqrt{\gamma} \epsilon_{ac} \epsilon_{bd} T^{ab} T^{cd}. \end{aligned} \quad (4.28)$$

Here ϵ_{ab} denotes the Levi–Civita tensor.

Performing the Hubbard–Stratonovich transformation, we obtain

$$e^{\frac{\Delta\mu}{2} \int d^2x \sqrt{\gamma_{[\mu]}} \epsilon_{ac} \epsilon_{bd} T_{[\mu]}^{ab} T_{[\mu]}^{cd}} \propto \int \mathcal{D}h \, e^{+ \int d^2x \sqrt{\gamma_{[\mu]}} \frac{1}{2} T_{[\mu]}^{ab} h_{ab} - \frac{1}{8\Delta\mu} \int d^2x \sqrt{\gamma_{[\mu]}} \epsilon_{[\mu]}^{ac} \epsilon_{[\mu]}^{bd} h_{ab} h_{cd}}. \quad (4.29)$$

Since $\Delta\mu$ is assumed to be small, we evaluate the h -integral using the saddle-point

approximation. The saddle-point configuration h_{ab}^* satisfies

$$\begin{aligned} T_{[\mu]}^{ab} &= -\frac{1}{2\Delta\mu} \epsilon_{[\mu]}^{ac} \epsilon_{[\mu]}^{bd} h_{cd}^*, \\ h_{ab}^* &= -2\Delta\mu (\hat{T}_{[\mu]})_{ab} = -2\Delta\mu (T_{[\mu]}_{ab} - (\gamma_{[\mu]})_{ab} T_{[\mu]}). \end{aligned} \quad (4.30)$$

Substituting the saddle-point solution back into the exponent reproduces the original $T\bar{T}$ interaction. We furthermore note that $h_{ab} \sim O(\Delta\mu)$, and thus

$$\begin{aligned} e^{-S_{[\mu+\Delta\mu]}^{\text{eff}}[\gamma_{[\mu+\Delta\mu]}]} &= \int \mathcal{D}X e^{-S_{[\mu]}[X; \gamma_{[\mu]}]} e^{-\frac{\Delta\mu}{2} \int d^2x \sqrt{\gamma_{[\mu]}} \epsilon_{ac} \epsilon_{bd} T_{[\mu]}^{ab} T_{[\mu]}^{cd}} \\ &\propto \int \mathcal{D}X \mathcal{D}h e^{-S_{[\mu]}[X; \gamma_{[\mu]} + h] - \frac{1}{8\Delta\mu} \int d^2x \sqrt{\gamma_{[\mu]}} \epsilon_{[\mu]}^{ac} \epsilon_{[\mu]}^{bd} h_{ab} h_{cd} + O((\Delta\mu)^2)} \\ &= \int \mathcal{D}h e^{-S_{[\mu]}^{\text{eff}}[\gamma_{[\mu]} + h] - \frac{1}{8\Delta\mu} \int d^2x \sqrt{\gamma_{[\mu]}} \epsilon_{[\mu]}^{ac} \epsilon_{[\mu]}^{bd} h_{ab} h_{cd} + O((\Delta\mu)^2)} \\ &\sim e^{-S_{[\mu]}^{\text{eff}}[\gamma_{[\mu]} + h^*] - \frac{1}{8\Delta\mu} \int d^2x \sqrt{\gamma_{[\mu]}} \epsilon_{[\mu]}^{ac} \epsilon_{[\mu]}^{bd} (h^*)_{ab} (h^*)_{cd} + O((\Delta\mu)^2)}. \end{aligned} \quad (4.31)$$

From this expression, we can read off

$$\begin{aligned} (\gamma_{[\mu+\Delta\mu]})_{ab} &= (\gamma_{[\mu]})_{ab} + (h^*)_{ab}, \\ \Leftrightarrow \frac{\partial}{\partial\mu} (\gamma_{[\mu]})_{ab} &= -2 \left(\hat{T}_{[\mu]} \right)_{ab}. \end{aligned} \quad (4.32)$$

This coincides precisely with the first equation in (4.45). To determine the flow of $T_{[\mu]}$, we must compare the effective actions. Using the saddle-point result, we find

$$\begin{aligned} S_{[\mu+\Delta\mu]}^{\text{eff}}[\gamma_{[\mu+\Delta\mu]}] &= S_{[\mu]}^{\text{eff}}[\gamma_{[\mu]}] - \frac{1}{2} \int d^2x \sqrt{\gamma_{[\mu]}} (T_{[\mu]})^{ab} (h^*)_{ab} + \frac{1}{8\Delta\mu} \int d^2x \sqrt{\gamma_{[\mu]}} \epsilon_{[\mu]}^{ac} \epsilon_{[\mu]}^{bd} (h^*)_{ab} (h^*)_{cd} \\ &= S_{[\mu]}^{\text{eff}}[\gamma_{[\mu]}] + \frac{\Delta\mu}{2} \int d^2x \sqrt{\gamma_{[\mu]}} \mathcal{O}_{T\bar{T}}. \end{aligned} \quad (4.33)$$

Next, consider a deformation of the metric $\gamma_{[\mu]} \rightarrow \gamma_{[\mu]} + \delta\gamma$ that leaves the flow equation (4.32) unchanged, namely

$$\frac{\partial}{\partial\mu} (\delta\gamma_{ab}) = -2\delta(\hat{T}_{[\mu]})_{ab} = 0. \quad (4.34)$$

Taking the variation of (4.33) along this direction gives

$$\delta(S_{[\mu+\Delta\mu]}^{\text{eff}} - S_{[\mu]}^{\text{eff}}) = -\frac{\Delta\mu}{2} \delta \left(\int d^2x \sqrt{\gamma_{[\mu]}} \epsilon_{[\mu]}^{ac} \epsilon_{[\mu]}^{bd} \left(\hat{T}_{[\mu]} \right)_{ab} \left(\hat{T}_{[\mu]} \right)_{cd} \right). \quad (4.35)$$

Since $\hat{T}$ is invariant under this variation, and using the definition of the stress tensor,

$\delta S_{[\mu]}^{\text{eff}} = \frac{1}{2} \int d^2x \sqrt{\gamma_{[\mu]}} (T_{[\mu]})_{ab} \delta \gamma^{ab} = -\frac{1}{2} \int d^2x \sqrt{\gamma_{[\mu]}} (T_{[\mu]})^{ab} \delta \gamma_{ab}$, we obtain

$$\frac{\partial}{\partial \mu} (\sqrt{\gamma_{[\mu]}} (T_{[\mu]})^{ab}) \delta (\gamma_{[\mu]})_{ab} = \delta (\sqrt{\gamma_{[\mu]}} \epsilon_{[\mu]}^{ac} \epsilon_{[\mu]}^{bd}) (\hat{T}_{[\mu]})_{ab} (\hat{T}_{[\mu]})_{cd}. \quad (4.36)$$

The right-hand side consists of two contributions. The first arises from the variation of $\sqrt{\gamma}$,

$$\delta(\sqrt{\gamma}) \left((\hat{T}^2 - \hat{T}_c^d \hat{T}_d^c) \right) = \frac{1}{2} \sqrt{\gamma} \gamma^{ab} \delta \gamma_{ab} (\hat{T}^2 - \hat{T}_c^d \hat{T}_d^c). \quad (4.37)$$

The second contribution is

$$\begin{aligned} 2\sqrt{\gamma} (\delta \gamma^{ab} \gamma^{cd} - \delta \gamma^{ad} \gamma^{bc}) \hat{T}_{ab} \hat{T}_{cd} &= 2\sqrt{\gamma} (\delta \gamma^{ab} \hat{T}_{ab} \hat{T} - \delta \gamma^{ad} \hat{T}_{ab} \hat{T}_d^b) \\ &= 2\sqrt{\gamma} \delta \gamma_{ab} (-\hat{T}^{ab} \hat{T} + \hat{T}_c^a \hat{T}^{cb}). \end{aligned} \quad (4.38)$$

Collecting terms, we read off the flow equation for the stress tensor,

$$\frac{\partial}{\partial \mu} (\sqrt{\gamma_{[\mu]}} T_{[\mu]}^{ab}) = \sqrt{\gamma_{[\mu]}} \left(\frac{1}{2} \gamma_{[\mu]}^{ab} (T_{[\mu]}^2 - T_{[\mu]}^{cd} T_{[\mu]cd}) + 2T_{[\mu]c}^a T_{[\mu]}^{bc} - T_{[\mu]}^{ab} T_{[\mu]} \right). \quad (4.39)$$

For the lowered indices, this implies

$$\begin{aligned} \frac{\partial}{\partial \mu} (\sqrt{\gamma_{[\mu]}} (T_{[\mu]})_{ab}) &= 2 \frac{\partial}{\partial \mu} (\gamma_{[\mu]ac}) \gamma_{[\mu]bd} \sqrt{\gamma_{[\mu]}} T_{[\mu]}^{cd} + \gamma_{[\mu]ac} \gamma_{[\mu]bd} \frac{\partial}{\partial \mu} (\sqrt{\gamma_{[\mu]}} T_{[\mu]}^{cd}) \\ &= \sqrt{\gamma_{[\mu]}} \left(\frac{1}{2} \gamma_{[\mu]ab} (T_{[\mu]}^2 - T_{[\mu]}^{cd} T_{[\mu]cd}) + 2(T_{[\mu]ab} T_{[\mu]} - T_{[\mu]ac} T_{[\mu]b}^c) \right). \end{aligned} \quad (4.40)$$

Finally, let us compare two different evaluations of the variation of the effective action $S_{[\mu+\Delta\mu]}^{\text{eff}}$,

$$\begin{aligned} \delta S_{[\mu+\Delta\mu]}^{\text{eff}} &= \delta S_{[\mu]}^{\text{eff}} + \Delta\mu \delta \left(\frac{1}{2} \int d^2x (\sqrt{\gamma} (T^{ab} T_{ab} - T^2))_{[\mu]} \right) \\ &= \frac{1}{2} \int d^2x (\sqrt{\gamma} T_{ab} \delta \gamma^{ab})_{[\mu]} + \Delta\mu \delta \left(\frac{1}{2} \int d^2x (\sqrt{\gamma} (T^{ab} T_{ab} - T^2))_{[\mu]} \right) \\ &= \frac{1}{2} \int d^2x (\sqrt{\gamma} T_{ab} \delta \gamma^{ab})_{[\mu+\Delta\mu]}. \end{aligned} \quad (4.41)$$

This leads to the identity

$$\frac{\partial}{\partial \mu} ((\sqrt{\gamma} T_{ab} \delta \gamma^{ab})_{[\mu]}) = \delta (\sqrt{\gamma} \mathcal{O}_{T\bar{T}})_{[\mu]}. \quad (4.42)$$

After some algebraic manipulations, this relation can be rewritten as

$$\begin{aligned} & \frac{\partial}{\partial \mu}(\sqrt{\gamma}T_{ab})\delta\gamma^{ab} + \sqrt{\gamma}T_{ab}\delta\left(\frac{\partial}{\partial \mu}\gamma^{ab}\right) \\ &= \sqrt{\gamma}\left(\left(-\frac{1}{2}\mathcal{O}_{T\bar{T}}\gamma_{ab} - 2T_{ac}T^{bc} + 2TT_{ab}\right)\delta\gamma^{ab} + 2T_{ab}\delta(T^{ab} - \gamma^{ab}T)\right). \end{aligned} \quad (4.43)$$

Here, for notational simplicity, we have suppressed the $_{[\mu]}$ labels, *i.e.*, $\gamma = \gamma_{[\mu]}$ and $T_{ab} = T_{[\mu]ab}$.⁵

By matching independently the terms proportional to $\delta\gamma^{ab}$ and to T_{ab} , we obtain

$$\begin{aligned} \frac{\partial}{\partial \mu}\gamma_{ab} &= -2\hat{T}_{ab} =: -2(T_{ab} - \gamma_{ab}T), \\ \frac{\partial}{\partial \mu}(\sqrt{\gamma}T_{ab}) &= \sqrt{\gamma}\left(-\frac{1}{2}\mathcal{O}_{T\bar{T}}\gamma_{ab} - 2T_{ac}T^{bc} + 2TT_{ab}\right). \end{aligned} \quad (4.44)$$

From these relations, we finally arrive at

$$\begin{aligned} \frac{\partial}{\partial \mu}\gamma_{ab} &= -2\hat{T}_{ab}, \\ \frac{\partial}{\partial \mu}\hat{T}_{ab} &= -\hat{T}_{ac}\hat{T}_b^c, \\ \frac{\partial}{\partial \mu}(\hat{T}_{ac}\hat{T}_b^c) &= 0. \end{aligned} \quad (4.45)$$

The third equation implies

$$(\hat{T}_{[\mu]})_{ac}(\hat{T}_{[\mu]})_b^c = (\hat{T}_{[0]})_{ac}(\hat{T}_{[0]})_b^c. \quad (4.46)$$

Consequently, the second equation integrates to

$$(\hat{T}_{[\mu]})_{ab} = (\hat{T}_{[0]})_{ab} - \mu(\hat{T}_{[0]})_{ac}(\hat{T}_{[0]})_b^c. \quad (4.47)$$

Finally, using the first equation, we find

$$\gamma_{[\mu]ab} = \gamma_{[0]ab} - 2\mu(\hat{T}_{[0]})_{ab} + \mu^2(\hat{T}_{[0]})_{ac}(\hat{T}_{[0]})_b^c. \quad (4.48)$$

Equivalently, one may write

$$\frac{\partial^3}{\partial \mu^3}(\gamma_{[\mu]})_{ab} = 0. \quad (4.49)$$

⁵When taking the variation $\delta\gamma^{ab}$, one must also account for the induced variation $\delta T_{[\mu]}^{ab}$. These two variations should be treated as independent, which ensures that the decomposition (4.43) is unique.

Moreover, we can establish the relations

$$\frac{\partial}{\partial \mu}(\sqrt{\gamma_{[\mu]}}R[\gamma_{[\mu]}]) = 0, \quad \frac{\partial}{\partial \mu}(\sqrt{\gamma_{[\mu]}}T_{[\mu]}) = \sqrt{\gamma_{[\mu]}}(\mathcal{O}_{T\bar{T}})_{[\mu]}. \quad (4.50)$$

Invoking the trace anomaly,

$$R[\gamma_{[0]}] = \frac{c}{24\pi}T_{[0]}, \quad (4.51)$$

we finally obtain

$$T_{[\mu]} = \frac{c}{24\pi}R[\gamma_{[\mu]}] - \mu\sqrt{\gamma_{[\mu]}}(\mathcal{O}_{T\bar{T}})_{[\mu]}. \quad (4.52)$$

This expression represents a natural generalization of the trace anomaly to $T\bar{T}$ -deformed theories.

■ Gravity Dual: Finite cut-off proposal

Now we discuss holographic dual of $T\bar{T}$ deformed holographic CFTs. Before discussing the $T\bar{T}$ deformation we summarize the geometry of AdS_3 and its holography. Suppose we have asymptotically AdS_3 spacetimes with the Fefferman-Graham coordinate (3.20). By using the Einstein equation, in three dimensional asymptotic AdS spacetime, we find that the FG expansion exactly end up in $g_{(4)}$ and $g_{(4)}$ is expressed in terms of $g_{(0)}, g_{(2)}$ by [70],

$$g_{(4)} = \frac{1}{4}g_{(2)}(g_{(0)})^{-1}g_{(2)}. \quad (4.53)$$

Also some components of the $g_{(2)}$ are determined but not all of the component are still determined.

$$g_{(2)} = -\frac{1}{2}R[g_{(0)}]g_{(0)} + t \quad (4.54)$$

where $t = (t_{ab})$ is the two rank tensor which satisfies

$$\nabla_a^{(0)}t^{ab} = 0, \quad g_{(0)}^{ab}t_{ab} = \frac{1}{2}R[g_{(0)}]. \quad (4.55)$$

,where $\nabla_a^{(0)}$ is covariant derivative with respect to $g_{(0)}$. These equations fix some components of the t but not all. Suppose, we take $g_{(0)}$ as

$$g_{(0)} = \gamma_{[0]} = \eta, \quad ds^2 = \eta_{ab}dx^a dx^b = du dv. \quad (4.56)$$

Then, we find that t_{ab} satisfies the same equations of the CFT_2 stress tensor on the flat space. Thus, $t_{uv} = 0$ and the algebras factorize into the chiral and anti-chiral,

$$t_{uu} = \mathcal{L}(u), \quad t_{vv} = \tilde{\mathcal{L}}(v) \quad (4.57)$$

The form of $\mathcal{L}(u)$ and $\tilde{\mathcal{L}}(v)$ are not determined from the Einstein equation and

related to the boundary quantities like Brown-York tensor. The, we find that

$$\begin{aligned} g_{(2)ab} dx^a dx^b &= \mathcal{L}(u) du^2 + \tilde{\mathcal{L}}(v) dv^2 \\ g_{(4)ab} dx^a dx^b &= \frac{1}{4} \mathcal{L}(u) \tilde{\mathcal{L}}(v) du dv. \end{aligned} \quad (4.58)$$

The total three dimensional metric is given by so called Banados metric,

$$G_{AB} dx^A dx^B = dr^2 + e^{2r} \left(1 + e^{-4r} \mathcal{L}(u) \tilde{\mathcal{L}}(v) \right) du dv + \mathcal{L}(u)^2 du^2 + \tilde{\mathcal{L}}(v)^2 dv^2 \quad (4.59)$$

Since three dimensional asymptotic AdS spacetime is locally equivalent to vacuum AdS spacetime. This means that the Banados metric are obtained by moderate coordinate transformation with chiral/ anti-chiral function $\mathcal{L}(u)$ and $\tilde{\mathcal{L}}(v)$. These coordinate transformations are corresponding to the local conformal transformations in the boundary CFTs.

Now let us going back to the problems of $T\bar{T}$ deformation and use the general background $g_{(0)}$. As in the standard prescription in the multi trace deformation in AdS/CFT [88], by using the relation for the un-deformed theory, we obtain the holographic dictionary for the $T\bar{T}$ deformed theory and its gravity dual. To this end, notice that

$$g_{(0)ab} = \gamma_{[0]ab}, \quad g_{(2)ab} = 8\pi G_N \left(\hat{T}_{[0]} \right)_{ab}. \quad (4.60)$$

Then we compare two results

1. The solutions of $T\bar{T}$ flow equation (4.48) is quadratic in μ .
2. The solutions of Einstein equation in pure three dimensional AdS gravity is quadratic in radial coordinate e^{-2r} .

Now suppose we consider a gravitational theories with finite cu-off surface $r = r_c$ such that

$$e^{-2r_c} = -\frac{\mu}{4\pi G_N}, \quad \mu < 0 \quad (4.61)$$

and assign the Dirichlet boundary conditions there. Then we observe that (4.48) and

$$\gamma(r_c, x) = g_{(0)} + e^{-2r_c} g_{(2)} + e^{-4r_c} g_{(4)} \quad (4.62)$$

completely matches. Notice that the effect of $T\bar{T}$ deformations are back reacting effect *i.e.*, effect in $O(1/G_N)$. Also corresponding to the fact that $T\bar{T}$ deformation is the irrelevant deformation the effect only appears around the conformal boundary.

■ $T\bar{T}$ deformation from holographic renormalization

Above we see the finite cut-off holography is related to $T\bar{T}$ deformed theories. We can see this relation also on the level of action by using the holographic renormalization in Section 3.2.2. Now we consider terms in recursion which we miss in large r_c limits. These correspond to the irrelevant deformations. In the same manner of our finding $\mathcal{L}_{(2)}, \pi_{(2)a}^b$, we find

$$\pi_{(4)} = -8\pi G_N (\pi_{(2)}^2 - \pi_{(2)a}^b \pi_{(2)b}^a), \quad (4.63)$$

$$\mathcal{L}_{(4)} = -\pi_{(4)} = -8\pi G_N (\pi_{(2)a}^b \pi_{(2)b}^a - \pi_{(2)}^2). \quad (4.64)$$

Finally we can write down the effective action for the boundary theory at $r = r_c$.

$$\begin{aligned} S_{\text{CFT}} &:= \lim_{r_c \rightarrow \infty} \int_{\Sigma_{r_c}} d^2x \sqrt{h} \mathcal{L}_{(2)}, \\ S_{\text{finite}} &:= \int_{\Sigma_{r_c}} d^2x \sqrt{h} (\mathcal{L}_{(2)} + \mathcal{L}_{(4)}). \end{aligned} \quad (4.65)$$

Then we have two stress tensors,

$$\begin{aligned} \langle T^{ab} \rangle_{g(0)}^{\text{CFT}} &:= -\frac{2}{\sqrt{g(0)}} \frac{\delta S_{\text{CFT}}}{\delta g_{(0)ab}} = -2 \lim_{r_c \rightarrow \infty} e^{2r_c} \pi^{(2)ab}, \\ \langle T^{ab} \rangle^{\text{finite}} &:= -e^{2r_c} \frac{2}{\sqrt{h}} \frac{\delta S_{\text{finite}}}{\delta h_{ab}} = -2e^{2r_c} (\pi^{(2)ab} + \pi^{(4)ab}). \end{aligned} \quad (4.66)$$

The finite cut-off action can be expanded as

$$S_{\text{finite}} = S_{\text{CFT}}[g_{(0)}] - 2\pi G_N e^{-2r_c} \int d^2x \sqrt{g_{(0)}} \left(\langle T^{ab} \rangle_{g(0)}^{\text{CFT}} \langle T_{ab} \rangle_{g(0)}^{\text{CFT}} - \left(\langle T_a^a \rangle_{g(0)}^{\text{CFT}} \right)^2 \right). \quad (4.67)$$

Comparing this to the definition of the $T\bar{T}$ deformation, we see the relation (4.61) again. Similar results are observed in [101]. We summarized the important points of usual $T\bar{T}$ deformation,

1. The three times derivatives of the metric is zero,

$$\partial_\mu^3 \gamma_{[\mu]ab} = 0. \quad (4.68)$$

2. The Einstein equation with Einstein Hilbert action shows that the Fefferman-Graham expansion ends in the order of $O(e^{-4r})$ and we can match this expansion and the solutions of the flow equation by which we see the finite cut-off interpretation.
3. From the holographic renormalization computation shows that the finite cut-off correction is actually equivalent to the $T\bar{T}$ deformation.

4.2.2 Generalized $T\bar{T}$ deformations, non-local gravity duals and wormhole

We now consider a broader class of $T\bar{T}$ -type deformations,

$$S_{T\bar{T}} = -\frac{1}{2} \int d^2x \sqrt{\gamma_{[\mu]}} \epsilon_{ac} \epsilon_{bd} T^{ab} K(-\square_\gamma) T^{cd}. \quad (4.69)$$

The kernel $K(x)$ is introduced to suppress contributions from high-energy modes. Even though the ordinary $T\bar{T}$ deformation is irrelevant, the present non-local deformation is engineered to act predominantly on infrared degrees of freedom. We postpone specifying an explicit form of $K(x)$ until later. The motivations for studying such non-local $T\bar{T}$ deformations can be summarized as follows.

- We would like to regard this setup as a generalization of the Gao–Jafferis–Wall wormhole mechanism with backreaction.
- As discussed above, the gravitational dual of the $T\bar{T}$ deformation inevitably involves backreaction effects.
- However, the usual $T\bar{T}$ deformation is an irrelevant deformation.
 - $\Rightarrow$ Accordingly, the resulting backreaction is confined to the vicinity of the asymptotic region.
 - $\Rightarrow$ By introducing a controlled non-locality, one may instead deform the infrared sector of the QFT.
 - $\Rightarrow$ This suggests the possibility of producing a wormhole that penetrates further into the bulk, analogously to what happens for relevant double-trace deformations.

We now analyze the non-local $T\bar{T}$ deformation.

We proceed as before, starting from a Hubbard–Stratonovich representation,

$$\begin{aligned} & e^{-S_{[\mu+\Delta\mu]}^{\text{eff}}} \\ &= \int \mathcal{D}X e^{-S_{[\mu]}[X; \gamma_{[\mu]}]} e^{\frac{\Delta\mu}{2} \int d^2x \sqrt{\gamma_{[\mu]}} \epsilon_{ab} \epsilon_{cd} T^{ab} K(-\square_\gamma) T^{cd}} \\ &= \int \mathcal{D}X \mathcal{D}h e^{-S_{[\mu]}[X; \gamma_{[\mu]}]} e^{-\frac{1}{8\Delta\mu} \int d^2x \sqrt{\gamma_{[\mu]}} \epsilon^{ab} \epsilon^{cd} h_{ab} K^{-1}(-\square_\gamma) h_{cd} + \frac{1}{2} \int d^2x \sqrt{\gamma_{[\mu]}} T^{ab} h_{ab}}. \end{aligned} \quad (4.70)$$

The saddle-point equations are then

$$\begin{aligned} -\frac{1}{4\Delta\mu} \epsilon^{ac} \epsilon^{bd} K^{-1}(-\square_\gamma) (h^*)_{cd} + \frac{1}{2} \hat{T}^{ab} &= 0, \\ (h^*)_{ab} &= -2\Delta\mu K(-\square_\gamma) \hat{T}_{ab}. \end{aligned} \quad (4.71)$$

This leads to the metric flow equation

$$\frac{\partial}{\partial \mu} \gamma_{[\mu]ab} = -2K(-\square_\gamma) \left(\hat{T}_{[\mu]} \right)_{ab}. \quad (4.72)$$

Likewise, the flow of the effective action is

$$\frac{\partial}{\partial \mu} S_{[\mu]}^{\text{eff}}[\gamma_{[\mu]}] = -\frac{1}{2} \int d^2x \sqrt{\gamma_{[\mu]}} \epsilon^{ac} \epsilon^{bd} (\hat{T}_{[\mu]})_{ab} K(-\square_\gamma) (\hat{T}_{[\mu]})_{cd}. \quad (4.73)$$

Next, we consider varying the metric $\gamma_{[\mu]ab}$ while keeping the flow equation (4.72) intact,

$$\frac{\partial}{\partial \mu} \delta \gamma_{[\mu]ab} = -2\delta \left(K(-\square_\gamma) \left(\hat{T}_{[\mu]} \right)_{ab} \right) = 0. \quad (4.74)$$

The right-hand side can be reorganized as

$$\delta \left(\hat{T}_{[\mu]} \right)_{ab} = -K^{-1}(-\square_\gamma) (\delta K(-\square_\gamma)) \left(\hat{T}_{[\mu]} \right)_{ab} = K^{-1}(-\square_\gamma) K'(-\square_\gamma) \delta(\square_\gamma) \left(\hat{T}_{[\mu]} \right)_{ab}.$$

Using this identity, and after an involved but direct computation, one obtains

$$\frac{\partial}{\partial \mu} \left(\hat{T}_{ab} \right) = -\frac{1}{2} \left(\hat{T}_{ac} \hat{Y}_b^c + \hat{T}_{bc} \hat{Y}_a^c - 3\hat{T} \hat{Y}_{ab} + 3\hat{T} \hat{Y}_{ab} \right) - N_{ab}, \quad (4.75)$$

where $\hat{Y}_{ab} = K(-\square_\gamma) \hat{T}_{ab}$. Moreover, by applying the variation formula summarized in Appendix A of [102], we find

$$\begin{aligned} U^{ab} := & \left[(\nabla^a \hat{T}_{cd}) (\nabla^b W^{cd}) + (\nabla^b \hat{T}_{cd}) (\nabla^a W^{cd}) - \gamma^{ab} \left((\square \hat{T}_{cd}) W^{cd} + (\nabla_e \hat{T}_{cd}) (\nabla^e W^{cd}) \right) \right. \\ & + \square (\hat{T}_c^a W^{bc}) + \square (\hat{T}_c^b W^{ac}) - 2\nabla_e (\hat{T}_c^a \nabla^e W^{bc}) - 2\nabla_e (\hat{T}_c^b \nabla^e W^{ac}) \\ & + \nabla_c \nabla^b (\hat{T}_d^a W^{cd}) + \nabla_c \nabla^a (\hat{T}_d^b W^{cd}) - 2\nabla_c (\hat{T}_d^a \nabla^b W^{cd}) - 2\nabla_c (\hat{T}_d^b \nabla^a W^{cd}) \\ & \left. - \nabla^e \nabla^a (\hat{T}_{ec} W^{bc}) - \nabla^e \nabla^b (\hat{T}_{ec} W^{ac}) + 2\nabla^e (\hat{T}_{ec} \nabla^a W^{bc}) + 2\nabla^e (\hat{T}_{ec} \nabla^b W^{ac}) \right], \end{aligned}$$

where $W_{ab} = K'(-\square_\gamma) (\hat{T}_{ab} - \gamma_{ab} \hat{T})$. For the standard $T\bar{T}$ deformation, one simply takes $Y_{ab} = T_{ab}$ and $U_{ab} = 0$, and substantial cancellations occur. By contrast, for the present general (non-local) deformation (4.69), we instead obtain

$$\frac{\partial^3}{\partial \mu^3} \gamma_{[\mu]ab} \neq 0. \quad (4.76)$$

■ Possible Gravity Dual and Wormhole

From the discussion above, we can contemplate several holographic scenarios for a non-local $T\bar{T}$ deformation:

- One possibility is that a finite cut-off holographic description exists within Einstein gravity, although the cut-off surface need not be of the simple form $r = \text{const.}$.
- Another possibility is to keep a cut-off surface at $r = \text{const.}$, but to replace the bulk dynamics by a non-Einstein theory. In such a setup, the Fefferman–Graham expansion need not truncate at order $O(e^{-4r})$. Hence it can remain consistent to identify μ with e^{-r} .
- A third possibility is that no finite cut-off interpretation exists at all.

The first possibility clashes with the following symmetry-based argument. Start from an undeformed boundary theory with a flat metric $g_{(0)ab} = \delta_{ab}$. If the deformed theory were dual to a description on a cut-off surface $r = \bar{r}(x^a)$, then the induced metric on that surface would generically fail to preserve the symmetries of δ_{ab} , namely the two-dimensional Poincaré symmetry. However, the non-local $T\bar{T}$ deformation does not break this symmetry, so the two pictures would not match. Furthermore, inserting an inhomogeneous cut-off surface into the holographic renormalization group analysis merely produces a $T\bar{T}$ deformation with a position-dependent coupling.

For these reasons, we focus on the second possibility. As we show below, the non-local $T\bar{T}$ deformation arises naturally as a finite cut-off effect in a gravitational theory whose Hamiltonian takes the form

$$\begin{aligned}
H &= \int_{\Sigma_r} d^2x (N\mathcal{H} + N_a\mathcal{H}^a), \\
\frac{1}{\sqrt{h}}\mathcal{H} &= 16\pi G_N (\pi_b^a K(-e^{2r}\square_h)\pi_a^b - \pi K(-e^{2r}\square_h)\pi) + \frac{1}{16\pi G_N}(R[h] + 2), \\
\mathcal{H}^a &= -2D_b\left(\frac{1}{\sqrt{h}}\pi^{ab}\right).
\end{aligned} \tag{4.77}$$

where D_a is the covariant derivative compatible with h_{ab} and

$$K(-e^{2r}\square_h) = \sum_{k=0} k_k (-e^{2r}\square_h)^k. \tag{4.78}$$

We will later return to the choice of coefficients k_k .

To study the gravity dual of a non-local $T\bar{T}$ deformation, we must refine the usual scaling estimates by taking into account the explicit r -dependence in the Hamiltonian constraint. Once the r -direction appears in this way, it is natural to modify

the dilation operator as

$$\delta_D = \int d^2x \left(\partial_r + 2h_{ab} \frac{\delta}{\delta h_{ab}} \right). \quad (4.79)$$

Here, ∂_r acts only on the explicit dependence on r in the Hamiltonian constraint. The combination $-e^{2r}\square_h$ is then invariant under this modified scaling, in the sense that for any field X obeying $\delta_D X_{ab} = -\lambda X_{ab}$,

$$\begin{aligned} \delta_D(K(-e^{2r}\square_h)X_{ab}) &= -\lambda K(-e^{2r}\square_h)X_{ab}, \\ \delta_D(K(-e^{2r}\square_h)X) &= -(\lambda + 2)K(-e^{2r}\square_h)X. \end{aligned} \quad (4.80)$$

As in the analysis of the ordinary $T\bar{T}$ deformation, we solve the Hamilton–Jacobi equation perturbatively. The first-order equation reads

$$0 = (16\pi G_N) \left(\pi_{(0)a}^b K(-e^{2r}\square_h) \pi_{(0)b}^a - \pi_{(0)} K(-e^{2r}\square_h) \pi_{(0)} \right) + \frac{1}{8\pi G_N}. \quad (4.81)$$

Assuming $\pi_{(0)ab} = Ph_{ab}$, we obtain

$$P = -\frac{1}{\sqrt{k_0}} \frac{1}{16\pi G_N}, \quad (4.82)$$

where the sign is chosen so that the spacetime remains asymptotically AdS.

The second-order equation becomes

$$\begin{aligned} &\pi_{(0)a}^b K(-e^{2r}\square_h) \pi_{(2)b}^a + \pi_{(2)a}^b K(-e^{2r}\square_h) \pi_{(0)b}^a \\ &- \pi_{(0)} K(-e^{2r}\square_h) \pi_{(2)} - \pi_{(2)} K(-e^{2r}\square_h) \pi_{(0)} = -\frac{1}{(16\pi G_N)^2} R[h] \end{aligned} \quad (4.83)$$

This is solved by

$$\pi_{(2)} = -\frac{1}{\sqrt{k_0}} \frac{1}{16\pi G_N} Q(-e^{2r}\square_h) R[h], \quad (4.84)$$

where $Q(-e^{2r}\square_h)$ is the inverse of $1 + \frac{1}{k_0} K(-e^{2r}\square_h)$ and

$$Q(-e^{2r}\square_h) = \sum_{k=0}^{\infty} q_k (-e^{2r}\square_h)^k, \quad q_0 = \frac{1}{2}, \quad q_1 = -\frac{k_1}{4k_0}, \dots \quad (4.85)$$

The resulting second-order Lagrangian is topological:

$$\int d^2x \sqrt{h} \tilde{\mathcal{L}}_{(2)} = \frac{1}{32\pi G_N} \frac{1}{\sqrt{k_0}} \int d^2x \sqrt{h} R[h], \quad (4.86)$$

and therefore

$$\tilde{\Pi}_{ab}^{(2)} = 0. \quad (4.87)$$

The fourth-order equation becomes

$$0 = \pi_{(0)a}^b K(-e^{2r}\square_h) \pi_{(4)b}^a + \pi_{(2)a}^b K(-e^{2r}\square_h) \pi_{(2)b}^a + \pi_{(4)a}^b K(-e^{2r}\square_h) \pi_{(0)b}^a - \pi_{(0)} K(-e^{2r}\square_h) \pi_{(4)} - \pi_{(2)} K(-e^{2r}\square_h) \pi_{(2)} - \pi_{(4)} K(-e^{2r}\square_h) \pi_{(0)}. \quad (4.88)$$

This is solved by

$$\pi_{(4)} = -\frac{16\pi G_N}{\sqrt{k_0}} Q(-e^{2r}\square_h) (\pi_{(2)a}^b K(-e^{2r}\square_h) \pi_{(2)b}^a - \pi_{(2)} K(-e^{2r}\square_h) \pi_{(2)}). \quad (4.89)$$

The associated fourth-order action is

$$\int d^2x \sqrt{h} \mathcal{L}_{(4)} = \frac{8\pi G_N}{\sqrt{k_0}} \int d^2x \sqrt{h} (\pi_{(2)a}^b K(-e^{2r}\square_h) \pi_{(2)b}^a - \pi_{(2)} K(-e^{2r}\square_h) \pi_{(2)}). \quad (4.90)$$

In boundary variables, the resulting effective action at $r = r_c$ is

$$\begin{aligned} S_{\text{finite}} &= \int_{\Sigma_{r_c}} d^2x \sqrt{h} (\mathcal{L}_{(2)} + \mathcal{L}_{(4)}) \\ &= S_{\text{CFT}}[\gamma] + \frac{2\pi G_N e^{-2r_c}}{\sqrt{k_0}} \int d^2x \sqrt{\gamma} \langle T^{ab} \rangle_{\gamma}^{\text{CFT}} K(-\square_{\gamma}) \langle \hat{T}_{ab} \rangle_{\gamma}^{\text{CFT}}. \end{aligned} \quad (4.91)$$

We now fix k_0 . As in the previous discussion, we want to regulate boundary UV degrees of freedom while suppressing the deformation near the conformal boundary, unlike the usual $T\bar{T}$ case. Thus we require $\frac{e^{-2r_c}}{\sqrt{k_0}} K(k^2) \rightarrow 0$ when the boundary momentum k^2 is large and r_c is large. We also demand that bulk physics is modified only deep in the interior, $r \ll 1$, while the asymptotic region $r \gg 1$ reproduces Einstein gravity. Hence we impose $K(e^{2r} k_{\text{bulk}}^2) \rightarrow 1$ for large r . Both requirements are met by choosing $k_0 = e^{4r_c} \bar{\mu}^2$, $\bar{\mu} = O(e^{-0 \cdot r_c})$, together with $K(x) \rightarrow 1$ as $x \rightarrow \infty$, and then taking $r_c \gg 1$ at the end. A concrete example is $K(x) = \frac{x/\Lambda^2 + e^{4r_c} \bar{\mu}^2}{x/\Lambda^2 + 1}$, with a UV scale Λ . This leads to the boundary effective action

$$S_{\text{finite}} = S_{\text{CFT}}[g_{(0)}] - \frac{2\pi G_N e^{-2r_c}}{\sqrt{k_0}} \int d^2x \sqrt{g_{(0)}} \langle T^{ab} \rangle_{g_{(0)}}^{\text{CFT}} K(-\square_{g_{(0)}}) \langle \hat{T}_{ab} \rangle_{g_{(0)}}^{\text{CFT}}. \quad (4.92)$$

This expression matches a non-local $T\bar{T}$ deformation. In other words, a generalized $T\bar{T}$ deformation with infinitely many derivatives appears as a finite cut-off correction to a modified bulk gravity theory whose Hamiltonian contains explicit non-local operators. For sufficiently large r_c , the cut-off effects become negligible, and the

deformation effectively acts only in the deep interior of the bulk.

We emphasize that the modified Hamiltonian (4.77) depends explicitly on r , and hence breaks diffeomorphism invariance in the radial direction. From the boundary point of view, the infrared fixed point of the deformed theory retains Poincaré and scale invariance, but no longer exhibits full conformal symmetry. This provides a loophole to the standard theorem that scale invariance in two dimensions typically implies conformal invariance under broad assumptions⁶ [103–105].

The bulk non-locality can be interpreted as an effective imprint of wormhole contributions. As emphasized by Coleman [106–109], wormholes can influence long-distance physics. If one constructs an effective theory valid at scales larger than the wormhole throat size, the resulting description naturally includes a non-local action of the form

$$\begin{aligned} I_{\text{eff}} &= \frac{1}{2} \int d^{d+1}x \sqrt{G(x)} \int d^{d+1}y \sqrt{G(y)} \sum_{i,j} \Delta_{ij} \Phi_i(x) G(x,y) \Phi_j(y) \\ &= \sum_{i,j} \int d^{d+1}x \sqrt{G(x)} \Phi_i(x) \frac{\Delta_{ij}}{\square} \Phi_j(x). \end{aligned} \tag{4.93}$$

Here Φ_i are low-energy composite operators in the bulk and $G(x,y)$ is the Green function for the $\square$ operator⁷. Turning this logic around, one may conjecture that the modified Hamiltonian (4.77) effectively captures the influence of certain wormhole geometries. The detailed features of such wormholes would then be encoded in the specific functional form of $K(-e^{2r}\square_h)$.

The characteristic separation scale of the wormhole mouths, l_{WH} , is reflected in the momentum dependence of K . Since $K(e^{2r}k_{\text{bulk}}^2)$ departs appreciably from its limiting values only when $e^{2r}k_{\text{bulk}}^2 \leq \Lambda$, bulk probes with wavelengths of order $l_{\text{WH}} \sim k_{\text{bulk}}^{-1} \sim e^r \Lambda$ are sensitive to wormhole effects. Near the asymptotic boundary (large r), only very small momenta can detect such effects, and in the limit $r \rightarrow \infty$ only zero-momentum modes remain sensitive. This is consistent with the boundary perspective, where the deformation acts only in the extreme infrared.

The explicit r -dependence may also be viewed as an effective breaking of radial diffeomorphism symmetry induced by the wormhole, leading to an effective description on the wormhole background without this symmetry. Other mechanisms beyond geometric wormholes could also generate similar non-local structures.

⁶We thank Yu Nakayama for emphasizing this point.

⁷Coleman’s original discussion effectively used $G(x,y) = \text{const.}$, whereas the present setup allows for a more general kernel.

4.2.3 T_1T_2 deformation: discussion

We now return to our main target: the possibility of obtaining a CFT dual via a non-local T_1T_2 deformation. The discussion below is necessarily heuristic. We begin with the local T_1T_2 deformation,

$$\frac{\partial}{\partial\mu}S_{[\mu]} = \frac{1}{2} \int d^2x \sqrt{\gamma_{[\mu]}} \left(T_{ab}^{(1)} T^{(2)ab} - T^{(1)} T^{(2)} \right). \quad (4.94)$$

Applying the Hubbard–Stratonovich method, one finds that the metric flow equations take the form

$$\begin{aligned} \frac{\partial}{\partial\mu} \gamma_{[\mu]ab}^{(1)} &= -2 \left(\hat{T}^{(2)} \right)_{ab}, \\ \frac{\partial}{\partial\mu} \gamma_{[\mu]ab}^{(2)} &= -2 \left(\hat{T}^{(1)} \right)_{ab}. \end{aligned} \quad (4.95)$$

To derive the stress-tensor flow, we must confront a key subtlety of the T_1T_2 deformation: after turning on the interaction, the two metrics that would otherwise be associated with the two CFTs are no longer independent. Since the interaction breaks part of the diffeomorphism invariance, one effectively has only a single conserved stress tensor. To sidestep this issue, we restrict attention to a finite-temperature setting where $\text{CFT}_{(1)}$ and $\text{CFT}_{(2)}$ are related by the KMS condition,

$$T^{(2)ab}(t, x) = T^{(1)ab} \left(t + i\frac{\beta}{2}, x \right). \quad (4.96)$$

In this case, one can write the stress-tensor flow equation as

$$\frac{\partial}{\partial\mu} \hat{T}_{ab}^{(1)} = -\frac{1}{2} \hat{T}_{ac}^{(1)} \hat{T}_b^{(2)c} - \frac{1}{2} \hat{T}_{ac}^{(2)} \hat{T}_b^{(1)c} - \frac{3}{2} \hat{T}^{(1)} \hat{T}_{ab}^{(2)} + \frac{3}{2} \hat{T}_{ab}^{(1)} \hat{T}^{(2)}, \quad (4.97)$$

where the right-hand side is expressed solely in terms of the metric $\gamma_{ab}^{(1)}$. A corresponding equation can also be written for $\hat{T}_{ab}^{(2)}$. Solving these coupled flow equations in general is nontrivial. However, if we impose $\gamma_{ab}^{(1)} = \gamma_{ab}^{(2)}$ and $\hat{T}_{ab}^{(1)} = \hat{T}_{ab}^{(2)}$ at $\mu = 0$, then the flow reduces to the standard $T\bar{T}$ case. In that situation, the bulk dual is the usual finite cut-off holography, at least at the level of the on-shell action and thermodynamics. In more general situations, we expect the two boundaries to be related in a nontrivial way near the asymptotic region. One possibility is to glue two surfaces in a manner similar to [4, 110]. Note that [4] studied brane-world holography via junction conditions, which may differ from the dual of a T_1T_2 deformation: junction conditions yield “dynamical” gravity on the brane and require imposing a Virasoro constraint on the sum of the two stress tensors.

We now extend this logic to the non-local $T_1 T_2$ deformation to probe effects deeper in the bulk. Concretely, we consider

$$\frac{\partial}{\partial \mu} S_{[\mu]} = \frac{1}{2} \int d^2 x \sqrt{-\gamma_{[\mu]}} \epsilon_{ab} \epsilon_{cd} (T_{[\mu]}^{(1)})^{ab} K(-\square_\gamma) (T_{[\mu]}^{(2)})^{cd}. \quad (4.98)$$

Although an analytic treatment of this deformation is difficult, we expect it to share qualitative features with the non-local $T\bar{T}$ deformation. For the bulk interpretation, it is useful to again focus on the finite-temperature setup and identify

$$T_{ab}^{(2)}(t, x) = T_{ab}^{(1)}\left(t + i\frac{\beta}{2}, x\right) = e^{i\frac{\beta}{2}\partial_t} T_{ab}^{(1)}(t, x). \quad (4.99)$$

In this viewpoint, the differential operator $e^{i\frac{\beta}{2}\partial_t}$ can be regarded as an additional source of non-locality, to be combined with $K(-\square_\gamma)$. By analogy with the non-local $T\bar{T}$ case, one may then imagine that the bulk dual is a non-local gravitational theory with additional factors of the form $e^{ie^{2r}\frac{\beta}{2}\partial_t}$. Such factors may be interpreted as encoding wormhole-like connections between Rindler wedges. This suggests that even when one begins with two disconnected AdS geometries, an appropriate double-trace deformation can create an IR connection between the two. Applied to the eternal AdS black hole setup discussed here, it is then natural to expect wormhole-like connections to develop in the near-horizon region. A more detailed analysis of this speculative picture is left for future work.

4.3 AdS/BCFT and AdS/ICFT

The AdS/BCFT correspondence is a generalization of the AdS/CFT correspondence with boundary. As a bottom-up model, the following correspondence was proposed [111, 112]: A BCFT on a d -dimensional manifold N is equivalent to $(d+1)$ -dimensional AdS gravity with a dynamical d -dimensional membrane attached. This dynamical membrane is called an end-of-the-world brane (EOW brane) [113]. More concretely, consider the gravitational action

$$I_{\text{grav}}[G] = \frac{1}{16\pi G_N^{(d+1)}} \int_M d^{d+1}x \sqrt{-G} (R[G] - 2\Lambda) \quad (4.100)$$

$$+ \frac{1}{8\pi G_N^{(d+1)}} \int_N d^d x \sqrt{-\gamma} K[\gamma] + I_{\text{ct}}[\gamma] \quad (4.101)$$

$$+ \frac{1}{8\pi G_N^{(d+1)}} \int_Q d^d x \sqrt{-h} (K[h] - \mathcal{T}). \quad (4.102)$$

Here $\partial M = N \cup Q$, γ_{ab} is the induced metric on the conformal boundary N on which Dirichlet boundary conditions are imposed, and h_{ab} is the induced metric on

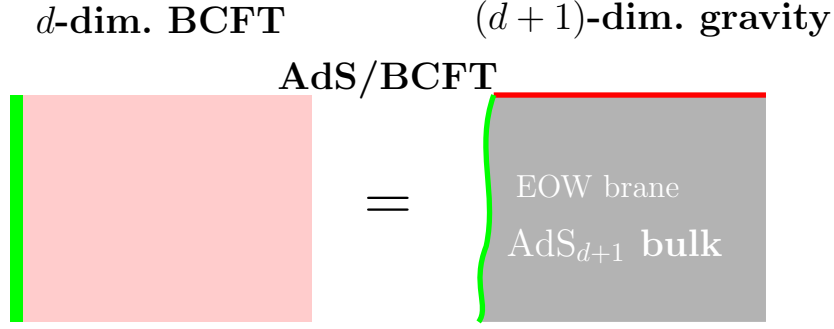


Figure 11: Geometry of the gravitational dual in AdS/BCFT. M is the $(d+1)$ -dimensional bulk spacetime. N is the d -dimensional conformal boundary where the BCFT lives. Q is the EOW brane.

the EOW brane Q , where the Israel boundary condition

$$K[h]_{ab} - K[h] h_{ab} + \mathcal{T} h_{ab} = 0 \quad (4.103)$$

is imposed. The parameter $\mathcal{T}$ is the brane tension.

The classical position of the EOW brane is determined by the boundary condition (4.103). Taking the trace of (4.103), we obtain

$$K = \frac{d}{d-1} \mathcal{T}. \quad (4.104)$$

This yields

$$K_{ab} = \frac{1}{d-1} \mathcal{T} h_{ab}, \quad (4.105)$$

which is the equation we must solve.

Let us now consider EOW branes allowed in the bulk when the bulk is three-dimensional with the Poincaré metric. Suppose the EOW brane is expressed as $z = F(x, t)$ in these coordinates. The components of (4.105) become

$$K_{xx} = \frac{1 + F'^2 + FF''}{F^2 \sqrt{1 - \dot{F}^2 + F'^2}} = \mathcal{T} \frac{1 + F'^2}{F^2} = \mathcal{T} h_{xx}, \quad (4.106)$$

$$K_{tt} = \frac{-1 + \dot{F}^2 + F\ddot{F}}{F^2 \sqrt{1 - \dot{F}^2 + F'^2}} = \mathcal{T} \frac{-1 + \dot{F}^2}{F^2} = \mathcal{T} h_{tt}, \quad (4.107)$$

$$K_{xt} = \frac{\dot{F}F' + F\dot{F}'}{F^2 \sqrt{1 - \dot{F}^2 + F'^2}} = \mathcal{T} \frac{F'\dot{F}}{F^2} = \mathcal{T} h_{xt}. \quad (4.108)$$

Here $F' = \partial_x F(x, t)$ and $\dot{F} = \partial_t F(x, t)$.

Assuming a static vacuum BCFT with boundary located at $X = 0$, we have

$\dot{F} = 0$. Then the second equation yields

$$\frac{1}{\sqrt{1+F'^2}} = \mathcal{T}. \quad (4.109)$$

From this we find that the EOW brane surface is uniquely given by

$$x + \lambda z = 0. \quad (4.110)$$

The induced metric on the EOW brane Q is the AdS_2 metric

$$ds^2|_{\text{ind}} = \frac{(1 + \lambda^2)dz^2 - dt^2}{z^2}, \quad (4.111)$$

and the relation between the brane tension and the coefficient λ is

$$\lambda = \frac{\mathcal{T}}{\sqrt{1 - \mathcal{T}^2}}. \quad (4.112)$$

Thus, for an AdS brane we must have $|\mathcal{T}| < 1$.

The form of (4.106) in Poincaré coordinates can be constructed using the Bañados transformation [13, 114]. Since any 3D AdS spacetime can be mapped into the Poincaré metric by a coordinate transformation, EOW branes in other spacetimes can be obtained by transforming the solutions (4.109) and (4.110).

■ Holographic Entanglement Entropy

We now extend the holographic entanglement entropy formula to the AdS/BCFT correspondence. The following prescription gives the entanglement entropy [111, 112]. For simplicity, we discuss the case $d = 2$ where the subsystem is an interval, since this is the only case we will use later.

Let $A \cup N$ be a subsystem of the BCFT. Define S_{con_A} and S_{dis_A} by

$$S_{\text{con}_A} := \frac{\text{Area}[\Gamma_A^{\text{con}}]}{4G_N}, \quad (4.113)$$

$$S_{\text{dis}_A} := \frac{\text{Area}[\Gamma_A^{\text{dis}}]}{4G_N}. \quad (4.114)$$

Here Γ_A^{con} is the geodesic connecting ∂A , and Γ_A^{dis} is the sum of two geodesics: one from each endpoint of A to a point on the EOW brane. This point is chosen to minimize the total length. We call Γ_A^{con} the connected geodesic and Γ_A^{dis} the disconnected geodesic. The holographic entanglement entropy is then

$$S_A = \min\{S_{\text{con}_A}, S_{\text{dis}_A}\}. \quad (4.115)$$

As an example, consider the vacuum state of a BCFT defined on the left half-line

$x \leq 0$. Let the subsystem be $A = [l_1, l_2]$. The bulk metric is the Poincaré metric. Since Γ_A^{con} is the same as in the case without a boundary, using the cutoff $\eta = \epsilon$ we obtain

$$S_{\text{con}A} = \frac{c}{3} \log \left(\frac{l_2 - l_1}{\epsilon} \right). \quad (4.116)$$

Next consider the disconnected geodesics. Take a point $R_1 = (z = z_R, x = x_R = -\lambda z_R, t)$ on the EOW brane and a boundary point $P_1 = (z_P = \epsilon, x = x_P = l_1, t)$. Then

$$-X_{P_1}^{\mathbf{M}} X_{R_1 \mathbf{M}} = \frac{L_{\text{AdS}}^2}{\epsilon} \left(\frac{1}{2z_R} ((1 + \lambda^2) z_R^2 + l_1^2) + l_1 \lambda \right). \quad (4.117)$$

This is minimized for $z_R = \frac{l_1}{\sqrt{1 + \lambda^2}}$, and we obtain

$$S_{\text{dis}A} = \frac{c}{6} \log \frac{2l_1}{\epsilon} + \frac{c}{6} \log \frac{2l_2}{\epsilon} + 2S_{\text{bdry}}, \quad (4.118)$$

$$S_{\text{bdry}} = \frac{c}{6} \log \left(\lambda + \sqrt{1 + \lambda^2} \right), \quad (4.119)$$

where S_{bdry} corresponds to the boundary entropy.

The difference is

$$S_{\text{con}A} - S_{\text{dis}A} = \frac{c}{6} \log \left(\frac{(l_2 - l_1)^2}{4l_1 l_2} \frac{1}{\lambda + \sqrt{1 + \lambda^2}} \right), \quad (4.120)$$

so for example when $l_2 \gg l_1$, the disconnected geodesic is preferred.

For the half-line $A = [l, \infty)$, the connected geodesic does not exist, and only the disconnected one contributes:

$$S_A = S_{\text{dis}A} = \frac{c}{6} \log \frac{2l}{\epsilon} + S_{\text{bdry}}. \quad (4.121)$$

This agrees with the field-theoretic calculation. Indeed, recalling the entanglement entropy of an interval in the vacuum of a 2D CFT, one obtains this formula from the one-point function of the twist operator in BCFT via the doubling method.

■ AdS/ICFT

The AdS/ICFT correspondence is a generalization of AdS/BCFT to the interface CFTs (ICFTs for short). There are many models of holographic ICFTs, typically the bulk duals possess some branes which dual to the conformal interface. Here, we discuss Janus deformations which includes a soft wall. We consider a three dimensional gravitational theory with massless dilaton fields. To show this, let us recall the AdS_3 Janus solution [115, 116]. For this we start with the three dimensional

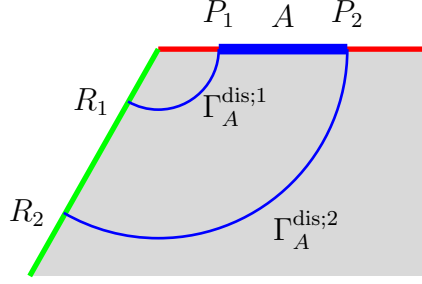


Figure 12: Holographic entanglement entropy in AdS/BCFT. The green curve is the connected RT surface; the yellow curve is the disconnected RT surface.

action for the 3d metric and the dilaton field φ :

$$I_{\text{grav}}[G, \varphi] = \frac{1}{16\pi G_N} \int d^3x \sqrt{G} [R[G] - G^{AB} \partial_A \varphi \partial_B \varphi + 2]. \quad (4.122)$$

The equation of motion is given by

$$\begin{aligned} R[G]_{AB} - \frac{1}{2} R[G] G_{AB} - G_{AB} &= 8\pi G_N T_{(\varphi)AB}, \\ T_{(\varphi)AB} &= \frac{1}{8\pi G_N} \left(\nabla_A \varphi \nabla_B \varphi - \frac{1}{2} G_{AB} (\nabla_C \varphi \nabla^C \varphi) \right), \\ \square_G \varphi &= 0. \end{aligned} \quad (4.123)$$

Now suppose we seek the solution with AdS_2 isometry similar to the previous AdS/BCFT subsection. The solution ansatz looks like

$$ds^2 = f(\mu)(d\mu^2 + ds_{\text{AdS}_2}^2), \quad \varphi = \varphi(\mu). \quad (4.124)$$

The equations of motion for the scalar field φ and the metric $f(\mu)$ leads to

$$\begin{aligned} \frac{1}{f(\mu)^{3/2}} \frac{d}{d\mu} \left(\sqrt{f(\mu)} \frac{d}{d\mu} \varphi \right) &= 0 \Rightarrow \frac{d\varphi(\mu)}{d\mu} = \frac{\gamma}{\sqrt{f(\mu)}}, \\ R[G] &= -\frac{2}{f} - \frac{2f''(\mu)}{f(\mu)} + \frac{3f'(\mu)}{2f(\mu)^3} = -6 + \frac{\gamma^2}{f(\mu)^2} \Leftrightarrow \frac{df(\mu)}{d\mu} = \sqrt{f(4f^2 - 4f + 2\gamma^2)}, \end{aligned} \quad (4.125)$$

where γ is a constant which parametrizes the Janus deformation. For generic γ we find a following solution

$$f(\mu) = \frac{\left(\frac{1+\chi}{2}\right)^2}{\text{sn}\left(\frac{1+\chi}{2}(\mu + \mu_0), \frac{1-\chi}{1+\chi}\right)} \quad (4.126)$$

where we introduced $\chi = \sqrt{1 - 2\gamma^2}$, μ_0 is explained right now. In the Janus solution, there is a parameter μ_0 . This is the such that $-\mu_0 \leq \mu \leq \mu_0$, given by

$$\mu_0 = \sqrt{\frac{2}{1+\chi}} \int_0^1 \frac{ds}{\sqrt{(1-s^2)\left(1 - \frac{1-\chi}{1+\chi}s^2\right)}} = \sqrt{\frac{2}{1+\chi}} K\left(\frac{1-\chi}{1+\chi}\right), \quad (4.127)$$

where $K(z)$ is the complete elliptic integral of the first kind. We find that γ should be $\gamma^2 < \frac{1}{2}$ for nice solution. If $\gamma^2 \geq \frac{1}{2}$, its is known that there is a naked singularity. At $\mu = \pm\mu_0$, $f(\mu)$ gets divergent and these two correspond to two different AdS boundaries. When $\gamma = 0$ i.e. without the Janus deformation, we have $f(\mu) = \frac{1}{\cos^2 \mu}$ and $\mu_0 = \frac{\pi}{2}$. There is another nice radial coordinate. The coordinate transformation $y = \int_0^\mu ds \sqrt{f(s)}$ rewrites the solution as follows

$$\begin{aligned} ds^2 &= dy^2 + f(y) ds_{\text{AdS}_2}^2, \\ f(y) &= \frac{1}{2}(1 + \sqrt{1 - 2\gamma^2} \cosh 2y), \\ \varphi(y) &= \varphi_0 + \frac{1}{\sqrt{2}} \log \left[\frac{1 + \sqrt{1 - 2\gamma^2} + \sqrt{2}\gamma \tanh y}{1 + \sqrt{1 - 2\gamma^2} - \sqrt{2}\gamma \tanh y} \right]. \end{aligned} \quad (4.128)$$

We are noticing that $\varphi(y \rightarrow \pm\infty) = \varphi_0 \pm \frac{1}{\sqrt{2}} \log \left[\frac{1 + \sqrt{1 - 2\gamma^2} + \sqrt{2}\gamma}{1 + \sqrt{1 - 2\gamma^2} - \sqrt{2}\gamma} \right]$. This two different asymptotic values of the dilaton indicate that the CFTs in $y \rightarrow \pm\infty$ are different CFTs. Two distinct CFTs are connected by a interface. Because of left AdS₂ isometry we can see that this interface is conformal interface since they keep half of conformal symmetry. One specific examples are D1-D5 CFTs and we see the deformed theories constructed from two different tori radius. The integration constant γ or χ is related to the energy of this conformal interface.

Depending on the choice of AdS₂ metric in (4.128), we have various physical situations of ICFTs. One simplest choice is the Poincare AdS₂ metric,

$$ds_{\text{AdS}_2}^2 = \frac{-dt^2 + d\xi^2}{\xi^2}, \quad \xi > 0. \quad (4.129)$$

In this case, we see that the corresponding ICFT lie in the plane and interface locates in $x = 0$. Similar to the EOW brane in the AdS/BCFT correspondence, there is a brane corresponds to this interface. Indeed, the solution of equation of motion is very similar to the kink solution of ϕ^4 theory and the bulk stress tensor is localized as we can see in the Figure 13. Thus the Janus solution is nothing but a soft wall solution.

Later, we consider another choice that $ds_{\text{AdS}_2}^2$ is blackhole which gives nontrivial

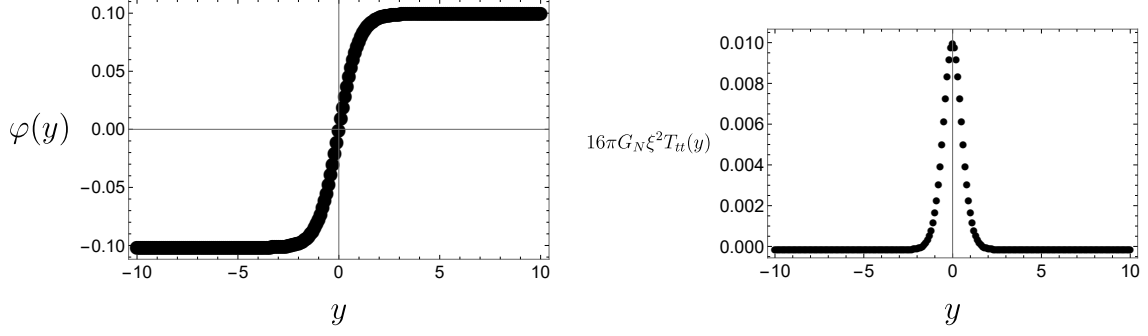


Figure 13: Plot of solution of $\varphi(y)$ and bulk stress tensor $T_{tt}(y)$. We can see that the plots are very similar to the kink solution.

non-equilibrium dynamics.

4.4 Braneworld holography

In the usual AdS/CFT correspondence, we assign the Dirichlet boundary conditions for bulk fields and boundary fields are frozen and non dynamical (Thus they are play a role of source). We can consider another boundary conditions for example the Neumann boundary conditions [117–119]. In this case, the boundary fields can be dynamical and some parts of the conformal boundary is EOW brane. As one specific example, we consider flat brane $\mathcal{T} = 1$ which is originally discussed by Randall-Sundrum. Then we consider to expand the Einstein Hilbert action with Hwaking-Gibbons boundary action. This procedure is exactly same as what we done in the holographic renormalization. Here we consider $d = 2$ for simplicity. Suppose we have $S_{\text{ren}}^{(\text{fn})}$ as a renormalized on-shell action. Then we discuss the GKPW relation for this gravitational action,

$$e^{-S_{\text{ren}}^{(\text{fn})}[\gamma]} = e^{-W_{\text{CFT}}^{(\text{fn})}[\gamma]} \quad (4.130)$$

,where

$$\begin{aligned} S_{\text{ren}}^{(\text{fn})} &= \int d^2x \sqrt{\gamma} (\mathcal{L}^{(2)} + \mathcal{L}^{(4)}) \\ &= S_{\text{reg}} + S_{\text{ct}} = W_{\text{CFT}}^{(\text{fn})} \end{aligned} \quad (4.131)$$

Then moving the counter term to the left and doing the integral since we consider dynamical fluctuation of the boundary fields

$$\int [d\gamma] e^{-S_{\text{EH}}[G;\gamma] - S_{\text{HG}}[\gamma]} = \int [d\gamma] e^{+S_{\text{ct}}[\gamma] - W_{\text{CFT}}^{(\text{fn})}[\gamma]} \quad (4.132)$$

The gravitational equation of motion is obtained as

$$\frac{\delta S_{\text{ct}}[\gamma]}{\delta \gamma_{ab}} - \frac{\delta W_{\text{CFT}}^{(\text{fin})}[\gamma]}{\delta \gamma_{ab}} = 0. \quad (4.133)$$

The concrete form of the counter term is

$$\begin{aligned} S_{\text{ct}} &= -\frac{1}{\kappa^2} \int_{\Sigma_r} d^2x \sqrt{\gamma} \left(-1 - \frac{1}{8} R[\gamma] \log e^{-2r} \right) \\ &= \frac{1}{\kappa^2} \int d^2x \sqrt{\gamma} + \frac{\pi \chi}{2\kappa^2} \log e^{-2r} \end{aligned} \quad (4.134)$$

,where χ is the Euler number of the surface Σ_r . Thus, the equation of motion can be rewritten as

$$\langle T^{ab} \rangle_{\text{ren}}^{(\text{fin})} + \frac{1}{\kappa^2} \gamma^{ab} = 0. \quad (4.135)$$

To compare to the convention of the paper [4], we use the relation

$$\begin{aligned} \tau^{ab} &:= K[\gamma]^{ab} - K[\gamma] \gamma^{ab} + \gamma^{ab} \\ \frac{1}{\sqrt{\gamma}} \Pi^{ab} &= -\frac{1}{2\kappa^2} (K[\gamma]^{ab} - K[\gamma] \gamma^{ab}) \\ &= -\frac{1}{2} \langle T^{ab} \rangle_{\text{ren}}^{(\text{fin})} \\ \langle T^{ab} \rangle_{\text{ren}}^{(\text{fin})} &= \frac{1}{\kappa^2} (\tau^{ab} - \gamma^{ab}). \end{aligned} \quad (4.136)$$

Then the (4.135) is equivalent to

$$\tau_{ab} = 0 \quad (4.137)$$

,which is by definition, equivalent to the junction condition.

Part II

Eigenstate Thermalization Hypothesis and Gravity

5 Thermalization and Equilibration in closed quantum many body systems

We study quantum many-body systems defined either on spatial lattices or as quantum field theories on $\mathbb{R} \times S^{d-1}$, and we assume that the energy spectrum is discrete within the energy window of interest. This discreteness is expected since the spatial manifold has finite volume. We write the Hamiltonian in its spectral decomposition as $H = \sum_n E_n |E_n\rangle \langle E_n|$. The theory is defined on a Hilbert space $\mathcal{H}$ associated with Cauchy slices, and we denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators acting on $\mathcal{H}$. As a preliminary step, we introduce an energy shell inside the full Hilbert space $\mathcal{H}$,

$$\mathcal{H}_{E,\Delta E} = \bigoplus_{E < E_n < E + \Delta E} \mathbb{C} |E_n\rangle \subset \mathcal{H}, \quad (5.1)$$

whose dimension is denoted by $D_{E,\Delta E} = \dim \mathcal{H}_{E,\Delta E}$. The corresponding microcanonical entropy is defined as $S_{\text{mc}}(E) := \log D_{E,\Delta E}$. The energy width ΔE is chosen to be much smaller than the central energy E , while still being sufficiently large so that the shell contains a large number of energy eigenstates. The microcanonical ensemble associated with this energy shell is defined by

$$\rho_{\text{mc}}^{(E,\Delta E)} = \frac{1}{D_{E,\Delta E}} \mathbb{P}_{\mathcal{H}_{E,\Delta E}}, \quad (5.2)$$

where $\mathbb{P}_{\mathcal{H}_{E,\Delta E}}$ denotes the projection operator onto $\mathcal{H}_{E,\Delta E}$. For an operator $\mathcal{O} \in \mathcal{B}(\mathcal{H})$, we denote its expectation value in the microcanonical ensemble by $\langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} = \text{Tr} \left[\mathcal{O} \rho_{\text{mc}}^{(E,\Delta E)} \right]$.

In practice, especially in the thermodynamic limit, one does not consider the full operator algebra $\mathcal{B}(\mathcal{H})$, but rather focuses on a distinguished subset of physically relevant operators that remain meaningful in that limit. We denote such a set by $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$. Depending on the choice of thermodynamic limit and of $\mathcal{A}$, several notions of thermal equilibrium can be formulated. Typical examples include

- the large volume limit, with $\mathcal{A}$ chosen as a set of macroscopic operator densities, such as the density of total magnetization [30, 120–123]. This choice leads to the concept of macroscopic thermal equilibrium (MATE). A notable feature is that this notion applies equally well to classical systems, in contrast to microscopic notions of equilibrium;
- the large volume limit with $\mathcal{A}$ taken to consist of local or few-body operators [124–127]. This setting gives rise to microscopic thermal equilibrium (MITE). This notion is closely tied to canonical typicality, namely the statement that typical pure states in an energy shell are locally indistinguishable

from the microcanonical ensemble. It is inherently quantum mechanical and has no direct classical analogue. Related ideas, often referred to as subsystem ETH, have been discussed for conformal field theories in [128]. For a detailed comparison between MITE and MATE, see [129];

- the large N limit, where $\mathcal{A}$ consists of light single-trace operators or a small number of multi-trace operators. This limit corresponds to the semiclassical regime of the gravitational dual in the AdS/CFT correspondence and plays a central role in black hole thermodynamics.⁸

Precise formulations of these notions of thermal equilibrium are given in Section 5.5.

In introductory courses on statistical physics, equilibrium states are typically described by mixed states such as the microcanonical or canonical ensembles. For isolated quantum systems, however, it is necessary to formulate what it means for a *pure* state to represent thermal equilibrium. For a given thermodynamic limit and a chosen set of observables $\mathcal{A}$, we adopt the following definition.

Definition

Definition 1. We say that a pure state $|\psi\rangle \in \mathcal{H}_{E,\Delta E}$ represents equilibrium with respect to a fixed thermodynamic limit and the operator set $\mathcal{A}$ if, for every $\mathcal{O} \in \mathcal{A}$, the inequality

$$\left| \langle \psi | \mathcal{O} | \psi \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right| \leq e^{-O(S_{\text{mc}}(E))} \quad (5.3)$$

holds in the regime of large but finite $S_{\text{mc}}(E)$.

A comment is in order regarding the exponentially small bound appearing in this definition. In realistic quantum many-body systems, the deviation from the microcanonical expectation value can depend sensitively on the choice of the energy width ΔE . The width is assumed to be chosen such that the energy shell contains many quantum states, but its precise scaling can influence quantitative estimates.

Nevertheless, equilibrium pure states form an overwhelming majority within the energy shell: they are typical states in the Hilbert space. Indeed, a typicality theorem guarantees that almost all pure states drawn from the energy shell represent thermal equilibrium in the sense defined above.

5.1 Typicality Theorems

Surprisingly, mathematically we can show that most of the quantum states in the energy shell represents the equilibrium defined in the Definition 5. This is basi-

⁸In some situations, a large volume limit is also considered simultaneously with the large N limit in AdS/CFT settings.

cally follows from the mathematical phenomena called measure concentration in probability theory. Typically people call this phenomena as typicality of thermal equilibrium. Following we see some concrete realization of typicality theorems.

5.1.1 Lloyd's Discussion

Here we review the Lloyd's typicality discussions and raise its problem in our purpose. We consider the random state $|\psi\rangle = \sum_n c_n |E_n\rangle \in \mathcal{H}_{E,\Delta E}$, $\sum_n |c_n|^2 = 1$. The nice measure on $\mathcal{H}_{E,\Delta E}$ is uniform measure. This is unique basis independent measure. We define the average over $\mathcal{H}_{E,\Delta E}$ as

$$\mathbb{E}[F(|\psi\rangle)] := \frac{\int (\prod_n dc_n) \delta(\sum_n |c_n|^2 - 1) F(\{c_n\})}{\int (\prod_n dc_n) \delta(\sum_n |c_n|^2 - 1)} \quad (5.4)$$

The formula on this measure is following

$$\mathbb{E}[c_n^* c_m] = \frac{1}{D_{E,\Delta E}} \delta_{nm} \quad (5.5)$$

$$\mathbb{E}[c_n^* c_m c_k^* c_l] = \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} + 1)} (\delta_{nm} \delta_{lk} + \delta_{nl} \delta_{mk}). \quad (5.6)$$

From these formulas we can derive for any $\mathcal{O} \in \mathcal{B}(\mathcal{H})$,

$$\mathbb{E}[\langle \psi | \mathcal{O} | \psi \rangle] = \sum_{n,m} \mathbb{E}[c_n^* c_m] \langle E_n | \mathcal{O} | E_m \rangle \quad (5.7)$$

$$= \frac{1}{D_{E,\Delta E}} \sum_n \langle E_n | \mathcal{O} | E_n \rangle = \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E}. \quad (5.8)$$

Also the square is

$$\begin{aligned} & \mathbb{E}[\langle \psi | \mathcal{O} | \psi \rangle^2] \\ &= \sum_{n,m,k,l} \mathbb{E}[c_n^* c_m c_k^* c_l] \langle E_n | \mathcal{O} | E_m \rangle \langle E_k | \mathcal{O} | E_l \rangle \\ &= \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} + 1)} \left(\left(\sum_n \langle E_n | \mathcal{O} | E_n \rangle \right)^2 + \sum_{n,m} \langle E_n | \mathcal{O} | E_m \rangle \langle E_m | \mathcal{O} | E_n \rangle \right) \\ &= \frac{D_{E,\Delta E}}{D_{E,\Delta E} + 1} \left(\langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right)^2 + \frac{1}{D_{E,\Delta E} + 1} \text{Tr} \left[(\rho_{\text{mc}}^{(E,\Delta E)})^{\frac{1}{2}} \mathcal{O} (\rho_{\text{mc}}^{(E,\Delta E)})^{\frac{1}{2}} \mathcal{O} \right]. \end{aligned} \quad (5.9)$$

Then the variance is evaluate as

$$\begin{aligned}
& \mathbb{V}[\langle \psi | \mathcal{O} | \psi \rangle] \\
&= \mathbb{E}[\langle \psi | \mathcal{O} | \psi \rangle^2] - \mathbb{E}[\langle \psi | \mathcal{O} | \psi \rangle]^2 \\
&= -\frac{1}{D_{E,\Delta E} + 1} \left(\langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right)^2 + \frac{1}{D_{E,\Delta E} + 1} \text{Tr} \left[\left(\rho_{\text{mc}}^{(E,\Delta E)} \right)^{\frac{1}{2}} \mathcal{O} \left(\rho_{\text{mc}}^{(E,\Delta E)} \right)^{\frac{1}{2}} \mathcal{O} \right] \\
&= \frac{1}{D_{E,\Delta E} + 1} \text{Tr} \left[\left(\rho_{\text{mc}}^{(E,\Delta E)} \right)^{\frac{1}{2}} \hat{\mathcal{O}}_{\text{mc}} \left(\rho_{\text{mc}}^{(E,\Delta E)} \right)^{\frac{1}{2}} \hat{\mathcal{O}}_{\text{mc}} \right] \\
&= \frac{1}{D_{E,\Delta E} + 1} \langle \text{mcTFD} | \hat{\mathcal{O}}_{\text{mc}}^L \otimes \hat{\mathcal{O}}_{\text{mc}}^R | \text{mcTFD} \rangle
\end{aligned} \tag{5.10}$$

where we define the one-point function subtracted operator $\hat{\mathcal{O}}_{\text{mc}} = \mathcal{O} - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E}$ and $|\text{mcTFD}\rangle = \frac{1}{\sqrt{D_{E,\Delta E}}} \sum_n |E_n\rangle_L |E_n\rangle_R$. We have another expression by using the projection on the energy shell,

$$\mathbb{V}[\langle \psi | \mathcal{O} | \psi \rangle] = \frac{1}{D_{E,\Delta E} + 1} \left(\langle \mathcal{O} \mathbb{P}_{E,\Delta E} \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} - \left(\langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right)^2 \right). \tag{5.11}$$

If $\mathbb{P}_{E,\Delta E} \mathcal{O} = \mathcal{O}$, so the operator $\mathcal{O}$ is high energy operator, then

$$\mathbb{V}[\langle \psi | \mathcal{O} | \psi \rangle] = \frac{1}{D_{E,\Delta E} + 1} \left(\langle \mathcal{O}^2 \rangle_{\text{mc}}^{E,\Delta E} - \left(\langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right)^2 \right). \tag{5.12}$$

By using the Chebishev's inequality, we have bound for the probability,

$$\text{Prob} \left[\left| \langle \psi | \mathcal{O} | \psi \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right| > \epsilon \right] < \frac{\mathbb{V}[\langle \psi | \mathcal{O} | \psi \rangle]}{\epsilon^2}. \tag{5.13}$$

If the right hand side is very small, we can say that for almost all state $|\psi\rangle$, we cannot distinguish the pure state and microcanonical ensemble with probe $\mathcal{O}$. From now we see one specific example, when $\mathcal{O}$ is local operator.

5.1.2 Other Typicality theorem

The canonical typicality ensures that for any local operator such that $\mathcal{O} = \mathcal{O}_X \otimes 1_{X^c}$,

$$\langle \psi | \mathcal{O} | \psi \rangle = \text{Tr}_X [\rho_\psi^X \mathcal{O}_X] \approx \text{Tr}_X \left[\left(\rho_{\text{mc}}^{(E,\Delta E)} \right)^X \mathcal{O}_X \right] = \text{Tr} [\rho_{\text{mc}}^{(E,\Delta E)} \mathcal{O}] \tag{5.14}$$

where $\left(\rho_{\text{mc}}^{(E,\Delta E)} \right)^X := \text{Tr}_{X^c} \left[\rho_{\text{mc}}^{(E,\Delta E)} \right]$. This ensure us that most of the single pure state reproduce the same expectation value as that of microcanonical one. Surprisingly, similar statement holds for bounded operator [130–132],

$$\text{Prob} \left[\left| \langle \psi | \mathcal{O} | \psi \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right| > \epsilon \right] < \frac{\|\mathcal{O}\|_\infty^2}{\epsilon^2 D_{E,\Delta E}} \tag{5.15}$$

where $\|\mathcal{O}\|_\infty := \sup_{|\psi\rangle \neq 0} \sqrt{\frac{\langle \psi | \mathcal{O}^\dagger \mathcal{O} | \psi \rangle}{\langle \psi | \psi \rangle}}$ denotes the operator norm. This theorem says that the locality of an operator is not important at the level of expectation value, when we take the total system sufficiently large or whose Hilbert space is large.

We have more tight bounds which may be useful in QFTs:

Theorem

Theorem 1. *For any bounded operators $\mathcal{O} \in \mathcal{B}(\mathcal{H})$ and for any $\epsilon > 0$, it holds that*

$$\text{Prob} \left[\left| \langle \psi | \mathcal{O} | \psi \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right| > \epsilon \right] < \frac{1}{\epsilon^2 (D_{E, \Delta E} + 1)} \langle \text{mcTFD} | \hat{\mathcal{O}}_{\text{mc}}^L \otimes \hat{\mathcal{O}}_{\text{mc}}^R | \text{mcTFD} \rangle. \quad (5.16)$$

Also this right hand side is bounded by the following quantities

•

$$\begin{aligned} & \langle \text{mcTFD} | \hat{\mathcal{O}}_{\text{mc}}^L \otimes \hat{\mathcal{O}}_{\text{mc}}^R | \text{mcTFD} \rangle \\ & < \frac{Z(\beta) e^{\beta(E + \Delta E)}}{D_{E, \Delta E}} \langle \text{TFD}(\beta) | \hat{\mathcal{O}}_{\text{can}}^L \otimes \hat{\mathcal{O}}_{\text{can}}^R | \text{TFD}(\beta) \rangle \end{aligned} \quad (5.17)$$

where we denote

$$\langle \mathcal{O} \rangle_{\text{can}}^\beta = \text{Tr}[\mathcal{O} \rho_{\text{can}}^{(\beta)}], \quad \hat{\mathcal{O}}_{\text{can}} = \mathcal{O} - \langle \mathcal{O} \rangle_{\text{can}}^\beta. \quad (5.18)$$

The optimal choice of β is chosen such that

$$\frac{\partial S(E)}{\partial E} = \beta, \quad S(E) = \log D_{E, \Delta E} \quad (5.19)$$

and

$$\frac{Z(\beta) e^{\beta(E + \Delta E)}}{D_{E, \Delta E}} = O(\log(D_{E, \Delta E})). \quad (5.20)$$

The canonical correlation functions are

$$\begin{aligned} \langle \text{TFD}(\beta) | \hat{\mathcal{O}}_{\text{can}}^L \otimes \hat{\mathcal{O}}_{\text{can}}^R | \text{TFD}(\beta) \rangle &= \text{Tr} \left[\hat{\mathcal{O}}_{\text{can}} (\rho_{\text{can}}^{(\beta)})^{\frac{1}{2}} \hat{\mathcal{O}}_{\text{can}} (\rho_{\text{can}}^{(\beta)})^{\frac{1}{2}} \right] \\ &= \text{Tr} \left[\rho_{\text{can}}^{(\beta)} \hat{\mathcal{O}}_{\text{can}} \left(\frac{\beta}{2} \right) \hat{\mathcal{O}}_{\text{can}}(0) \right]. \end{aligned} \quad (5.21)$$

•

$$(\text{RHS}) < \frac{4 \|\mathcal{O}\|_\infty^2}{\epsilon^2 (D_{E, \Delta E} + 1)}. \quad (5.22)$$

This is similar to (5.15)

•

$$\langle \text{mcTFD} | \hat{\mathcal{O}}_{\text{mc}}^L \otimes \hat{\mathcal{O}}_{\text{mc}}^R | \text{mcTFD} \rangle \leq \|\mathcal{O}\|_{\infty}^2 \sqrt{2I(L : R)} \quad (5.23)$$

where $I(L : R)$ is a mutual information between the left and right subsystems of the (micro) canonical TFD state

In these bounds the first one is most tight and UV finite.

Proof. From the Chebyshev's inequality we have

$$\text{Prob} \left[\left| \langle \psi | \mathcal{O} | \psi \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right| > \epsilon \right] < \frac{\mathbb{V}[\langle \psi | \mathcal{O} | \psi \rangle]}{\epsilon^2}. \quad (5.24)$$

From (5.10), we have (5.16).

Then, we try to prove the bound with canonical ensemble (5.17). Firstly, we consider the following inequality,

$$\begin{aligned} & \langle \text{mcTFD} | \hat{\mathcal{O}}_{\text{mc}}^L \otimes \hat{\mathcal{O}}_{\text{mc}}^R | \text{mcTFD} \rangle \\ & \leq \langle \text{mcTFD} | (\hat{\mathcal{O}}_{\text{mc}}^L + a1_L) \otimes (\hat{\mathcal{O}}_{\text{mc}}^L + a1_R) | \text{mcTFD} \rangle \end{aligned} \quad (5.25)$$

for any $a \in \mathbb{R}$. The proof is very simple,

$$\begin{aligned} & \langle \text{mcTFD} | (\hat{\mathcal{O}}_{\text{mc}}^L + a1_L) \otimes (\hat{\mathcal{O}}_{\text{mc}}^L + a1_R) | \text{mcTFD} \rangle \\ & = \langle \text{mcTFD} | \hat{\mathcal{O}}_{\text{mc}}^L \otimes \hat{\mathcal{O}}_{\text{mc}}^L | \text{mcTFD} \rangle + a \langle \text{mcTFD} | \hat{\mathcal{O}}_{\text{mc}}^L \otimes 1_R | \text{mcTFD} \rangle \\ & + a \langle \text{mcTFD} | 1_L \otimes \hat{\mathcal{O}}_{\text{mc}}^R | \text{mcTFD} \rangle + a^2 \langle \text{mcTFD} | 1_L \otimes 1_R | \text{mcTFD} \rangle \\ & = \langle \text{mcTFD} | \hat{\mathcal{O}}_{\text{mc}}^L \otimes \hat{\mathcal{O}}_{\text{mc}}^L | \text{mcTFD} \rangle + a^2 \geq \langle \text{mcTFD} | \hat{\mathcal{O}}_{\text{mc}}^L \otimes \hat{\mathcal{O}}_{\text{mc}}^L | \text{mcTFD} \rangle \end{aligned} \quad (5.26)$$

where we use

$$\langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} = \langle \text{mcTFD} | \mathcal{O}^L \otimes 1_R | \text{mcTFD} \rangle = \langle \text{mcTFD} | 1_R \otimes \mathcal{O}^R | \text{mcTFD} \rangle \quad (5.27)$$

and $\langle \hat{\mathcal{O}}_{\text{mc}} \rangle_{\text{mc}}^{E, \Delta E} = 0$ by definition. Then, we set $a = \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} - \langle \mathcal{O} \rangle_{\text{can}}^{\beta}$ with some

$\beta > 0$. Then, by denoting $\hat{\mathcal{O}}_{\text{can}} = \mathcal{O} - \langle \mathcal{O} \rangle_{\text{can}}^\beta$,

$$\begin{aligned}
& \langle \text{mcTFD} | \hat{\mathcal{O}}_{\text{can}}^L \otimes \hat{\mathcal{O}}_{\text{can}}^L | \text{mcTFD} \rangle \\
&= \frac{1}{D_{E,\Delta E}} \sum_{|E_n\rangle, |E_m\rangle \in \mathcal{H}_{E,\Delta E}} \left| \langle E_n | \hat{\mathcal{O}}_{\text{can}} | E_m \rangle \right|^2 \\
&= \frac{1}{D_{E,\Delta E}} \sum_{|E_n\rangle, |E_m\rangle \in \mathcal{H}_{E,\Delta E}} e^{\frac{\beta E_n}{2}} e^{-\frac{\beta E_n}{2}} e^{\frac{\beta E_m}{2}} e^{-\frac{\beta E_m}{2}} \left| \langle E_n | \hat{\mathcal{O}}_{\text{can}} | E_m \rangle \right|^2 \\
&\leq \frac{e^{\beta(E+\Delta E)}}{D_{E,\Delta E}} \sum_{|E_n\rangle, |E_m\rangle \in \mathcal{H}_{E,\Delta E}} e^{-\frac{\beta E_n}{2}} e^{-\frac{\beta E_m}{2}} \left| \langle E_n | \hat{\mathcal{O}}_{\text{can}} | E_m \rangle \right|^2 \\
&= \frac{e^{\beta(E+\Delta E)}}{D_{E,\Delta E}} \sum_{|E_n\rangle, |E_m\rangle \in \mathcal{H}_{E,\Delta E}} \langle E_n | e^{-\frac{\beta H}{2}} \hat{\mathcal{O}}_{\text{can}} e^{-\frac{\beta H}{2}} | E_m \rangle \langle E_m | \hat{\mathcal{O}}_{\text{can}} | E_n \rangle \quad (5.28) \\
&\leq \frac{e^{\beta(E+\Delta E)}}{D_{E,\Delta E}} \sum_{|E_n\rangle, |E_m\rangle \in \mathcal{H}} \langle E_n | e^{-\frac{\beta H}{2}} \hat{\mathcal{O}}_{\text{can}} e^{-\frac{\beta H}{2}} | E_m \rangle \langle E_m | \hat{\mathcal{O}}_{\text{can}} | E_n \rangle \\
&= \frac{Z(\beta) e^{\beta(E+\Delta E)}}{D_{E,\Delta E}} \text{Tr} \left[(\rho_{\text{can}}^{(\beta)})^{\frac{1}{2}} \hat{\mathcal{O}}_{\text{can}} (\rho_{\text{can}}^{(\beta)})^{\frac{1}{2}} \hat{\mathcal{O}}_{\text{can}} \right] \\
&= \frac{Z(\beta) e^{\beta(E+\Delta E)}}{D_{E,\Delta E}} \langle \text{TFD}(\beta) | \hat{\mathcal{O}}_{\text{can}}^L \otimes \hat{\mathcal{O}}_{\text{can}}^R | \text{TFD}(\beta) \rangle.
\end{aligned}$$

Notice that so far β is introduced just a positive parameter. With some technical assumption, we can show that [133] with setting

$$\frac{\partial S(E)}{\partial E} = \beta(E) = \beta \quad (5.29)$$

we have

$$\frac{Z(\beta) e^{\beta(E+\Delta E)}}{D_{E,\Delta E}} = O(\log D_{E,\Delta E}). \quad (5.30)$$

Intuitively, since $Z(\beta) \approx e^{-\beta F(\beta)}$,

$$\frac{Z(\beta) e^{\beta(E+\Delta E)}}{D_{E,\Delta E}} \approx \exp\{-\beta(F(\beta) - E - TS(E))\} \quad (5.31)$$

and the right hand side is not huge in the thermodynamic limit with thermodynamic relation.

Next we try to show (5.22). This is easy if $\mathcal{O}$ is hermitian⁹. The derivation is

⁹The hermitian condition will be just for simplification

follows,

$$\begin{aligned}
& \langle \text{mcTFD} | \hat{\mathcal{O}}_{\text{mc}}^L \otimes \hat{\mathcal{O}}_{\text{mc}}^R | \text{mcTFD} \rangle \\
&= \text{Tr} \left[\hat{\mathcal{O}}_{\text{mc}} \left(\rho_{\text{mc}}^{(E, \Delta E)} \right)^{\frac{1}{2}} \hat{\mathcal{O}}_{\text{mc}} \left(\rho_{\text{mc}}^{(E, \Delta E)} \right)^{\frac{1}{2}} \right] \\
&\leq \sqrt{\text{Tr} \left[\hat{\mathcal{O}}_{\text{mc}} \left(\rho_{\text{mc}}^{(E, \Delta E)} \right)^{\frac{1}{2}} \left(\hat{\mathcal{O}}_{\text{mc}} \left(\rho_{\text{mc}}^{(E, \Delta E)} \right)^{\frac{1}{2}} \right)^\dagger \right] \text{Tr} \left[\hat{\mathcal{O}}_{\text{mc}} \left(\rho_{\text{mc}}^{(E, \Delta E)} \right)^{\frac{1}{2}} \left(\hat{\mathcal{O}}_{\text{mc}} \left(\rho_{\text{mc}}^{(E, \Delta E)} \right)^{\frac{1}{2}} \right)^\dagger \right]} \\
&= \text{Tr} \left[\rho_{\text{mc}}^{(E, \Delta E)} \hat{\mathcal{O}}_{\text{mc}}^2 \right] = \frac{1}{D_{E, \Delta E}} \sum_{|E_n\rangle \in \mathcal{H}_{E, \Delta E}} \langle E_n | \hat{\mathcal{O}}_{\text{mc}}^2 | E_n \rangle \leq \|\hat{\mathcal{O}}\|_\infty^2 = 4\|\mathcal{O}\|_\infty^2.
\end{aligned} \tag{5.32}$$

Next we consider (5.23). Define $C_{\rho_{AB}}[\mathcal{O}_A, \mathcal{O}_B]$ as

$$C_{\rho_{AB}}[\mathcal{O}_A, \mathcal{O}_B] := \text{Tr}\{\rho_{AB} \mathcal{O}_A \otimes \mathcal{O}_B\} - \text{Tr}\{\rho_A \mathcal{O}_A\} \text{Tr}\{\rho_B \mathcal{O}_B\}. \tag{5.33}$$

There is a useful formula

Lemma 1. *For any bounded operators $\mathcal{O}_A, \mathcal{O}_B$ and quantum state ρ_{AB}*

$$\frac{1}{2} \left(\frac{C_{\rho_{AB}}[\mathcal{O}_A, \mathcal{O}_B]}{\|\mathcal{O}_A\|_\infty \|\mathcal{O}_B\|_\infty} \right)^2 \leq I_{\rho_{AB}}(A : B) \tag{5.34}$$

, where $A = \text{supp} \mathcal{O}_A, B = \text{supp} \mathcal{O}_B$ and $I(A : B)$ is the quantum mutual information,

$$I_{\rho_{AB}}(A : B) = S(A) + S(B) - S(AB) \tag{5.35}$$

If we set $A = \mathcal{H}_L = \mathcal{H}, B = \mathcal{H}_R = \mathcal{H}, \rho_{AB} = \rho_{\text{mc}}^{(E, \Delta E)}$, we see (5.23).

□

5.2 Equilibration and Thermalization

5.2.1 Typicality is kinematical

In the previous subsection, we argue that the most of the quantum states in the shell represent thermal equilibrium. From this fact, we may consider following paradigm for non-equilibrium dynamics: as an initial state, we have a non-equilibrium and atypical state. It is natural to consider, the unitary time evolution then move this non-equilibrium atypical state to typical equilibrium state. See Figure 14.

However, we need to notice that the typicality theorems does not say about the time evolutions. They are just a mathematical theorem which do not depends on the detail of Hamiltonian and thus kinematical. We know that the actual dynamics highly depends on the structure of Hamiltonian like energy eigenstates and spectrum.

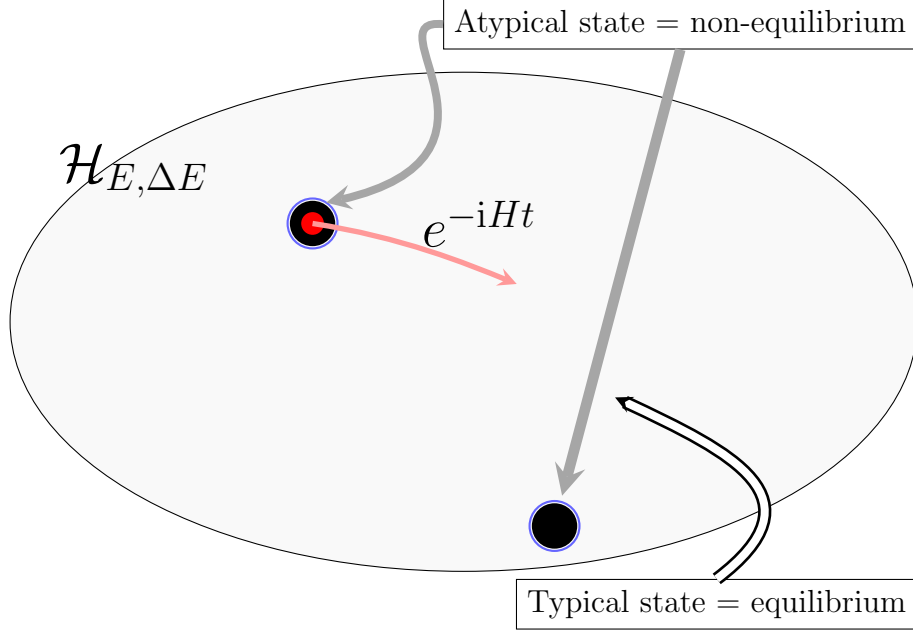


Figure 14: Illustration of non-equilibrium dynamics which we "naively" expect from typicality theorem. The red points express the initial state $|\psi(0)\rangle$.

This is non-trivial and we need to go beyond the typicality theorem. Thus, to discuss the thermalization behavior from non equilibrium to equilibrium in closed systems we need to more about the information of Hamiltonian and initial states. From now we discuss this part.

5.2.2 Equilibration and Thermalization

Now we would like to ask a question, for example, for states $|\psi(0)\rangle$ and time evolution $|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$, there exists a "equilibration time t_{eq} " such that for any time $t \geq t_{\text{eq}}$, we see

$$\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right| = e^{-O(S_{\text{mc}}(E))}. \quad (5.36)$$

Empirically this will be true. However, this never happens. There is a theorem called Poincare recurrence which states that for closed systems with finite dimensional energy shell some time evolved states are very close to the initial states. In practical, the recurrence time is very long compared to the equilibration time. With this observation, we ask another question: for states $|\psi(0)\rangle$ and time evolution $|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$, can we confirm that for almost all time t the equation (5.2.2) holds?

To formulate this question in more practical manner, we consider an uniform distribution of time μ_{time} . Then the above question is evaluation of the probability

$$\text{Prob}_{\mu_{\text{time}}} \left[\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right| > \epsilon \right] \quad (5.37)$$

is small or not for $\epsilon = e^{-O(S_{\text{mc}}(E))}$. This probability is bounded by above

$$\begin{aligned}
& \text{Prob}_{\mu_{\text{time}}} \left[\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right| > \epsilon \right] \\
&= \text{Prob}_{\mu_{\text{time}}} \left[\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right|^2 > \epsilon^2 \right] \\
&\leq \frac{1}{\epsilon^2} \mathbb{E}_{\mu_{\text{time}}} \left[\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right|^2 \right] \\
&= \frac{1}{\epsilon^2} \overline{\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right|^2}^T
\end{aligned} \tag{5.38}$$

where we denote the long-time average as

$$\overline{A(t)}^T = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt A(t). \tag{5.39}$$

by using the triangle inequality we can decompose the time average by two parts

$$\begin{aligned}
\overline{\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right|^2}^T &\leq \overline{\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T \right|^2}^T \\
&\quad + \overline{\left| \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right|^2}^T \\
&= \overline{\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T \right|^2}^T \\
&\quad + \overline{\left| \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right|^2}^T.
\end{aligned} \tag{5.40}$$

The first term represents the fluctuation of $\langle \psi(t) | \mathcal{O} | \psi(t) \rangle$ from the steady long-time average value $\overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T$ is thermodynamically small. If this part is small, then we call the initial state $|\psi(0)\rangle$ shows equilibration.

Definition

Definition 2. For a given initial state $|\psi_0\rangle$, we say the time evolution $|\psi(t)\rangle = e^{-iHt} |\psi_0\rangle$ undergoes equilibration if it holds that for any $\mathcal{O} \in \mathcal{A}$

$$\overline{\left(\langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T \right)^2}^T \leq e^{-O(S_{\text{mc}}(E))} \tag{5.41}$$

As an important remark, in the definition of the equilibration, we just have some steady state. In concrete, the relevant quantum state is given by

$$\rho_D := \overline{|\psi(t)\rangle \langle \psi(t)|}^T = \sum_n |c_n|^2 |E_n\rangle \langle E_n| \tag{5.42}$$

in the last equality we assume that there is no energy eigenstates degeneracy. Indeed, we have

$$\overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T = \text{Tr}[\rho_D \mathcal{O}]. \quad (5.43)$$

ρ_D is called diagonal ensemble and initial state dependent state. Thus to discuss, the state reaches to thermal equilibrium state, we need to discuss that the steady state in equilibration is close to microcanonical ensemble. This is called thermalization and related to the second term of (5.40).

Definition

Definition 3. For a given initial state $|\psi_0\rangle$, we say the time evolution $|\psi(t)\rangle = e^{-iHt} |\psi_0\rangle$ undergoes thermalization if it holds that for any $\mathcal{O} \in \mathcal{A}$,

$$\left| \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right| = \left| \text{Tr}[\rho_D \mathcal{O}] - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} \right| \leq e^{-O(S_{\text{mc}}(E))} \quad (5.44)$$

If a given initial state $|\psi_0\rangle$ and theory undergoes equilibration and thermalization, we see that the time-evolved state $|\psi(t)\rangle$ represents equilibrium for most of the time t larger than some relaxation time like a Figure 15.

Now We consider the sufficient conditions for equilibration and thermalization. We can discuss the upper bound for the left-hand side of (5.41) and (5.44) independently. These discussion involves dependence of 1) initial states and 2) theory or Hamiltonian.

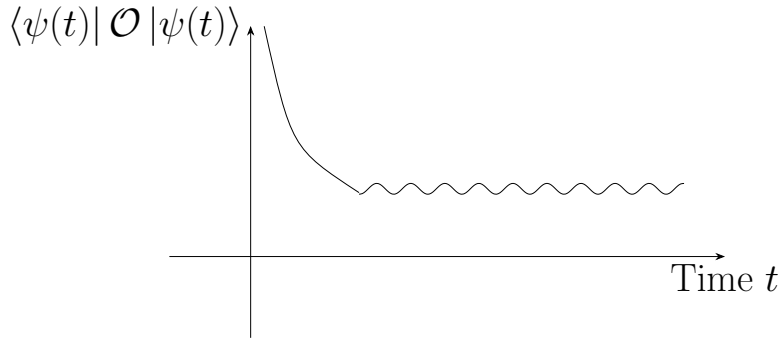


Figure 15: Typical behavior of the one point function $\langle \psi(t) | \mathcal{O} | \psi(t) \rangle$.

5.3 Initial State Independent Upper Bound: Strong ETH

We begin by examining an upper bound that does not depend on the choice of initial states. This situation arises in sufficiently chaotic systems. For simplicity, let us impose the following two assumptions on the energy spectrum within the chosen energy shell:

1. Non-degeneracy:

$$E_n = E_m \Rightarrow n = m \quad (5.45)$$

2. Non-resonance:

$$E_n - E_m = E_l - E_k \Rightarrow n = l, m = k \quad (5.46)$$

These assumptions may be relaxed to some degree. Under them, the left-hand side of (5.41) can be rewritten as

$$\begin{aligned} & \overline{\left(\langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T \right)^2} \\ &= \sum_{n \neq m} |c_n|^2 |c_m|^2 |\langle E_n | \mathcal{O} | E_m \rangle|^2 \\ &< \left(\max_{n \neq m} |\langle E_n | \mathcal{O} | E_m \rangle| \right)^2 \sum_{n \neq m} |c_n|^2 |c_m|^2 \\ &< \left(\max_{n \neq m} |\langle E_n | \mathcal{O} | E_m \rangle| \right)^2. \end{aligned} \quad (5.47)$$

Thus, if the following condition known as the off-diagonal ETH is satisfied, every initial state in the shell equilibrates:

$$\begin{aligned} \text{Off-diagonal ETH: } \forall \mathcal{O} \in \mathcal{A}, \forall |E_n\rangle, |E_m\rangle \in \mathcal{H}_{E, \Delta E}, n \neq m, \\ |\langle E_n | \mathcal{O} | E_m \rangle| \leq e^{-O(S_{\text{mc}}(E))}. \end{aligned} \quad (5.48)$$

Next, consider the upper bound that appears on the left-hand side of (5.44). Suppose that the system additionally satisfies the (diagonal) ETH:

$$\begin{aligned} \text{Diagonal ETH: } \forall \mathcal{O} \in \mathcal{A}, \forall |E_n\rangle \in \mathcal{H}_{E, \Delta E}, \\ |\langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E}| \leq e^{-O(S_{\text{mc}}(E))}. \end{aligned} \quad (5.49)$$

Under this assumption, the left-hand side of (5.44) becomes small for arbitrary initial states, leading to thermalization throughout the energy shell.

We refer to the combination of the off-diagonal and diagonal ETH conditions (5.48), (5.49) as the strong ETH [31, 32]:

Definition

Definition 4. A system is said to satisfy the strong eigenstate thermalization hypothesis (ETH) on an energy shell $\mathcal{H}_{E, \Delta E}$ with respect to a chosen thermodynamic limit and a class of observables $\mathcal{A}$, if both the diagonal and off-diagonal ETH conditions hold. When this is the case, the system exhibits equilibration

and thermalization for any initial state belonging to the shell.

There exist numerical studies supporting the validity of strong ETH [39, 134]. Nevertheless, establishing strong ETH analytically for specific chaotic Hamiltonians remains notoriously challenging, as one typically lacks explicit expressions for energy eigenstates and cannot directly compute matrix elements such as $\langle E_n | \mathcal{O} | E_m \rangle$.

5.4 Initial State Dependent Upper Bound

As an alternative upper bound for the left-hand sides of (5.41) and (5.44), we may consider bounds that explicitly depend on the choice of initial state. These bounds will later play an important role in our analysis. To proceed, let us introduce the set of energy eigenstates within the shell, denoted by $\mathcal{E}_{E,\Delta E}$, equipped with the uniform probability measure. Using this, we define the following probability:

$$\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}] := \text{Prob}_{\mathcal{E}_{E,\Delta E}} \left[\left| \langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right| > \epsilon \right], \quad (5.50)$$

where $\epsilon = e^{-O(S_{\text{mc}}(E))}$ is taken to be a small positive parameter¹⁰. This quantity measures the fraction of energy eigenstates in the shell for which the diagonal ETH is violated by more than ϵ .

A similar quantity can be introduced for matrix elements between different eigenstates. Define a set $\mathcal{E}'_{E,\Delta E}$ of ordered pairs of distinct energy eigenstates:

$$\mathcal{E}'_{E,\Delta E} := \{ (|E_n\rangle, |E_m\rangle) \in \mathcal{H}_{E,\Delta E} \times \mathcal{H}_{E,\Delta E} \mid |E_n\rangle \neq |E_m\rangle \text{ up to an unphysical phase} \}. \quad (5.51)$$

Equipping this set with the uniform probability measure, we define

$$\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}] := \text{Prob}_{\mathcal{E}'_{E,\Delta E}} [|\langle E_n | \mathcal{O} | E_m \rangle| > \epsilon]. \quad (5.52)$$

Our goal is to derive state-dependent upper bounds involving these probabilities. Throughout this discussion, we assume the non-degeneracy condition (5.45) and the non-resonance condition (5.46).

Let us begin with thermalization. A useful inequality established in [123] is the following:

¹⁰To obtain a meaningful upper bound, one needs to be careful in choosing ϵ . We will clarify this point in a later section.

Theorem

Theorem 2. For any $\epsilon > 0$, any initial state $|\psi_0\rangle = \sum_{|E_n\rangle \in \mathcal{H}_{E,\Delta E}} c_n |E_n\rangle$, and any bounded operator $\mathcal{O} \in \mathcal{B}(\mathcal{H})$, the following bound holds:

$$\left| \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right| < \|\mathcal{O}\|_\infty \left(\epsilon + \sqrt{\frac{2D_{E,\Delta E}}{D_{\text{eff}}}} \Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}] \right), \quad (5.53)$$

where the effective dimension is given by [123, 135]

$$D_{\text{eff}} := \left(\sum_n |c_n|^4 \right)^{-1} = (\text{Tr}[\rho_D^2])^{-1}. \quad (5.54)$$

The quantity D_{eff} characterizes how many eigenstates meaningfully contribute to the initial state. For instance, a single energy eigenstate yields $D_{\text{eff}} = 1$, whereas a maximally coherent superposition, $|\psi\rangle = \sum_{|E_n\rangle \in \mathcal{H}_{E,\Delta E}} \frac{1}{\sqrt{D_{E,\Delta E}}} |E_n\rangle$, gives $D_{\text{eff}} = D_{E,\Delta E}$. It is well known that typical states in high-dimensional Hilbert spaces satisfy $D_{\text{eff}} = O(D_{E,\Delta E})$ [136]. We put this discussion in up-coming subsection Section 5.4.2.

Since local operators often suffer from UV divergences in their operator norms due to short-distance singularities, one typically smears them using a test function f , $\mathcal{O}[f](t) := \int dx f(x) \mathcal{O}(x, t)$, $\int dx f(x) = 1$, or alternatively performs Euclidean time evolution, $\mathcal{O}(t + ib) := e^{bH} \mathcal{O} e^{-bH}$. For brevity, such smearing will be left implicit in what follows.

An important feature of (5.53) is that the dependence on the initial state and the dependence on the Hamiltonian factorize: the former appears solely via D_{eff} , while the latter enters only through $\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$.

From the inequality, we conclude that for a given initial state, if for sufficiently small ϵ ,

$$\frac{D_{E,\Delta E}}{D_{\text{eff}}} \sup_{\mathcal{O} \in \mathcal{A}} \Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}] \leq e^{-O(S_{\text{mc}}(E))}, \quad (5.55)$$

then thermalization is guaranteed. Furthermore, since $1 \leq D_{\text{eff}} \leq D_{E,\Delta E}$, the most conservative requirement ensuring thermalization for all initial states is

$$D_{E,\Delta E} \sup_{\mathcal{O} \in \mathcal{A}} \Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}] \leq e^{-O(S_{\text{mc}}(E))}. \quad (5.56)$$

The scaling behavior of $\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$ in the thermodynamic limit is therefore crucial. For instance, applying large deviation estimates in translation-invariant quantum

spin systems on a d -dimensional lattice, one obtains [137]

$$\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}] \leq e^{-\gamma(\epsilon) N_{\text{dof}}}, \quad (5.57)$$

for some positive coefficient $\gamma(\epsilon)$. Other upper bounds for operators with large support are discussed in [138, 139]. These inequalities apply to both integrable and non-integrable models and therefore do not by themselves imply thermalization. However, if we assume the scaling

$$D_{E,\Delta E} = e^{s(E) \text{Vol}(S^{d-1})}, \quad D_{\text{eff}} = e^{\eta \text{Vol}(S^{d-1})}, \quad (5.58)$$

with sufficiently large volume, then the condition $s(E) < \eta + \gamma(\epsilon)$ ensures thermalization of states with effective dimension satisfying (5.58).

As discussed in [140], the actual scaling of $\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$ may differ drastically between integrable and chaotic systems. For integrable models, it is conjectured that

$$\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}] \approx e^{-O(S_{\text{mc}}(E))}, \quad (5.59)$$

whereas for genuinely non-integrable systems the conjectured scaling is doubly exponential:

$$\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}] \approx e^{-\# e^{O(S_{\text{mc}}(E))}}, \quad \# > 0. \quad (5.60)$$

These behaviors have been numerically observed in spin-chain models: the XX model (integrable) and the XXX model with perturbations (non-integrable). If the doubly exponential suppression indeed holds, condition (5.56) is automatically satisfied, implying thermalization from any initial state—equivalent to the diagonal strong ETH. In contrast, if there exist ETH-violating eigenstates such as quantum many-body scars [141, 142], the doubly exponential suppression cannot occur, since $D_{E,\Delta E} \Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$ would then fail to be doubly exponentially small.

A similar analysis applies to the off-diagonal sector. For any quantum many-body Hamiltonian satisfying non-resonance, one has:

Theorem

Theorem 3. *Let $\epsilon > 0$, $|\psi_0\rangle$ any initial state, and $\mathcal{O} \in \mathcal{B}(\mathcal{H})$ a bounded operator. Under the assumptions of non-degeneracy and the non-resonance condition (5.46), the following inequality holds:*

$$\left| \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle} - \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T \right|^2 \leq \|\mathcal{O}\|_\infty^2 \left(\epsilon^2 + \frac{(D_{E,\Delta E})^{3/2}}{D_{\text{eff}}} \sqrt{\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}]} \right). \quad (5.61)$$

Here $\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}]$ is the probability defined in (5.52).

The proof of this result is presented in Section 5.4.1. As before, equilibration follows for a given initial state if

$$\frac{(D_{E,\Delta E})^{3/2}}{D_{\text{eff}}} \sqrt{\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}]} = e^{-O(S_{\text{mc}})}. \quad (5.62)$$

Moreover, if the Hamiltonian satisfies a doubly exponential suppression analogous to (5.60),

$$\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}] \approx e^{-\#'^{O(S_{\text{mc}})}}, \quad \#' > 0, \quad (5.63)$$

then equilibration holds for every initial state.

5.4.1 Proof of theorems 2

In this section, we present a mathematical derivation of the theorems introduced above.

■ Proof of Theorem 2

We begin by establishing the inequality (5.53). Define the subset of energy eigenstates satisfying $|\langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E}| < \epsilon \|\mathcal{O}\|_\infty$ as $\mathcal{E}_{E,\Delta E}^{(\epsilon)}$. With this notation, a straightforward manipulation yields

$$\begin{aligned} & \left| \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right| \\ &= \left| \sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E}} |c_n|^2 \left(\langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right) \right| \\ &\leq \sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E}} |c_n|^2 \left| \langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right| \\ &\leq \sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E} - \mathcal{E}_{E,\Delta E}^{(\epsilon)}} |c_n|^2 \left| \langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right| \\ &\quad + \sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E}^{(\epsilon)}} |c_n|^2 \left| \langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right| \\ &\leq \sqrt{\sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E} - \mathcal{E}_{E,\Delta E}^{(\epsilon)}} |c_n|^4 \sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E} - \mathcal{E}_{E,\Delta E}^{(\epsilon)}} \left| \langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right|^2} + \epsilon \|\mathcal{O}\|_\infty \\ &\leq \sqrt{\sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E}} |c_n|^4 \sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E} - \mathcal{E}_{E,\Delta E}^{(\epsilon)}} \left| \langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right|^2} + \epsilon \|\mathcal{O}\|_\infty. \end{aligned} \quad (5.64)$$

Noting that $\sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E}} |c_n|^4 = (D_{\text{eff}})^{-1}$, $\left| \langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right| \leq 2\|\mathcal{O}\|_\infty$, and using $\sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E} - \mathcal{E}_{E,\Delta E}^{(\epsilon)}} 1 = D_{E,\Delta E} \Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$, we arrive at the asserted inequality.

■ Proof of Theorem 2

We now turn to Theorem 2, which concerns the contribution from the off-diagonal matrix elements.

Proof. Assuming the non-degeneracy of the spectrum, we can write

$$\overline{\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T \right|^2} = \sum_{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E,\Delta E}} |c_n|^2 |c_m|^2 |\langle E_n | \mathcal{O} | E_m \rangle|^2. \quad (5.65)$$

Introduce the subset

$$\mathcal{E}'_{E,\Delta E}^{(\epsilon)} = \{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E,\Delta E}; |\langle E_n | \mathcal{O} | E_m \rangle|^2 \leq \epsilon^2 \|\mathcal{O}\|_\infty^2\}. \quad (5.66)$$

Splitting the sum into contributions from $\mathcal{E}'_{E,\Delta E}^{(\epsilon)}$ and its complement, we obtain

$$\begin{aligned} & \sum_{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E,\Delta E}} |c_n|^2 |c_m|^2 |\langle E_n | \mathcal{O} | E_m \rangle|^2 \\ &= \sum_{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E,\Delta E}^{(\epsilon)}} |c_n|^2 |c_m|^2 |\langle E_n | \mathcal{O} | E_m \rangle|^2 \\ &+ \sum_{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E,\Delta E} - \mathcal{E}'_{E,\Delta E}^{(\epsilon)}} |c_n|^2 |c_m|^2 |\langle E_n | \mathcal{O} | E_m \rangle|^2 \\ &\leq \epsilon^2 \|\mathcal{O}\|_\infty^2 + \sqrt{\left(\sum_{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E,\Delta E} - \mathcal{E}'_{E,\Delta E}^{(\epsilon)}} |c_n|^4 |c_m|^4 \right) \left(\sum_{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E,\Delta E} - \mathcal{E}'_{E,\Delta E}^{(\epsilon)}} |\langle E_n | \mathcal{O} | E_m \rangle|^4 \right)}. \end{aligned} \quad (5.67)$$

The first factor is bounded above by $(D_{\text{eff}})^{-2}$. To bound the second, observe that

$$\begin{aligned} |\langle E_n | \mathcal{O} | E_m \rangle| &= |\text{Tr}_{\mathcal{H}_{E,\Delta E}}[(|E_m\rangle \langle E_n|) \mathcal{O}]| \\ &\leq \sqrt{\text{Tr}_{\mathcal{H}_{E,\Delta E}}[(|E_m\rangle \langle E_n|)(|E_n\rangle \langle E_m|)] \text{Tr}_{\mathcal{H}_{E,\Delta E}}[\mathcal{O}^\dagger \mathcal{O}]} \\ &= \sqrt{\text{Tr}_{\mathcal{H}_{E,\Delta E}}[\mathcal{O}^\dagger \mathcal{O}]} = \sqrt{D_{E,\Delta E}} \|\mathcal{O}\|_\infty. \end{aligned} \quad (5.68)$$

Thus,

$$\sum_{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E, \Delta E} - \mathcal{E}'_{E, \Delta E}^{(\epsilon)}} |\langle E_n | \mathcal{O} | E_m \rangle|^4 \leq (D_{E, \Delta E})^3 \|\mathcal{O}\|_\infty^4 \Delta_{\text{off, ETH}}^\epsilon[\mathcal{O}]. \quad (5.69)$$

Combining the above bounds yields the desired inequality. $\square$

5.4.2 Typicality for the large effective dimension

We can show that the pure states whose effective dimension is large is typical in the energy shell in the thermodynamic limit.

Theorem

Theorem 4. *For any positive number δ and for the Haar measure on the energy shell following inequality holds.*

$$\text{Prob}_{|\psi_0\rangle \sim \text{Haar}} \left[\sum_n |c_n|^4 \geq \delta \right] \leq \frac{1}{\delta} \frac{2}{D_{E, \Delta E} + 1}$$

Especially by putting $\delta = e^{-\eta N_{\text{dof}}}$, $D_{E, \Delta E} = e^{s N_{\text{dof}}}$,

$$\text{Prob}_{|\psi_0\rangle \sim \text{Haar}} [D_{\text{eff}}(|\psi_0\rangle) \leq e^{\eta N_{\text{dof}}}] \leq \frac{2e^{\eta N_{\text{dof}}}}{e^{s N_{\text{dof}}} + 1} \sim e^{-N_{\text{dof}}(s-\eta)} \rightarrow 0 \text{ as } N_{\text{dof}} \rightarrow \infty$$

Proof. By using the Markov's inequality, we have

$$\begin{aligned} \text{Prob}_{|\psi_0\rangle \sim \text{Haar}} \left[\sum_n |c_n|^4 \geq \delta \right] &\leq \frac{\mathbb{E}[\sum_n |c_n|^4]}{\delta} \\ &= \frac{1}{\delta} \sum_{n,m,k,l} \mathbb{E}[c_n^* c_m c_k^* c_l] \\ &= \frac{2}{\delta(D_{E, \Delta E} + 1)}. \end{aligned} \quad (5.70)$$

$\square$

The last equations follow from (5.5) Thus, we find that the typical initial states get thermalized. However, for a given theory, we should concretely examine which states have the large effective states. For example, it is quite non-trivial whether non-equilibrium states prepared from quantum quench have the large effective dimension.

5.5 MATE and MITE

5.5.1 MATE

First, in a macroscopic system, we consider a set of mutually commuting macroscopic physical quantities $\{M_i\}_{i=1,\dots,k}$. These are extensive quantities for some degree of freedom of the system, we denote N_{dof} . Example of N_{dof} is system volume, lattice number, central charge and so on. For an integer tuple $\nu = (m_1, m_2, \dots, m_k)$, we define the Hilbert space $\mathcal{H}_\nu$ by

$$\mathcal{H}_\nu := \text{Span}\{|E_n\rangle \mid |E_n\rangle \in \mathcal{H}_{E,\Delta E}, M_i |E_n\rangle = m_i \Delta M_i\}. \quad (5.71)$$

These subspaces provide a direct-sum decomposition of the energy shell:

$$\mathcal{H}_{E,\Delta E} = \bigoplus_{\nu} \mathcal{H}_\nu. \quad (5.72)$$

A representative vector in each $\mathcal{H}_\nu$ corresponds to a macroscopic state. For a physically reasonable statistical-mechanical system, there exists a particular macroscopic Hilbert space $\mathcal{H}_{\text{eq}} := \mathcal{H}_{\nu_{\text{eq}}}$ describing thermal equilibrium. Let $D_{\text{eq}} = \dim \mathcal{H}_{\text{eq}}$. Then, in the thermodynamic limit, one expects that for sufficiently small ϵ ,

$$1 - \frac{D_{\text{eq}}}{D_{E,\Delta E}} = \epsilon. \quad (5.73)$$

This condition is called *thermodynamic typicality* [30, 121, 129]. For example, in a translation-invariant quantum spin system in $(1+1)$ dimensions with short-range interactions, thermodynamic typicality has been proven for a single extensive operator [143]. Let $\mathbb{P}_{\text{eq}}$ denote the projection operator onto $\mathcal{H}_{\text{eq}}$. We now define what it means for a pure state to represent thermal equilibrium in the sense of MATE.

Definition

Definition 5. A pure state $|\psi\rangle$ within the energy shell $\mathcal{H}_{E,\Delta E}$ is said to represent a macroscopic thermal equilibrium (MACROscopic Thermal Equilibrium, MATE) if, in the thermodynamic limit, for a sufficiently small positive quantity δ ,

$$1 - \langle \psi | \mathbb{P}_{\text{eq}} | \psi \rangle < \delta \quad (5.74)$$

is satisfied [121].

If thermodynamic typicality (5.73) holds, then pure states that represent thermal

equilibrium in the sense of MATE are typical in the energy shell. Indeed,

$$\overline{\langle \psi | \mathbb{P}_{\text{eq}} | \psi \rangle} = \text{Tr } \rho_{\text{mc}} \mathbb{P}_{\text{eq}} = \frac{D_{\text{eq}}}{D_{u, N_{\text{dof}}}} = 1 - \epsilon. \quad (5.75)$$

Define the nonequilibrium projection operator by $\mathbb{P}_{\text{neq}} := 1 - \mathbb{P}_{\text{eq}}$. For a MATE state,

$$\langle \psi | \mathbb{P}_{\text{neq}} | \psi \rangle = \delta \ll 1. \quad (5.76)$$

■ When the set of macroscopic operators does not commute

Given a set of noncommuting macroscopic operators $\{M_i\}$, one introduces a set of approximately commuting operators $\{\widetilde{M}_i\}$ that satisfy

$$[\widetilde{M}_i, \widetilde{M}_j] = 0, \quad \lim_{N_{\text{dof}} \rightarrow \infty} \frac{1}{N_{\text{dof}}} \|M_i - \widetilde{M}_i\|_{\infty} = 0. \quad (5.77)$$

For translation-invariant lattice quantum spin systems where $M_i = \sum_{a \in \text{lattice}} m_a^{(i)}$, the existence of such operators is known [144].

5.5.2 TMATE

A different definition of macroscopic equilibrium associated with macroscopic operators was proposed in [123]. This definition does not assume a macroscopic Hilbert subspace nor approximate commutativity.

■ Single macroscopic observable

Consider a single macroscopic observable M . Define

$$m(E) := \lim_{N_{\text{dof}} \rightarrow \infty} \frac{1}{N_{\text{dof}}} \langle M \rangle_{\text{mc}}^{E, \Delta E}. \quad (5.78)$$

Fix $\delta > 0$ independent of N_{dof} , and define

$$\mathbb{P}_{\text{neq}} := P \left[\left| \frac{M}{N_{\text{dof}}} - m(E) \right| \geq \delta \right]. \quad (5.79)$$

For a Hermitian operator A diagonalized by $|a\rangle$, define

$$P[A \geq a_0] := \sum_{a \geq a_0} |a\rangle \langle a|. \quad (5.80)$$

Definition

Definition 6. A pure state $|\psi\rangle \in \mathcal{H}_{N_{\text{dof}}, E}$ satisfies TMATE (Tasaki's Macro-

scopic Thermal Equilibrium) if

$$\langle \psi | \mathbb{P}_{\text{neq}} | \psi \rangle \leq e^{-\alpha N_{\text{dof}}} \quad (5.81)$$

for some model-dependent constant $\alpha > 0$.

■ Multiple macroscopic observables

Let $M^{(i)}$, $i = 1, \dots, n$, be n extensive observables with $n = o(N_{\text{dof}})$. Define

$$m^{(i)} := \lim_{N_{\text{dof}} \rightarrow \infty} \frac{1}{N_{\text{dof}}} \langle M^{(i)} \rangle_{\text{mc}}^{E, \Delta E}. \quad (5.82)$$

Choose positive constants $\delta^{(i)}$ independent of N_{dof} .

Definition

Definition 7. A pure state $|\psi\rangle \in \mathcal{H}_{N_{\text{dof}}, E}$ satisfies TMATE if

$$\langle \psi | P \left[\left| \frac{M^{(i)}}{N_{\text{dof}}} - m^{(i)}(E) \right| \geq \delta^{(i)} \right] | \psi \rangle \leq e^{-\alpha N_{\text{dof}}} \quad (5.83)$$

for all $i = 1, \dots, n$.

One may also define

$$\mathbb{P}_{\text{neq}} := \sum_{i=1}^n P \left[\left| \frac{M^{(i)}}{N_{\text{dof}}} - m^{(i)}(E) \right| \geq \delta^{(i)} \right], \quad (5.84)$$

which is no longer a projection. The condition $\langle \psi | \mathbb{P}_{\text{neq}} | \psi \rangle \leq e^{-\alpha N_{\text{dof}}}$ follows from (5.83) if $n = o(e^{N_{\text{dof}}})$.

■ Thermodynamic bound

A key tool to show that TMATE is typical is the *thermodynamic bound*.

Definition

Definition 8. A quantum system satisfies the thermodynamic bound (at energy density E) if there exist constants $\gamma > 0$ and $(N_{\text{dof}})_0$ such that

$$\langle \mathbb{P}_{\text{neq}} \rangle_{\text{mc}}^{E, \Delta E} \leq e^{-\gamma N_{\text{dof}}} \quad (5.85)$$

for all $N_{\text{dof}} \geq (N_{\text{dof}})_0$.

■ Typicality of TMATE

Once the thermodynamic bound holds, TMATE becomes typical.

Theorem

Theorem 5. Assume the thermodynamic bound (5.85) and $\gamma > \alpha$. For sufficiently large $N_{\text{dof}} \geq (N_{\text{dof}})_0$,

$$\text{Prob}[\langle \psi | \mathbb{P}_{\text{neq}} | \psi \rangle > e^{-\alpha N_{\text{dof}}}] \leq e^{-(\gamma-\alpha)N_{\text{dof}}}. \quad (5.86)$$

Proof. By using the Markov's inequality,

$$\begin{aligned} \text{Prob}[\langle \psi | \mathbb{P}_{\text{neq}} | \psi \rangle > e^{-\alpha N_{\text{dof}}}] &\leq e^{\alpha N_{\text{dof}}} \mathbb{E}[\langle \psi | \mathbb{P}_{\text{neq}} | \psi \rangle] \\ &= e^{\alpha N_{\text{dof}}} \langle \mathbb{P}_{\text{neq}} \rangle_{\text{mc}}^{E, \Delta E} \\ &\leq e^{-(\gamma-\alpha)N_{\text{dof}}}. \end{aligned}$$

□

■ MATE and Thermodynamics

Define the Boltzmann entropy of a macroscopic state ν by $S_\nu = \log \dim \mathcal{H}_\nu$. Then,

$$S_{\text{eq}} = \log \dim \mathcal{H}_{\text{eq}} = \log D_{E, \Delta E} = S_{\text{mc}}(E, N_{\text{dof}}) \quad (5.87)$$

in the thermodynamic limit, reproducing the thermodynamic entropy.

5.5.3 Thermalization for MATE

We discuss thermalization in terms of (T)MATE [121, 123]. First, we consider the following lemma.

Theorem

Theorem 6. Assume that there exist a sufficiently large T and a constant $\nu > 0$ such that

$$\frac{1}{T} \int_0^T \langle \psi(t) | \mathbb{P}_{\text{neq}} | \psi(t) \rangle dt \leq e^{-(\alpha+\nu)N_{\text{dof}}}. \quad (5.88)$$

This assumption will later be shown to hold under certain conditions. Then, for most times $t \in [0, T]$, the state $|\psi(t)\rangle$ satisfies (T)MATE:

$$\langle \psi(t) | \mathbb{P}_{\text{neq}} | \psi(t) \rangle \leq e^{-\alpha N_{\text{dof}}}. \quad (5.89)$$

Proof. Let $\mathcal{B} \subset [0, T]$ be the set of times t for which

$$\langle \psi(t) | \mathbb{P}_{\text{neq}} | \psi(t) \rangle \geq e^{-\alpha N_{\text{dof}}}. \quad (5.90)$$

Then

$$\frac{|\mathcal{B}|}{T} = \frac{1}{T} \int_{t \in \mathcal{B}} dt \quad (5.91)$$

$$\leq e^{\alpha N_{\text{dof}}} \frac{1}{T} \int_0^T \langle \psi(t) | \mathbb{P}_{\text{neq}} | \psi(t) \rangle dt \quad (5.92)$$

$$\leq e^{-\nu N_{\text{dof}}}. \quad (5.93)$$

Thus, in the thermodynamic limit, the set of times violating (T)MATE has negligible measure. $\square$

Next, we analyze conditions under which (5.88) holds.

■ Specific initial state

Theorem

Theorem 7. *Assume the following:*

- *The thermodynamic bound (5.85) holds.*
- *The energy eigenvalues are nondegenerate.*
- *There exists $\eta > 0$ such that $D_{\text{eff}} \geq e^{-\eta N_{\text{dof}}} D_{E, \Delta E}$.*
- *$\gamma - \eta > 2(\alpha + \nu)$.*

Then (5.88) holds for the initial state.

Proof. Using nondegeneracy,

$$\frac{1}{T} \int_0^T \langle \psi(t) | \mathbb{P}_{\text{neq}} | \psi(t) \rangle dt \leq \overline{\langle \psi(t) | \mathbb{P}_{\text{neq}} | \psi(t) \rangle}^T \quad (5.94)$$

$$= \sum_n |c_n|^2 \langle E_n | \mathbb{P}_{\text{neq}} | E_n \rangle. \quad (5.95)$$

Applying Cauchy–Schwarz,

$$\sum_n |c_n|^2 \langle E_n | \mathbb{P}_{\text{neq}} | E_n \rangle \leq \sqrt{\left(\sum_n |c_n|^4 \right) \left(\sum_n \langle E_n | \mathbb{P}_{\text{neq}} | E_n \rangle^2 \right)} \quad (5.96)$$

$$\leq \sqrt{\frac{1}{D_{\text{eff}}} \sum_n \langle E_n | \mathbb{P}_{\text{neq}} | E_n \rangle} \quad (5.97)$$

$$= \sqrt{\frac{D_{E,\Delta E}}{D_{\text{eff}}} \langle \mathbb{P}_{\text{neq}} \rangle_{\text{mc}}^{E,\Delta E}} \leq e^{-\frac{\gamma-\eta}{2} N_{\text{dof}}}. \quad (5.98)$$

This implies (5.88) under $\gamma - \eta > 2(\alpha + \nu)$. $\square$

■ Arbitrary initial state

Definition

Definition 9. A system satisfies MATE-ETH if every eigenstate $|E_n\rangle \in \mathcal{H}_{E,\Delta E}$ satisfies (T)MATE; i.e., there exists $\kappa > 0$ such that

$$\langle E_n | \mathbb{P}_{\text{neq}} | E_n \rangle \leq e^{-\kappa N_{\text{dof}}} \leq e^{-(\alpha+\nu) N_{\text{dof}}}. \quad (5.99)$$

Theorem

Theorem 8. Assume MATE-ETH, nondegeneracy of $|E_n\rangle$, and $\kappa > \alpha + \nu$. Then any initial state satisfies (5.88).

Proof.

$$\frac{1}{T} \int_0^T \langle \psi(t) | \mathbb{P}_{\text{neq}} | \psi(t) \rangle dt \leq \overline{\langle \psi(t) | \mathbb{P}_{\text{neq}} | \psi(t) \rangle}^T \quad (5.100)$$

$$\leq e^{-\kappa N_{\text{dof}}}, \quad (5.101)$$

which implies (5.88) because $\kappa > \alpha + \nu$. $\square$

5.5.4 MITE

In this section, we discuss a notion of thermal equilibrium that is different from MATE and TMATE and is intrinsic to quantum systems, namely Microscopic Thermal Equilibrium (MITE). Roughly speaking, MITE means that a pure state is locally indistinguishable from the microcanonical ensemble. For arbitrary quantum systems, MITE is known to be a typical property of pure states in an energy shell. This fact is referred to as Canonical Typicality [124].

Definition

Definition 10. A pure state $|\psi\rangle \in \mathcal{H}_{E,\Delta E}$ is said to represent microscopic thermal equilibrium of length l (Microscopic Thermal Equilibrium, MITE $_l$) if, for every subsystem $X_i^{(l)}$ of size l , $i = 1, 2, \dots, V$,

$$\rho_\psi^{X_i^{(l)}} \sim \rho_{\text{mc}}^{X_i^{(l)}} \quad \text{for all } i = 1, 2, \dots, V.$$

This definition is equivalent to requiring that, for any local operator $\mathcal{O} \in \mathcal{S}_{\text{loc}}^{(l)}$ supported on a region of size l ,

$$\langle \psi | \mathcal{O} | \psi \rangle \sim \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E}.$$

The condition MITE $_l$ can be naturally extended to few-body operators. Therefore, one expects that strict geometric locality does not play a crucial role in the notion of microscopic thermal equilibrium.

Canonical typicality states that, for almost all pure states in a given energy shell, the reduced state on a sufficiently small subsystem X is locally equivalent to the microcanonical (or canonical) ensemble:

$$\rho_\psi^X \sim \rho_{\text{mc}}^X, \quad \text{for most } |\psi\rangle \in \mathcal{H}_{E,\Delta E}. \quad (5.102)$$

More precisely, the following theorem holds [145, 146].

Theorem

Theorem 9. Let $|\psi\rangle \in \mathcal{H}_{E,\Delta E}$ be sampled at random from the energy shell $\mathcal{H}_{E,\Delta E}$ according to the uniform (Haar) measure. For a proposition C about $|\psi\rangle$, let $\text{Prob}[C]$ denote the probability that C holds. Then, for any $\epsilon > 0$ and any subsystem X , one has

$$\text{Prob}[\|\rho_\psi^X - \rho_{\text{mc}}^X\|_1 > \eta] < \eta', \quad (5.103)$$

where

$$\eta = \epsilon + \sqrt{\frac{D_X^2}{D_{E,\Delta E}}}, \quad (5.104)$$

$$\eta' = 4 \exp\left(-\frac{\epsilon^2}{18\pi^2} D_{E,\Delta E}\right), \quad (5.105)$$

$$D_{E,\Delta E} := \dim \mathcal{H}_{E,\Delta E}, \quad D_X := \dim \mathcal{H}_X. \quad (5.106)$$

The proof is in Appendix B. The physical interpretation of this theorem is as follows. If the subsystem X is chosen so that η is sufficiently small, then for almost all pure states $|\psi\rangle$ in the energy shell, the reduced density matrix ρ_ψ^X is indistinguishable from ρ_{mc}^X . In particular, in typical many-body systems one has

$$D_{E,\Delta E} = e^{O(V)}, \quad D_X = e^{O(|X|)}, \quad (5.107)$$

and hence

$$\frac{D_X^2}{D_{E,\Delta E}} \sim e^{2|X|-V}. \quad (5.108)$$

Therefore, as long as the subsystem size $|X|$ is much smaller than half of the total system size V , almost all pure states $|\psi\rangle$ are locally indistinguishable from the microcanonical ensemble.

Moreover, in systems characterized by a parameter such as the central charge c in two-dimensional conformal field theory, one often has

$$D_{E,\Delta E} = e^{csO(V)}, \quad D_X = e^{cs_X O(|X|)} \quad (5.109)$$

where we consider entropy density for each central charge by $s = \frac{1}{c} \lim_{V \rightarrow \infty} \frac{1}{V} S_{\text{mc}}$ and $s_X = \frac{1}{c} \frac{1}{|X|} S_X$. S_X is thermodynamic entropy for subsystems. In this case, in the large- c limit, as long as the size of the subsystem X is at most half of the total system, one finds $\frac{D_X^2}{D_{E,\Delta E}} \ll 1$.

■ MITE and thermodynamics

When the interaction between a subsystem X and its complement X^c is sufficiently weak, one may write the Hamiltonian as

$$H = H_X + H_{X^c} + H_{\text{int}}, \quad (5.110)$$

with $|H_{\text{int}}| \ll |H_X|, |H_{X^c}|$. In this situation, the reduced microcanonical state on X

is approximately canonical:

$$\rho_{\text{mc}}^X \sim \rho_{\text{can}}. \quad (5.111)$$

The temperature is determined by the thermodynamic relation

$$\beta = \frac{\partial S_{\text{mc}}(V)}{\partial E}. \quad (5.112)$$

For a state $|\psi\rangle$ that satisfies MITE, the entanglement entropy of the subsystem X then satisfies

$$S_X[|\psi\rangle] \sim S_X[\rho_{\text{mc}}] = S_{\text{v.N.}}[\rho_{\text{can}}], \quad (5.113)$$

which reproduces the thermodynamic entropy of the subsystem X .

5.5.5 Application of typicality theorem: Page curve

One possible application of typicality theorem is page curve which is related to information loss paradox.

Let us divide the total Hilbert space into that of Hawking radiation $\mathcal{H}_R$ and that of the black hole $\mathcal{H}_B$. Following a unitary time evolution, we consider the entanglement between Hawking radiation and the black hole [147]. The key concept is canonical typicality. Although the definition of energy in gravitational theories generally contains ambiguities, here we assume that the energy is conserved in an appropriate sense so that an energy shell can be defined. If energy conservation is not assumed, one may simply take the entire Hilbert space as the “energy shell.” In this case the microcanonical ensemble is replaced by $\frac{\mathbb{I}}{\dim \mathcal{H}_{\text{tot}}}$ and we consider this case for simplicity.

Before black hole formation there is no Hawking radiation, so $S_R = 0$. Shortly after formation, when Hawking radiation begins to be emitted, we have $\dim \mathcal{H}_R \ll \dim \mathcal{H}_B$. By canonical typicality, the reduced density matrix for typical states satisfies

$$\rho_{\psi(t)}^R = \rho_{\text{mc}}^R + \delta\rho_R, \quad (5.114)$$

with

$$\delta\rho_R = O\left(\sqrt{\frac{(\dim \mathcal{H}_R)^2}{D_{E,\Delta E}}}\right) \quad (5.115)$$

which follows from the canonical typicality theorem. The entanglement entropy of the Hawking radiation is then

$$S_R[|\psi(t)\rangle] = S_{\text{v.N.}}[\rho_{\text{mc}}^R] - \text{Tr}[\delta\rho_R \log \rho_{\text{mc}}^R] + O(\delta\rho_R^2). \quad (5.116)$$

The first term, $S_{\text{v.N.}}[\rho_{\text{mc}}^R]$, is the thermodynamic entropy of the Hawking radiation,

and it dominates when $(\dim \mathcal{H}_R)^2 \ll D_{E,\Delta E}$. Next, let us consider the final stage of evaporation. At this time $\dim \mathcal{H}_B \ll \dim \mathcal{H}_R$. Again, by canonical typicality, the reduced density matrix of the black hole satisfies

$$\rho_{\psi(t)}^B = \rho_{\text{mc}}^B + \delta\rho_B, \quad (5.117)$$

where

$$\delta\rho_B = O\left(\sqrt{\frac{\dim \mathcal{H}_B^2}{D_{E,\Delta E}}}\right). \quad (5.118)$$

Since the total system is in a pure state,

$$S_R[|\psi(t)\rangle] = S_B[|\psi(t)\rangle] \approx S_{\text{v.N.}}[\rho_{\text{mc}}^B] - \text{Tr}[\delta\rho_B \log \rho_{\text{mc}}^B] + O(\delta\rho_B^2). \quad (5.119)$$

Here $S_{\text{v.N.}}[\rho_{\text{mc}}^B]$ represents the thermodynamic entropy of the black hole. From the above considerations, during the unitary process of black hole formation and evaporation, the time evolution of the entanglement entropy of Hawking radiation is expected to follow the curve known as the *Page curve* as in Figure 16. Deriving the Page curve correctly is an important step toward resolving the information loss problem. We will go back to this problem later.

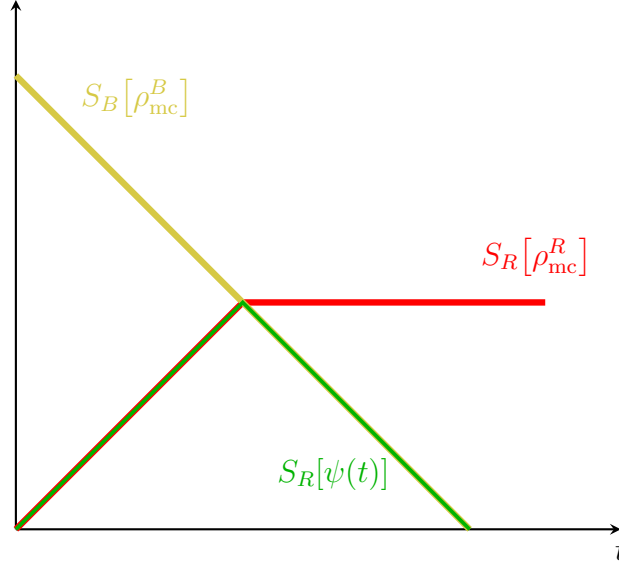


Figure 16: (Green) Page curve: the time evolution of the entanglement entropy of Hawking radiation. (Red) Thermodynamic entropy of Hawking radiation. (Yellow) Thermodynamic entropy of the black hole.

5.5.6 MATE vs. MITE

We have so far introduced two notions of thermal equilibrium: (T)MATE and MITE. A natural question is how these two concepts are related. In general, (T)MATE and MITE are independent notions, although in certain situations one can establish a relation between them [148, 149].

- If a system satisfies MATE, then a single measurement of a macroscopic observable already yields the corresponding equilibrium value. Thus, MATE is meaningful only for macroscopic systems in which quantum fluctuations are sufficiently suppressed. In contrast, for MITE, it is not required that the outcome of a single measurement be close to the microcanonical prediction. In this sense, MITE is a weaker notion than MATE.
- On the other hand, MITE is a condition of *state equivalence*: the quantum state locally resembles the equilibrium ensemble. By contrast, MATE only requires that the expectation values of a set of macroscopic observables match their equilibrium values, which is a strictly weaker requirement.
- Suppose a state $|\psi\rangle$ represents MITE of length l . In a translationally invariant system, assume that the operator defining MATE is of the form $M = \sum_i h_i$, where each h_i has support of size at most l . Then $|\psi\rangle$ automatically satisfies MATE. Thus, in such situations, MITE implies MATE.
- However, there also exist states that satisfy MATE but do *not* satisfy MITE. As a concrete example, consider a system of $N \gg 1$ spins with Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$, and take the Hamiltonian to be $H = 0$. Then the energy shell is the full Hilbert space $\mathcal{H}_{E,\Delta E} = \mathcal{H}$, and the microcanonical ensemble is $\rho_{\text{mc}} = 2^{-N} \mathbb{I}_{\mathcal{H}}$.

Let $X_1, X_2, \dots, X_N \in \{-1, 1\}$ and consider the product state

$$|\psi\rangle = |\sigma_z = X_1\rangle |\sigma_z = X_2\rangle \cdots |\sigma_z = X_N\rangle. \quad (5.120)$$

If the variables $\{X_k\}$ are chosen randomly, then for example the magnetization $M = \sum_{i=1}^N \sigma_z^{(i)}$ of $|\psi\rangle$ satisfies MATE with high probability. Nevertheless, the reduced density matrix of any small subsystem S is

$$\rho_S^\psi = |\sigma_z\rangle_S \langle \sigma_z|_S \neq \rho_S^{\text{mc}} = \mathbb{I}_S / 2^{|S|} \quad (5.121)$$

so $|\psi\rangle$ is clearly *not* MITE. Such behavior can occur in many-body localized phases. More generally, any state with low entanglement in small subsystems

will fail to satisfy MITE. This example highlights that MITE is sensitive to the entanglement structure and is thus a genuinely quantum notion, whereas MATE has a classical analogue.

5.5.7 Local ETH and Sub-system ETH

In this subsection, we consider a d -dimensional conformal field theory (CFT). In a conformal field theory, all the dynamical information of the theory is encoded in the spectrum of primary operators and their OPE coefficients. Therefore, whether ETH holds or not must also be determined solely by this data [150]. Motivated by this, we first introduce the following ETH ansatz formulated only in terms of primary operators and primary states.

Definition

Definition 11. For all energy eigenstates $|E_n\rangle, |E_m\rangle$ within an energy shell and for any primary operator $\mathcal{O}_p$, we assume

$$\langle E_n | \mathcal{O}_p | E_m \rangle = \mathcal{O}_p(E) \delta_{nm} + \mathcal{O}_{nm}^p, \quad (5.122)$$

where $\mathcal{O}_p(E)$ is a smooth function of

$$E = \frac{E_n + E_m}{2},$$

and the off-diagonal term $\mathcal{O}_{nm}^p$ behaves as

$$\mathcal{O}_{nm}^p \sim e^{-O(S_{\text{mc}}(E))}$$

in the thermodynamic limit.

We call this condition *local ETH*. For simplicity, in the following we assume that the energy eigenstates $|E_n\rangle$ are translationally invariant and that the primary operators have vanishing spin. As will be discussed later, the local ETH condition (5.122) for primary states determines the behavior for descendant states as well. If a given conformal field theory satisfies this local ETH, then the following sub-system ETH can be derived.

Definition

Definition 12. For any subsystem $X \subset S^d$, the following conditions hold:

1. There exists a density matrix $\rho^X(E)$ on X , which depends only on the energy E of the energy shell and on the choice of X , such that for any

energy eigenstate $|E_n\rangle$ in the energy shell one has

$$\|\rho_n^X - \rho^X(E)\|_1 \sim e^{-O(S_{\text{mc}}(E))}, \quad (5.123)$$

where we have abbreviated

$$\rho_n^X = \rho_{E_n}^X := \text{Tr}_{X^c}[|E_n\rangle \langle E_n|].$$

2. Defining

$$\tau_{nm} := \text{Tr}_{X^c}[|E_n\rangle \langle E_m|],$$

one has, for any $\alpha \in \mathbb{R}$,

$$\|e^{i\alpha}\tau_{nm} + e^{-i\alpha}\tau_{mn}\|_1 \sim e^{-O(S_{\text{mc}}(E))}. \quad (5.124)$$

In the second condition we introduced an extra phase α because under unphysical basis changes of the form $|E_n\rangle \rightarrow e^{i\phi}|E_n\rangle$, the phase of τ_{nm} can be changed arbitrarily. Therefore, a condition such as $|\text{Re } \tau_{nm}| \sim \exp\{-O(S_{\text{mc}}(E))\}$ would not be meaningful.

In a conformal field theory, sub-system ETH is equivalent to local ETH, as we now explain.

■ Sub-system ETH $\Rightarrow$ Local ETH

For any operator Δ and any $\mathcal{O} \in B(X)$, the Cauchy–Schwarz inequality implies

$$\text{Tr}[\Delta \mathcal{O}] \leq \sqrt{\text{Tr}[\Delta^\dagger \Delta] \text{Tr}[\mathcal{O}^\dagger \mathcal{O}]} = \|\Delta\|_2 \|\mathcal{O}\|_2 \leq \|\Delta\|_1 \|\mathcal{O}\|_2. \quad (5.125)$$

Taking $\Delta = \rho_n^X - \rho^X(E)$ and $\mathcal{O} = \mathcal{O}_p$, and defining

$$\mathcal{O}_p(E) := \text{Tr}[\rho^X(E) \mathcal{O}_p],$$

the first condition of sub-system ETH (5.123) implies the diagonal part of the local ETH (5.122). Here we assume that the smearing is such that $\|\mathcal{O}_p\|_2$ grows at most as $O(L^x)$ for some finite exponent x .

Similarly, taking $\Delta = e^{i\alpha}\tau_{nm} + e^{-i\alpha}\tau_{mn}$ and using the second condition (5.124), we obtain the off-diagonal part of the local ETH.

■ Local ETH $\Rightarrow$ Sub-system ETH

We now assume local ETH (5.122). Since we are not interested in real-time dynamics here, we carry out the analysis in Euclidean signature.

We first perform the conformal mapping from the cylinder to the plane,

$$ds_{\text{cylinder}}^2 = dt_c^2 + L^2 d\Omega_{d-1}^2 = \Lambda^2 (dr^2 + r^2 d\Omega_{d-1}^2), \quad (5.126)$$

$$\Lambda = \frac{L}{r}, \quad t_c = L \log r, \quad (5.127)$$

which maps the cylinder to the radially quantized plane. Under this map, the left-hand side of the local ETH (5.122) can be expressed, in terms of the primary operators Ψ_n and $\Psi_m^\dagger$ associated to the primary states $|E_n\rangle$ and $\langle E_m|$ and a primary operator $\mathcal{O}_p$, as

$$\langle E_m | \mathcal{O}_p | E_n \rangle = L^{-h_p} \langle \Psi_m^\dagger(\infty) \mathcal{O}_p(1) \Psi_n(0) \rangle = C_{nm}^p L^{-h_p}, \quad (5.128)$$

where C_{nm}^p are the corresponding OPE coefficients and h_p is the conformal dimension of $\mathcal{O}_p$. The energy is related to the conformal dimension via $h_n = E_n L$.

Next we discuss the scaling of the conformal dimension h_n in the thermodynamic limit. The energy density of the state $|E_n\rangle$ is

$$\epsilon_n \sim \frac{E_n}{L^{d-1}} = \frac{h_n}{L^d}.$$

In a CFT at temperature T , the energy density at high temperature is determined solely by the temperature (since there is no other intrinsic length scale), and scales as $\epsilon_n \sim c T^d$, where c is a parameter such as the central charge in two-dimensional CFT. Assuming that an energy eigenstate $|E_n\rangle$ can be characterized by an effective temperature-like scale, we can write

$$h_n \sim c (LT)^d. \quad (5.129)$$

On the other hand, the conformal dimension h_p of the primary operator $\mathcal{O}_p$ is of order $O(L^0)$.

Using the local ETH (5.122), we can rewrite the OPE coefficient as

$$C_{nm}^p = L^{h_p} (\mathcal{O}_p(E) \delta_{nm} + \mathcal{O}_{nm}^p) = h_n^{\frac{h_p}{d}} \delta_{nm} f_p(E) + R_{nm}^p, \quad (5.130)$$

where

$$f_p(E) = T^{h_p} (c \text{Vol}(S^d))^{-\frac{h_p}{d}} \mathcal{O}_p(E), \quad (5.131)$$

$$R_{nm}^p = L^{h_p} \mathcal{O}_{nm}^p \sim \exp \left[\frac{h_p}{d} \log h_n - O\left(h_n^{\frac{d-1}{d}}\right) \right], \quad (5.132)$$

$$S_{\text{mc}} \sim L^d. \quad (5.133)$$

We now proceed to discuss the reduced density matrix of a subsystem that has finite size even in the thermodynamic limit. On the cylinder, consider a ball-shaped region B on a constant- t_c slice $\mathbb{S}^{d-1}$ that contains the north pole. The size of the ball is specified by the angular coordinate θ_0 on $\mathbb{S}^{d-1}$ via

$$\theta_0 = \tan^{-1} \left(\frac{R}{2L} \right).$$

We take the thermodynamic limit $L \rightarrow \infty$ while keeping R fixed. In this limit, the ball B becomes a ball of radius R in $\mathbb{R}^{d-1}$.

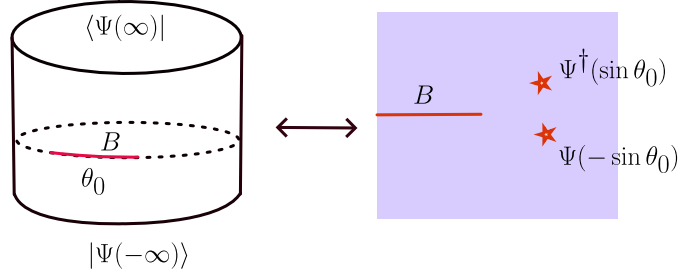


Figure 17: Choice of the ball region B . The left panel shows B on the cylinder, and the right panel shows the corresponding region and operator insertions in Rindler coordinates.

The trace norm is invariant under unitary transformations and, in particular, under conformal mappings. We therefore perform a conformal transformation to a more convenient Rindler frame. First, we map the cylinder to the radially quantized plane via (5.126). We then perform the following transformation from the coordinates $(r, \theta_1, \theta_2, \dots, \theta_{d-1})$ on the plane to Rindler coordinates $(X^0, X^1, \dots, X^{d-1})$:

$$\frac{l(r^2 - 1)}{1 + r^2 + 2r \cos \theta_1} = \frac{X^0}{1 - 2X^1 + X \cdot X}, \quad (5.134)$$

$$\frac{2l \sin \theta_1 \cos \theta_2}{1 + r^2 + 2r \cos \theta_1} = \frac{1 - X \cdot X}{2(1 - 2X^1 + X \cdot X)}, \quad (5.135)$$

$$\frac{2l \sin \theta_1 \sin \theta_2 \cdots \sin \theta_i}{1 + r^2 + 2r \cos \theta_1} = \frac{X^i}{2(1 - 2X^1 + X \cdot X)}, \quad i = 2, \dots, d-1, \quad (5.136)$$

$$l = \frac{1}{2} \cot \theta_0. \quad (5.137)$$

Under this map, the metric becomes

$$ds_{\text{plane}}^2 = dr^2 + r^2 d\Omega_{d-1}^2 = \Lambda(X) dX^i dX^i, \quad (5.138)$$

$$\Lambda(X) = \left(X^0 - \frac{lV_-}{2} - \frac{V_+}{8l} \right)^{-1}, \quad (5.139)$$

$$V_{\pm} = 1 \pm 2X^1 + X \cdot X. \quad (5.140)$$

The original ball region B on the cylinder is mapped by (5.126) to the region $r = 1$, $0 \leq \theta_1 \leq \theta_0$ on the plane, and then by (5.138) to the half-space $X^1 \leq 0$ in Rindler coordinates. The origin $r = \epsilon \rightarrow 0$ is mapped to

$$X_- = (-\sin \theta_0, \cos \theta_0, 0),$$

whereas $r = 1/\epsilon \rightarrow \infty$ is mapped to

$$X_+ = (\sin \theta_0, -\cos \theta_0, 0),$$

and in the thermodynamic limit these two points approach

$$X_* = (0, 1, 0).$$

We can thus apply an OPE expansion around this point.

Let ρ_n^B be the reduced density matrix of the region B for the energy eigenstate $|E_n\rangle$. Denote by $\tilde{\rho}_n^B$ its image under the conformal map to the Rindler frame. Then

$$\text{Tr}_B[\tilde{\rho}_n^B \dots] = \frac{\langle \Psi_n^\dagger(X_+) \Psi_n(X_-) \dots \rangle}{\langle \Psi_n^\dagger(X_+) \Psi_n(X_-) \rangle} \quad (5.141)$$

$$= \sum_p (2 \sin \theta_0)^{h_p} C_{nn}^p \langle \mathcal{O}_p(X_*) \dots \rangle \quad (5.142)$$

$$= \sum_p \left(\frac{R}{L} \right)^{h_p} C_{nn}^p \langle \mathcal{O}_p(X_*) \dots \rangle. \quad (5.143)$$

Using the expression (5.130) for the OPE coefficients, we can write $\tilde{\rho}_n^B$ as the sum of two pieces:

$$\tilde{\rho}_n^B = \tilde{\rho}^B(E) + R_B^n, \quad (5.144)$$

$$\tilde{\rho}^B(E) = \sum_p \mathcal{O}_p(E) R^{h_p} \mathcal{O}_p(X_*) + (\text{descendants}), \quad (5.145)$$

$$R_B^n = \sum_p \mathcal{O}_{nn}^p R^{h_p} \mathcal{O}_p(X_*) + (\text{descendants}), \quad (5.146)$$

where “descendants” denotes contributions from descendant operators.

We now verify the first condition of the sub-system ETH (5.123). In continuum field theory, the trace norm is often difficult to compute explicitly. We instead

consider the relative entropy, which is more tractable:

$$S_{\text{rel}}[\tilde{\rho}_n^B \|\tilde{\rho}^B(E)] = \int_0^\infty \frac{ds}{2} \text{Tr} \left[(\tilde{\rho}^B(E) + s)^{-1} R_B^n (\tilde{\rho}^B(E) + s)^{-1} R_B^n \right] + O((R_B^n)^3) = e^{-O(S_{\text{mc}}(E))}. \quad (5.147)$$

From this we obtain

$$\|\rho_n^B - \rho^B(E)\|_1 = \|\tilde{\rho}_n^B - \tilde{\rho}^B(E)\|_1 \leq S_{\text{rel}}[\tilde{\rho}_n^B \|\tilde{\rho}^B(E)] \sim e^{-O(S_{\text{mc}}(E))}. \quad (5.148)$$

For a more general subsystem X , we choose a ball B such that $X \subset B$. Using the fact that the trace distance is monotonically non-increasing under CPTP maps, we obtain

$$\|\rho_n^X - \rho^X(E)\|_1 \leq \|\rho_n^B - \rho^B(E)\|_1. \quad (5.149)$$

This proves the first condition of sub-system ETH. A similar argument applied to

$$\tau_{nm} = \text{Tr}_{X^c}[|E_n\rangle \langle E_m|]$$

establishes the second condition.

■ Descendant states

From the local ETH (5.122) for primary states, we have already determined the thermodynamic behavior of the OPE coefficients. The local ETH for descendant states is similarly determined by these OPE coefficients and the conformal dimensions. Thus, once local ETH holds for primary states in a given conformal family, it also holds for their descendants.

For simplicity, we focus on spinless descendants of the form $(P_i P^i)^r \Psi_n$. Consider two descendant states $|E_{n,r}\rangle$ and $|E_{m,s}\rangle$ and a primary operator $\mathcal{O}_p$:

$$\langle E_{m,s} | \mathcal{O}_p | E_{n,r} \rangle = L^{-h_p} \frac{\langle \Psi_m(\infty) (K_a K^a)^s \mathcal{O}_p(1) (P_b P^b)^r \Psi_n(0) \rangle}{\sqrt{A_{n,r} A_{m,s}}}, \quad (5.150)$$

where we must use the fact that in radial quantization one has $(P_i)^\dagger = K_a$. Here

$$A_{n,r} = \langle \Psi_n(\infty) (K_a K^a)^r (P_b P^b)^r \Psi_n(0) \rangle. \quad (5.151)$$

Since we are working within a fixed energy shell, the descendant levels are much smaller than the conformal dimensions, i.e. $r, s \ll h_n, h_m$. Without loss of generality we may assume $r \geq s$.

Recalling the conformal algebra, we first evaluate the denominator $A_{n,r}$. Using the fact that K_a annihilates a primary state, we have

$$(K_a K^a)^r (P_b P^b)^r \Psi_n(0) = [(K_a K^a)^r, (P_b P^b)^r] \Psi_n(0). \quad (5.152)$$

From the commutator $[K_a, P_b] = 2(\delta_{ij}D - M_{ij})$, terms with the dilatation operator D appear, and when D acts on the primary, it produces factors of h_n . The Lorentz generators M_{ij} contribute only terms of order $O(L^0)$, while h_n scales as $h_n \sim L^d$ in the thermodynamic limit. Hence, terms in which D acts as many times as possible on the primary dominate, and we obtain

$$A_{n,r} = d^r (2h_n)^{2r} (1 + O(h_n^{-1})). \quad (5.153)$$

Next, consider the numerator. It can be decomposed into two contributions:

$$\langle \Psi_m(\infty) (K_a K^a)^s \mathcal{O}_p(1) (P_b P^b)^r \Psi_n(0) \rangle \quad (5.154)$$

$$= \langle \Psi_m(\infty) \mathcal{O}_p(1) (K_a K^a)^s (P_b P^b)^r \Psi_n(0) \rangle \quad (5.155)$$

$$+ \langle \Psi_m(\infty) [(K_a K^a)^s, \mathcal{O}_p(1)] (P_b P^b)^r \Psi_n(0) \rangle. \quad (5.156)$$

For the first term, by the same reasoning as for the denominator, the dominant contribution comes from those terms in which the commutators $[K_a, P_b]$ produce the maximum number of D operators, and hence the maximum power of h_n :

$$\langle \Psi_m(\infty) \mathcal{O}_p(1) (K_a K^a)^s (P_b P^b)^r \Psi_n(0) \rangle \quad (5.157)$$

$$\sim d^s (2h_n)^{2s} \langle \Psi_m(\infty) \mathcal{O}_p(1) (P_i P^i)^{r-s} \Psi_n(0) \rangle \quad (5.158)$$

$$= d^s (2h_n)^{2s} \lim_{x \rightarrow 0} (\partial_i \partial^i)_x^{r-s} \langle \Psi_m(\infty) \mathcal{O}_p(1) \Psi_n(x) \rangle \quad (5.159)$$

$$= d^s (2h_n)^{2s} C_{nm}^p f_{r-s}(h_p), \quad (5.160)$$

where $f_{r-s}(h_p)$ is a polynomial of degree $2(r-s)$ in h_p .

In the second term, at least one of the $(K_a K^a)$ must act on $\mathcal{O}_p$. The action of K_a on $\mathcal{O}_p$ generates a special conformal transformation and thus produces at most derivatives of $\mathcal{O}_p$. The remaining $(K_a K^a)^u$ that do *not* act on $\mathcal{O}_p$ with $u < s$ contribute terms of order h_n^u . Therefore, the first term dominates in the thermodynamic limit. Putting this together, we obtain

$$\langle E_{m,s} | \mathcal{O}_p | E_{n,r} \rangle = L^{-h_p} C_{nm}^p d^{s-r} (2h_n)^{s-r} f_{r-s}(h_p) (1 + O(h_p^{-1})). \quad (5.161)$$

The behavior of C_{nm}^p is already determined by (5.130). A similar computation shows that sub-system ETH also holds for the descendant states $|E_{n,r}\rangle$. Thus, once local ETH (5.122) holds for primary states, it naturally extends to their descendants.

■ Sub-system ETH vs. MITE-ETH

Sub-system ETH is essentially equivalent to MITE-ETH and to the MITE-ETH ansatz (5.182). The only difference is that, in the first condition, we use a general density matrix $\rho^X(E)$ rather than the microcanonical reduced density matrix.

It is known that the entanglement entropy of $\rho^B(E)$ coincides with that of the reduced canonical ensemble

$$\rho_B(T) := \text{Tr}_{B^c} \left[\frac{e^{-\beta H}}{Z(\beta)} \right] \quad (5.162)$$

for a suitably chosen temperature T [151]. Using the relative entropy formula, one can show that, by choosing β so that the expectation value of the modular Hamiltonian with respect to $\rho_B(T)$ matches that of $\rho^B(E)$, the two states actually coincide. Therefore, sub-system ETH is, in essence, equivalent to MITE-ETH.

5.6 ETH from Random Hamiltonian

We consider a highly complex, nonintegrable system. Suppose that the unitary transformation connecting the energy eigenbasis and the eigenbasis of an operator $\mathcal{O}$ that does not commute with the Hamiltonian can be modeled by a random unitary ensemble [152, 153]:

$$\mathcal{O} = \sum_{\alpha} \mathcal{O}_{\alpha} |\alpha\rangle \langle \alpha|, \quad (5.163)$$

$$U_{n\alpha} = \langle E_n | \alpha \rangle, \quad (5.164)$$

$$\mathcal{O}_{nm} = \sum_{\alpha} \mathcal{O}_{\alpha} U_{n\alpha} U_{m\alpha}^*. \quad (5.165)$$

Here the unitary matrix U is Haar-random. This ensemble can be regarded as an ensemble of Hamiltonians $H = \sum_n E_n |E_n\rangle \langle E_n|$ with the energy spectrum $\{E_n\}$ held fixed. The basic Haar averages over the random unitary ensemble are:

$$\int d\mu[U] = 1, \quad (5.166)$$

$$\int d\mu[U] U_{n\alpha} U_{m\beta}^* = \frac{1}{D_{E,\Delta E}} \delta_{nm} \delta_{\alpha\beta}, \quad (5.167)$$

$$\begin{aligned} \int d\mu[U] U_{n\alpha} U_{m\beta} U_{p\gamma}^* U_{q\delta}^* &= \frac{1}{D_{E,\Delta E}^2 - 1} (\delta_{np} \delta_{mq} \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{nq} \delta_{mp} \delta_{\alpha\delta} \delta_{\beta\gamma}) \\ &\quad - \frac{1}{D_{E,\Delta E} (D_{E,\Delta E}^2 - 1)} (\delta_{np} \delta_{\gamma\beta} \delta_{mq} \delta_{\delta\alpha} + \delta_{nq} \delta_{\delta\beta} \delta_{mp} \delta_{\gamma\alpha}). \end{aligned} \quad (5.168)$$

Using these identities, we study the behavior of the matrix elements of $\mathcal{O}$ in the

energy eigenbasis. The average is

$$\int d\mu[U] \mathcal{O}_{nm} = \sum_{\alpha} \mathcal{O}_{\alpha} \int d\mu[U] U_{m\alpha} U_{n\alpha}^* = \frac{1}{D_{E,\Delta E}} \sum_{\alpha} \mathcal{O}_{\alpha} \delta_{mn} = \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \delta_{mn}. \quad (5.169)$$

Moreover,

$$\int d\mu[U] |\mathcal{O}_{mn}|^2 = \frac{1}{D_{E,\Delta E}^2 - 1} \sum_{\alpha,\beta} \mathcal{O}_{\alpha} \mathcal{O}_{\beta}^* (\delta_{\alpha\beta} + \delta_{mn}) - \frac{1}{D_{E,\Delta E}(D_{E,\Delta E}^2 - 1)} \sum_{\alpha,\beta} (1 + \delta_{\alpha\beta} \delta_{mn}). \quad (5.170)$$

Hence the variance of the diagonal elements is

$$\int d\mu[U] |\mathcal{O}_{nn}|^2 - \left(\int d\mu[U] \mathcal{O}_{nn} \right)^2 = \frac{1}{D_{E,\Delta E} + 1} \left(\frac{1}{D_{E,\Delta E}} \sum_{\alpha} \mathcal{O}_{\alpha}^2 - \left(\frac{1}{D_{E,\Delta E}} \sum_{\alpha} \mathcal{O}_{\alpha} \right)^2 \right), \quad (5.171)$$

while the variance of the off-diagonal elements is

$$\int d\mu[U] |\mathcal{O}_{nm}|^2 - \left| \int d\mu[U] \mathcal{O}_{nm} \right|^2 = \frac{D_{E,\Delta E}}{D_{E,\Delta E}^2 - 1} \left(\frac{1}{D_{E,\Delta E}} \sum_{\alpha} \mathcal{O}_{\alpha}^2 - \left(\frac{1}{D_{E,\Delta E}} \sum_{\alpha} \mathcal{O}_{\alpha} \right)^2 \right). \quad (5.172)$$

Therefore, for sufficiently large $D_{E,\Delta E}$, one expects

$$\mathcal{O}_{nm} = \frac{1}{D_{E,\Delta E}} \left(\sum_{\alpha} \mathcal{O}_{\alpha} \right) \delta_{nm} + \sqrt{\frac{1}{D_{E,\Delta E}}} \sqrt{\frac{1}{D_{E,\Delta E}} \sum_{\alpha} \mathcal{O}_{\alpha}^2 - \left(\frac{1}{D_{E,\Delta E}} \sum_{\alpha} \mathcal{O}_{\alpha} \right)^2} R_{nm}, \quad (5.173)$$

where R_{nm} is a random variable with $\int d\mu[U] R_{nm} = 0$ and $\int d\mu[U] |R_{nm}|^2 = 1$. If $\mathcal{O}$ is Hermitian, one can choose $R_{mn} = R_{nm}^*$. Then it follows that a typical non-integrable system satisfies the ETH ansatz. However, there is no guarantee that a “typical” Hamiltonian drawn from Haar-random unitaries corresponds to a physically reasonable local Hamiltonian; moreover, whether the Hamiltonian of a given theory satisfies ETH is a question that must be investigated on its own terms. Recently, it is discussed that by considering the ensemble of local Hamiltonians, we can construct local RMT. In the papers, it is numerically discussed that almost all local Hamiltonians satisfying the ETH [154, 155].

Also we can deduce this kind of relations from our definition of ETH (5.49) and (5.48). Essential point is that ETH states that each energy eigenstate is “close” to the microcanonical distribution. Here we examine how close they actually are

[149, 156]. Consequently, the eigenstates within the shell are also equivalent to one another with probes $\mathcal{O} \in \mathcal{A}$, meaning that

$$\mathcal{O}_{nn} \approx \mathcal{O}_{mm}, \quad \text{for } |E_n\rangle, |E_m\rangle \in \mathcal{H}_{E, \Delta E}. \quad (5.174)$$

Extending this idea to the off-diagonal components as well, we write

$$|\mathcal{O}_{nm}| \approx \mathcal{O}_{\bar{E}, \omega}. \quad (5.175)$$

Here,

$$\bar{E} = (E_n + E_m)/2, \quad \omega = E_m - E_n. \quad (5.176)$$

From this we expect

$$\sum_{m(\neq n): E_m - E_n \sim \omega} |\mathcal{O}_{nm}|^2 \approx (\mathcal{O}_{\bar{E}, \omega})^2 e^{S_{\text{mc}}(\bar{E} + \omega/2)}. \quad (5.177)$$

The quantity $\mathcal{O}_{\bar{E}, \omega}$ can be bounded using the operator norm. For any energy eigenstate, the definition of the operator norm gives

$$\langle E_n | \mathcal{O} \mathcal{O}^\dagger | E_n \rangle \leq \|\mathcal{O}\|_\infty^2. \quad (5.178)$$

Thus,

$$\sum_{m(\neq n)} |\mathcal{O}_{nm}|^2 \leq \sum_m |\mathcal{O}_{nm}|^2 \leq \|\mathcal{O}\|_\infty^2. \quad (5.179)$$

From this inequality, there exists a function $F_{\mathcal{O}}(\bar{E}, \omega)$ satisfying $0 \leq F_{\mathcal{O}}(\bar{E}, \omega) \leq 1$ and $\int d\omega F_{\mathcal{O}}(\bar{E}, \omega)^2 = 1$, such that

$$\sum_{m(\neq n): E_m - E_n \sim \omega} |\mathcal{O}_{nm}|^2 \leq F_{\mathcal{O}}(\bar{E}, \omega)^2 \|\mathcal{O}\|_\infty^2. \quad (5.180)$$

Combining this bound with (5.177), we obtain

$$\mathcal{O}_{\bar{E}, \omega} \leq \|\mathcal{O}\|_\infty F_{\mathcal{O}}(\bar{E}, \omega) e^{-S_{\text{mc}}(\bar{E} + \omega/2)/2}. \quad (5.181)$$

This suggests that the matrix elements of energy eigenstates take the form

$$\mathcal{O}_{nm} = \mathcal{O}(\bar{E}) \delta_{nm} + e^{-\frac{1}{2} S_{\text{mc}}(\bar{E})} F_{\mathcal{O}}(\bar{E}, \omega) R_{nm}, \quad (5.182)$$

as proposed in [156]. Here, $\mathcal{O}(\bar{E}) = \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} + \mathcal{O}\left(e^{-\frac{1}{2} S_{\text{mc}}(E)}\right)$ and $F_{\mathcal{O}}(\bar{E}, \omega)$ are smooth functions of their arguments. The matrix R_{nm} has properties similar to those of $\mathcal{O}_{nm}$; for example, if $\mathcal{O}$ is Hermitian, then $R_{nm} = R_{mn}^*$. By comparing

to the results from random Hamiltonian, we assume that R_{nm} is pseudo random variables. Equation (5.182) is referred to as the ETH ansatz, and it simultaneously expresses both the diagonal ETH and the off-diagonal ETH.

5.7 Free probability theory and higher order generalization of ETH

In this section, I will review some aspects of ETH with free probability and matrix models, which give one formulation of a higher order generalization of the ETH ansatz. The original reference is [157]. The notion of generalized ETH is also discussed in [158, 159]. As useful references for an introduction to this topic, see the conference “Free probability, between maths and physics” [160], and [161]. We also discuss asymptotic freeness [162].

5.7.1 Free Probability

■ Introduction to free probability

Here we introduce the notion of free probability. Roughly speaking, free probability is a probability theory on a non-commutative space. Applications include operator algebras, probability theory, combinatorics, random matrix theory, machine learning, ETH, and so on. See also [163–165] for pedagogical introductions. The important key ingredients of free probability theory are non-commutative probability spaces and free independence. We introduce these two notions quickly.

■ Non-commutative probability space

Let us start from a classical probability space. A classical probability space is specified by three ingredients $(\Omega, \mathcal{F}, \mathbb{P})$. Here Ω denotes the sample space, $\mathcal{F}$ denotes the σ -algebra, and $\mathbb{P}$ denotes the probability measure. These are very important, but in typical situations we regard the following two as the essence of probability theory: random variables X and the expectation value $\mathbb{E}$. Typically random variables live in some nice topological or normed space, especially L^∞ . So, roughly speaking, the probability space is characterized by the pair $(L^\infty, \mathbb{E})$. The essence is that L^∞ is some nice normed space and $\mathbb{E}$ is a unital linear map on L^∞ . We may generalize these to more general nice spaces and linear maps, and one example is the non-commutative probability space.

Before this, we introduce the notion of a $*$ -algebra.

Definition

Definition 13. We say that $\mathcal{A}$ is a unital associative algebra if

- $\mathcal{A}$ includes the identity for the operator product.

- $\mathcal{A}$ is a vector space over $\mathbb{C}$.
- For $\lambda \in \mathbb{C}$ and $\mathcal{O}, \mathcal{O}' \in \mathcal{A}$, it holds that

$$\lambda(\mathcal{O}\mathcal{O}') = (\lambda\mathcal{O})\mathcal{O}' = \mathcal{O}(\lambda\mathcal{O}'). \quad (5.183)$$

We also introduce a unital linear functional (or state) on a $*$ -algebra.

Definition

Definition 14. Let $\mathcal{A}$ be a $*$ -algebra. We say that $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ is a unital linear functional or state if

- φ is a linear functional.
- $\varphi(1) = 1$ for $1 \in \mathcal{A}$.

Then we consider the following non-commutative probability space.

Definition

Definition 15. $(\mathcal{A}, \varphi)$ is a non-commutative probability space if

- $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$.
- $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ is a unital linear functional.

We also call elements of $\mathcal{A}$ non-commutative random variables.

■ Free Independence

Another important notion in free probability is free independence. To this end, let us recall the notion of independence in classical probability theory.

Definition

Definition 16. The (classical) random variables X, Y are independent if

$$\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)] \quad (5.184)$$

for any bounded continuous functions f, g . In some nice cases, for example when we have an absolutely convergent moment generating function, we can rephrase this as

$$\mathbb{E}[X^m Y^n] = \mathbb{E}[X^m]\mathbb{E}[Y^n], \quad m, n \in \mathbb{Z}. \quad (5.185)$$

We would like to do something similar for the non-commutative space. There are two problems for a straightforward generalization.

- For $\mathcal{O} \in \mathcal{A}$, typically we do not have a well-defined $f(\mathcal{O})$ for a generic continuous function f . For example, we do not have $\sin(\mathcal{O})$ for a general $\mathcal{O}$.
- We need to take care of the ordering in products. For example, $\varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_1\mathcal{O}_2) \neq \varphi(\mathcal{O}_1^2\mathcal{O}_2^2)$ in general.

For these reasons, we introduce the following definition:

Definition

Definition 17. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. We say sub-unital associative algebras $(\mathcal{A}_i)_{i \in I}$ are freely independent if the following condition holds. For any $(\mathcal{O}_j)_j$ such that

- $\forall \mathcal{O}_j \in \mathcal{A}_{i(j)}, i(1) \neq i(2) \neq \dots \neq i(k).$
- $\forall j \in \{1, 2, \dots, k\}, \varphi(\mathcal{O}_j) = 0,$

it holds that

$$\varphi(\mathcal{O}_1\mathcal{O}_2 \cdots \mathcal{O}_k) = 0. \quad (5.186)$$

We also say that non-commutative random variables $(\mathcal{O}_i)_{i \in I}$ are freely independent if $(\text{Span}\{1, \mathcal{O}_i\})_{i \in I}$ are freely independent. Here $\text{Span}\{1, \mathcal{O}_1, \mathcal{O}_2, \dots\}$ denotes the algebra generated by $1, \mathcal{O}_1, \mathcal{O}_2, \dots$.

The first condition ensures that the vanishing condition holds for products of operators belonging to different subalgebras. If we encounter a product of operators which all belong to the same subalgebra, we regard them as a single “big” operator.

By using free independence we can simplify operator products. Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are freely independent. Then, we have

$$\varphi(\mathcal{O}_1\mathcal{O}_2) = \varphi(\mathcal{O}_1)\varphi(\mathcal{O}_2), \quad \forall \mathcal{O}_1 \in \mathcal{A}_1, \quad \forall \mathcal{O}_2 \in \mathcal{A}_2. \quad (5.187)$$

The proof is very simple. We consider the operators $\hat{\mathcal{O}}_i = \mathcal{O}_i - \varphi(\mathcal{O}_i)1$. Then, trivially $\varphi(\hat{\mathcal{O}}_i) = 0$. From the definition of free independence, we see

$$0 = \varphi(\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2) = \varphi(\mathcal{O}_1\mathcal{O}_2) - \varphi(\mathcal{O}_1)\varphi(\mathcal{O}_2), \quad (5.188)$$

where we use the linearity of φ .

The 4th order case is more non-trivial. We consider $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{A}_1$ and $\mathcal{O}_3, \mathcal{O}_4 \in \mathcal{A}_2$. Then

$$\varphi(\mathcal{O}_1\mathcal{O}_3\mathcal{O}_2\mathcal{O}_4) = \varphi(\mathcal{O}_1\mathcal{O}_2)\varphi(\mathcal{O}_3)\varphi(\mathcal{O}_4) + \varphi(\mathcal{O}_1)\varphi(\mathcal{O}_2)\varphi(\mathcal{O}_3\mathcal{O}_4) - \varphi(\mathcal{O}_1)\varphi(\mathcal{O}_2)\varphi(\mathcal{O}_3)\varphi(\mathcal{O}_4). \quad (5.189)$$

On the other hand,

$$\varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) = \varphi(\mathcal{O}_1\mathcal{O}_2)\varphi(\mathcal{O}_3\mathcal{O}_4). \quad (5.190)$$

The proof is the same as the second-order case.

■ Distribution

In classical probability, one natural object is the distribution which characterizes random variables. In general, the distribution of a random variable X is characterized by the set $\{\mathbb{E}[f(X)] : f \text{ is a bounded continuous function}\} \Leftrightarrow f \in C_b$. We say two random variables X, Y have the same distribution if

$$\forall f \in C_b, \mathbb{E}[f(X)] = \mathbb{E}[f(Y)]. \quad (5.191)$$

For good random variables, for example when the series of moments is absolutely convergent, we can characterize the distribution by the series of moments $\{\mathbb{E}[X^n]\}_{n \in \mathbb{N}}$. Following this notion we define the distribution of a non-commutative random variable as follows:

Definition

Definition 18. We define the distribution of a non-commutative random variable $\mathcal{O} \in \mathcal{A}$ as the series of its moments $\{\varphi(\mathcal{O}^n)\}_{n \in \mathbb{N}}$.

■ Non-crossing partition and freeness

The above definition of freeness is a little bit abstract and we would like to understand it more conceptually. Actually, freeness is almost parallel to classical probability theory if we consider moments and cumulants. Indeed, freeness is equivalent to the vanishing of mixed cumulants. Also, moments and cumulants are given by sums over non-crossing (planar) partitions.

Definition

Definition 19.

- A partition of $\{1, 2, \dots, n\}$ is a decomposition $\pi = \{V_1, \dots, V_r\}$ such that

$$V_i \neq \emptyset, V_i \cap V_j = \emptyset, \bigcup_{i=1}^r V_i = \{1, 2, \dots, n\}. \quad (5.192)$$

- Each V_i is a block of $\pi \in \mathcal{P}(n)$, where $\mathcal{P}(n)$ denotes the set of all partitions of $\{1, \dots, n\}$.
- We say $\pi \in \mathcal{P}(n)$ is crossing if there exist elements p_1, p_2, q_1, q_2 such that $p_1 < q_1 < p_2 < q_2$ and p_1, p_2 belong to the same block V_i while q_1, q_2 belong to the same block V_j , with $V_i \neq V_j$.
- We say $\pi \in \mathcal{P}(n)$ is non-crossing if it is not crossing. We denote the set of non-crossing partitions as $\text{NC}(n)$.

The set $\text{NC}(n)$ is actually a lattice with the refinement order.

- Example $n = 2$: We have only the trivial element $\pi = \{(1, 2)\}$ which is non-crossing. Pictorially,

$$\pi = \{(1, 2)\} = \begin{array}{cc} 1 & 2 \\ \hline \end{array} \quad (5.193)$$

- Example $n = 3$: We have five elements in $\mathcal{P}(3)$

$$\begin{aligned} \pi = \{(1, 2, 3)\} &= \begin{array}{ccc} 1 & 2 & 3 \\ \hline \end{array} \\ , \{(1, 2), (3)\} &= \begin{array}{ccc} 1 & 2 & 3 \\ \hline \end{array} , \{(1), (2, 3)\} = \begin{array}{ccc} 1 & 2 & 3 \\ | & \hline \end{array} , \{(1, 3), (2)\} = \begin{array}{ccc} 1 & 2 & 3 \\ \hline | \end{array} \\ , \{(1), (2), (3)\} &= \begin{array}{ccc} 1 & 2 & 3 \\ | & | & | \end{array} \end{aligned} \quad (5.194)$$

From the line pictures, we see that every element is non-crossing.

- Example $n = 4$: all elements of $\text{NC}(4)$ are given by

$$\begin{aligned}
\pi = \{(1, 2, 3, 4)\} &= \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline & & & \end{array}, \\
\{(1, 2, 3), (4)\} &= \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline & & & | \end{array}, \quad \{(1, 2, 4), (3)\} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline & & | & \end{array}, \\
\{(1, 3, 4), (2)\} &= \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline | & & & \end{array}, \quad \{(1), (2, 3, 4)\} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline | & & & \end{array}, \\
\{(1, 2), (3, 4)\} &= \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline & & & \end{array}, \quad \{(1, 4), (2, 3)\} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline & & & \end{array}, \\
\{(1, 2), (3), (4)\} &= \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline & & | & | \end{array}, \quad \{(1, 3), (2), (4)\} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline & & | & | \end{array}, \\
, \{(1, 4), (2), (3)\} &= \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline & & | & | \end{array}, \quad \{(1), (2, 3), (4)\} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline | & & & | \end{array}, \\
\{(1), (2, 4), (3)\} &= \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline | & & | & \end{array}, \quad \{(1), (2), (3, 4)\} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline | & | & & \end{array}, \\
\{(1), (2), (3), (4)\} &= \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline | & | & | & | \end{array},
\end{aligned} \tag{5.195}$$

On the other hand, there is one crossing element,

$$\pi = \{(1, 3), (2, 4)\} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline & & & \end{array}, \tag{5.196}$$

Then we define the free cumulants as follows:

Definition

Definition 20. For a unital linear functional $\varphi : \mathcal{A} \rightarrow \mathbb{C}$, we define cumulant functionals $\kappa_n : \mathcal{A}^n \rightarrow \mathbb{C}$ as multilinear functionals by the family of expansions

$$\varphi(\mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n) = \sum_{\pi \in \text{NC}(n)} \kappa_\pi[\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n]. \tag{5.197}$$

Here κ_π is related to the κ_n 's through

$$\kappa_\pi[\mathcal{O}_1, \dots, \mathcal{O}_n] = \prod_{V \in \pi} \kappa_{\#V}[\{\mathcal{O}_q\}_{q \in V}]. \tag{5.198}$$

This is the analogue of the classical cumulant expansion where we have the

expansion with $\mathcal{P}(n)$,

$$\mathbb{E}[X_1 X_2 \cdots X_n] = \sum_{\pi \in \mathcal{P}(n)} \kappa_{\pi}^{\text{cl}}[X_1, X_2, \dots, X_n]. \quad (5.199)$$

The higher order cumulants are defined recursively through the lower ones.

- The starting point is $n = 1$:

$$\kappa_1(\mathcal{O}_1) = \varphi(\mathcal{O}_1). \quad (5.200)$$

- $n = 2$:

$$\varphi(\mathcal{O}_1 \mathcal{O}_2) = \varphi(\mathcal{O}_1 \mathcal{O}_2) + \varphi(\mathcal{O}_1 \mathcal{O}_2) = \kappa_2(\mathcal{O}_1, \mathcal{O}_2) + \kappa_1(\mathcal{O}_1) \kappa_1(\mathcal{O}_2). \quad (5.201)$$

$\begin{array}{ccc} & \square & \\ & | & | \end{array}$

Then, we see

$$\kappa_2(\mathcal{O}_1, \mathcal{O}_2) = \varphi(\mathcal{O}_1 \mathcal{O}_2) - \varphi(\mathcal{O}_1) \varphi(\mathcal{O}_2). \quad (5.202)$$

- $n = 3$:

$$\begin{aligned} \varphi(\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3) &= \varphi(\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3) \\ &\quad \begin{array}{ccc} & \square & \square \end{array} \\ &+ \varphi(\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3) + \varphi(\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3) + \varphi(\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3) \\ &\quad \begin{array}{ccc} | & \square & \square \end{array} \quad \begin{array}{ccc} \square & | & \end{array} \quad \begin{array}{ccc} \square & & | \end{array} \\ &+ \varphi(\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3) \\ &\quad \begin{array}{ccc} | & | & | \end{array} \\ &= \kappa_3(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) \\ &+ \kappa_1(\mathcal{O}_1) \kappa_2(\mathcal{O}_2, \mathcal{O}_3) + \kappa_1(\mathcal{O}_2) \kappa_2(\mathcal{O}_1, \mathcal{O}_3) + \kappa_1(\mathcal{O}_3) \kappa_2(\mathcal{O}_1, \mathcal{O}_2) \\ &+ \kappa_1(\mathcal{O}_1) \kappa_1(\mathcal{O}_2) \kappa_1(\mathcal{O}_3). \end{aligned} \quad (5.203)$$

This determines $\kappa_3(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ in terms of moments:

$$\begin{aligned} \kappa_3(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) &= \varphi(\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3) - \varphi(\mathcal{O}_1) \varphi(\mathcal{O}_2 \mathcal{O}_3) - \varphi(\mathcal{O}_2) \varphi(\mathcal{O}_1 \mathcal{O}_3) \\ &\quad - \varphi(\mathcal{O}_3) \varphi(\mathcal{O}_1 \mathcal{O}_2) + 2\varphi(\mathcal{O}_1) \varphi(\mathcal{O}_2) \varphi(\mathcal{O}_3). \end{aligned} \quad (5.204)$$

- $n = 4$:

$$\begin{aligned}
\varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) &= \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) \\
&\quad \begin{array}{c} \square \square \square \square \end{array} \\
&+ \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) + \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) \\
&\quad \begin{array}{c} | \quad \square \square \end{array} \quad \begin{array}{c} \square \quad | \quad \square \end{array} \\
&+ \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) + \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) \\
&\quad \begin{array}{c} \square \square \quad | \end{array} \quad \begin{array}{c} \square \square \quad | \end{array} \\
&+ \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) + \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) \\
&\quad \begin{array}{c} \square \quad \square \end{array} \quad \begin{array}{c} \square \quad \square \end{array} \\
&+ \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) + \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) + \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) \\
&\quad \begin{array}{c} | \quad | \quad \square \end{array} \quad \begin{array}{c} | \quad \square \quad | \end{array} \quad \begin{array}{c} \square \quad | \quad | \end{array} \\
&+ \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) + \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) + \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) \\
&\quad \begin{array}{c} | \quad \square \quad | \end{array} \quad \begin{array}{c} \square \quad | \quad | \end{array} \quad \begin{array}{c} \square \quad | \quad | \end{array} \\
&+ \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) \\
&\quad \begin{array}{c} | \quad | \quad | \quad | \end{array} \\
&= \kappa_4(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4) \\
&+ \kappa_1(\mathcal{O}_1)\kappa_3(\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4) + \kappa_1(\mathcal{O}_2)\kappa_3(\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_4) \\
&+ \kappa_1(\mathcal{O}_3)\kappa_3(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4) + \kappa_1(\mathcal{O}_4)\kappa_3(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) \\
&+ \kappa_2(\mathcal{O}_1, \mathcal{O}_2)\kappa_2(\mathcal{O}_3, \mathcal{O}_4) + \kappa_2(\mathcal{O}_1, \mathcal{O}_3)\kappa_2(\mathcal{O}_2, \mathcal{O}_4) \\
&+ \kappa_1(\mathcal{O}_1)\kappa_1(\mathcal{O}_2)\kappa_2(\mathcal{O}_3, \mathcal{O}_4) + \kappa_1(\mathcal{O}_1)\kappa_2(\mathcal{O}_2, \mathcal{O}_3)\kappa_1(\mathcal{O}_4) \\
&+ \kappa_2(\mathcal{O}_1, \mathcal{O}_2)\kappa_1(\mathcal{O}_3)\kappa_1(\mathcal{O}_4) \\
&+ \kappa_1(\mathcal{O}_1)\kappa_2(\mathcal{O}_2, \mathcal{O}_4)\kappa_1(\mathcal{O}_3) + \kappa_2(\mathcal{O}_1, \mathcal{O}_4)\kappa_1(\mathcal{O}_2)\kappa_1(\mathcal{O}_3) \\
&+ \kappa_2(\mathcal{O}_1, \mathcal{O}_3)\kappa_1(\mathcal{O}_2)\kappa_1(\mathcal{O}_4) \\
&+ \kappa_1(\mathcal{O}_1)\kappa_1(\mathcal{O}_2)\kappa_1(\mathcal{O}_3)\kappa_1(\mathcal{O}_4).
\end{aligned} \tag{5.205}$$

We can invert this relation and define κ_4 in terms of moments. Here we write the expression in the simple case $\varphi(\mathcal{O}_j) = 0$:

$$\begin{aligned}
\kappa_4[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4] &= \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) - \kappa_2[\mathcal{O}_1, \mathcal{O}_2]\kappa_2[\mathcal{O}_3, \mathcal{O}_4] - \kappa_2[\mathcal{O}_1, \mathcal{O}_4]\kappa_2[\mathcal{O}_2, \mathcal{O}_3] \\
&= \varphi(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4) - \varphi(\mathcal{O}_1\mathcal{O}_2)\varphi(\mathcal{O}_3\mathcal{O}_4) - \varphi(\mathcal{O}_1\mathcal{O}_4)\varphi(\mathcal{O}_2\mathcal{O}_3).
\end{aligned} \tag{5.206}$$

With these definitions, we can see that freeness is expressed as the vanishing of mixed cumulants.

Theorem

Theorem 10. *The fact that $\mathcal{O}_1, \dots, \mathcal{O}_m$ are free is equivalent to the fact that*

$$\kappa_n[\mathcal{O}_{i(1)}, \mathcal{O}_{i(2)}, \dots, \mathcal{O}_{i(n)}] = 0 \quad (5.207)$$

for any map $i : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, m\}$ whenever

- $1 \leq i(1), \dots, i(n) \leq m$.
- *there exist p, q such that $i(p) \neq i(q)$ for $n \geq 2$.*

As an equivalent and important lemma, we have the following.

Theorem

Theorem 11. *Let $\{\mathcal{A}_i\}_{i=1}^n$ be freely independent unital subalgebras. We take free variables $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k$ such that $\mathcal{O}_j \in \mathcal{A}_{i(j)}$ where $i : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, n\}$. Then, we define the decomposition $\ker(i) \in \mathcal{P}(k)$ such that when $i(p) = i(q)$ then p, q belong to the same block. If $\ker(i) \in \text{NC}(k)$,*

$$\varphi(\mathcal{O}_1 \cdots \mathcal{O}_k) = \prod_{V \in \ker(i)} \varphi\left(\overset{\rightarrow}{\prod}_{j \in V} \mathcal{O}_j\right). \quad (5.208)$$

Here $\overset{\rightarrow}{\prod}_{j \in V} \mathcal{O}_j$ means that we take the product in the order such that j increases.

This theorem means that if the decomposition $\ker(i)$ is non-crossing then we can factor the moment into a product of blocks. For example:

- Suppose $n = 4$, $k = 7$ and $\mathcal{O}_1, \mathcal{O}_5, \mathcal{O}_7 \in \mathcal{A}_1$, $\mathcal{O}_2, \mathcal{O}_4 \in \mathcal{A}_2$, $\mathcal{O}_3 \in \mathcal{A}_3$, $\mathcal{O}_6 \in \mathcal{A}_4$. Then, $\ker(i) \in \text{NC}(7)$ and

$$\begin{aligned} \varphi(\mathcal{O}_1 \cdots \mathcal{O}_7) &= \varphi(\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \mathcal{O}_5 \mathcal{O}_6 \mathcal{O}_7) \\ &\quad \begin{array}{c} \boxed{} \quad \boxed{} \quad \boxed{} \quad \boxed{} \\ \end{array} \\ &= \varphi(\mathcal{O}_1 \mathcal{O}_5 \mathcal{O}_7) \varphi(\mathcal{O}_2 \mathcal{O}_4) \varphi(\mathcal{O}_3) \varphi(\mathcal{O}_6). \end{aligned} \quad (5.209)$$

We can easily check from the definition of freeness that

$$\begin{aligned} \kappa_1[\mathcal{O}_j] &= \varphi(\mathcal{O}_j) = 0, & \kappa_2[\mathcal{O}_1, \mathcal{O}_2] &= \varphi(\mathcal{O}_1 \mathcal{O}_2) = 0, \\ \kappa_3[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3] &= \varphi(\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3) = 0, \\ \kappa_4[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4] &= \varphi(\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4) - \varphi(\mathcal{O}_1 \mathcal{O}_2) \varphi(\mathcal{O}_3 \mathcal{O}_4) - \varphi(\mathcal{O}_1 \mathcal{O}_4) \varphi(\mathcal{O}_2 \mathcal{O}_3) = 0. \end{aligned} \quad (5.210)$$

We can check that all other mixed free cumulants also vanish.

A simple application is the computation of mixed moments of two operators $\mathcal{O}, \mathcal{O}' \in \mathcal{A}$. When $\mathcal{O}$ and $\mathcal{O}'$ are free, their mixed moments can be computed from their individual moments only:

Theorem

Theorem 12. *Suppose we have free unital subalgebras $\mathcal{A}_1, \mathcal{A}_2$. Then, the mixed moments of $\mathcal{O} \in \mathcal{A}_1$ and $\mathcal{O}' \in \mathcal{A}_2$ have a simple expression written only in terms of the single-operator moments:*

$$\varphi((\mathcal{O}\mathcal{O}')^n) = \sum_{\pi \in \text{NC}(n)} \kappa_\pi[\mathcal{O}, \mathcal{O}, \dots, \mathcal{O}] \varphi_{\pi^*}((\mathcal{O}')^n). \quad (5.211)$$

Here $\varphi_{\pi^*}((\mathcal{O}')^n)$ is the product of moments, one for each block of the partition π^* , known as the Kreweras complement of the partition π .

■ Free probability and ETH

In the papers, a generalization of ETH is discussed; we consider the matrix elements of operators $\omega_{nm} = E_n - E_m$ and $\mathcal{O}_{nm} := \langle E_n | \mathcal{O} | E_m \rangle$. Suppose these matrix elements follow the ETH ansatz (5.182). Then, for distinct indices $n_1 \neq n_2 \neq \dots \neq n_p$, we obtain following scaling

$$\mathbb{E}_R[\mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_3} \dots \mathcal{O}_{n_{p-1} n_p}] = (D_{E, \Delta E})^{1-p} F^{(p)}(\vec{\omega}) \quad (5.212)$$

where $\mathbb{E}_R[\dots]$ represents average over pseudo random variables R_{nm} . For repeated indices we have the factorization,

$$\begin{aligned} & \mathbb{E}_R[\mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_3} \dots \mathcal{O}_{n_{q-1} n_q} \mathcal{O}_{n_1 n_{q+1}} \dots \mathcal{O}_{n_{p-1} n_p}] \\ &= \mathbb{E}_R[\mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_3} \dots \mathcal{O}_{n_{q-1} n_q}] \mathbb{E}_R[\mathcal{O}_{n_1 n_{q+1}} \dots \mathcal{O}_{n_{p-1} n_p}] + O\left(\frac{1}{D_{E, \Delta E}}\right). \end{aligned} \quad (5.213)$$

Then, we take $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\varphi : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ by

$$\varphi(\mathcal{O}) := \langle \mathcal{O} \rangle_{\text{mc}}^{E, \Delta E} = \frac{1}{D_{E, \Delta E}} \text{Tr}[\rho_{\text{mc}}^{(E, \Delta E)} \mathcal{O}]. \quad (5.214)$$

We can write the correlation functions in terms of the energy eigenbasis,

$$\langle \mathcal{O}(t_1) \mathcal{O}(t_2) \dots \mathcal{O}(t_q) \rangle_{\text{mc}}^{E, \Delta E} = \frac{1}{D_{E, \Delta E}} \sum_{n_1, n_2, \dots, n_q} e^{i\vec{\omega} \cdot \vec{t}} \mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_3} \dots \mathcal{O}_{n_{p-1} n_q}, \quad (5.215)$$

with $\vec{\omega} = (\omega_{n_1 n_2}, \dots, \omega_{n_{p-1} n_p})$, $\vec{t} = (t_1, \dots, t_p)$.

Then, we consider decomposing the sum in the correlation function: this can be split into a sum with all different indices ($\sum_{n_1 \neq n_2 \neq \dots \neq n_p}$), plus sums in which only one index is repeated ($\sum_{n_1 = n_2 \neq \dots \neq n_p}$), plus sums with two repeated indices, and so on. We have a nice pictorial expression for these sums. The algorithm is as follows:

1. For each sum, we draw one circle and place p dots on it clockwise in the order of the matrix elements.
2. If there are repeated indices, then we draw lines connecting the corresponding dots.
3. We then have three fundamental objects:
 - (a) Simple loops: no lines.
 - (b) Non-crossing loops: there are lines, but they do not cross.
 - (c) Crossing loops: there are lines crossing.
4. The sum corresponding to a simple loop is given by

$$\mathcal{K}_q(\vec{t}) = \frac{1}{D_{E, \Delta E}} \sum_{n_1 \neq n_2 \neq \dots \neq n_q} e^{i\vec{\omega} \cdot \vec{t}} \mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_3} \cdots \mathcal{O}_{n_q n_1}. \quad (5.216)$$

By using the ETH ansatz (5.212), we see

$$\mathcal{K}_q(\vec{t}) = \int d\vec{\omega} e^{i\vec{\omega} \cdot \vec{t}} F^{(q)}(\vec{\omega}). \quad (5.217)$$

5. For the sums with repeated indices, by using the ETH ansatz (5.213), we find that the sums decompose into products of simple loops.
6. The crossing loops are suppressed compared to the other terms.

We show some concrete examples.

- Example $p = 2$:

$$\begin{aligned}
\langle \mathcal{O}(t_1) \mathcal{O}(t_2) \rangle_{\text{mc}}^{E, \Delta E} &= \frac{1}{D_{E, \Delta E}} \sum_{n_1, n_2} e^{i\omega_{n_1 n_2} t} \mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_1} \\
&= \frac{1}{D_{E, \Delta E}} \sum_{n_1 \neq n_2} e^{i\omega_{n_1 n_2} t} \mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_1} + \frac{1}{D_{E, \Delta E}} \sum_{n_1 = n_2} e^{i\omega_{n_1 n_2} t} \mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_1} \\
&= \sum \text{Diagram 1} + \sum \text{Diagram 2}.
\end{aligned}
\tag{5.218}$$

By using the ETH ansatz, we see

$$\mathbb{E}_R \left[\langle \mathcal{O}(t_1) \mathcal{O}(t_2) \rangle_{\text{mc}}^{E, \Delta E} \right] = \mathcal{K}_2(t_1, t_2) + \mathcal{K}_1(t_1) \mathcal{K}_1(t_2).
\tag{5.219}$$

- Example $p = 3$:

$$\begin{aligned}
\langle \mathcal{O}(t_1) \mathcal{O}(t_2) \mathcal{O}(t_3) \rangle_{\text{mc}}^{E, \Delta E} &= \frac{1}{D_{E, \Delta E}} \sum_{n_1, n_2, n_3} e^{i\vec{\omega} \cdot \vec{t}} \mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_3} \mathcal{O}_{n_3 n_1} \\
&= \frac{1}{D_{E, \Delta E}} \sum_{n_1 \neq n_2 \neq n_3} e^{i\vec{\omega} \cdot \vec{t}} \mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_3} \mathcal{O}_{n_3 n_1} \\
&\quad + \frac{1}{D_{E, \Delta E}} \left(\sum_{n_1 = n_2 \neq n_3} + \sum_{n_1 = n_3 \neq n_2} + \sum_{n_2 = n_3 \neq n_1} \right) e^{i\vec{\omega} \cdot \vec{t}} \mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_3} \mathcal{O}_{n_3 n_1} \\
&\quad + \frac{1}{D_{E, \Delta E}} \sum_{n_1 = n_2 = n_3} e^{i\vec{\omega} \cdot \vec{t}} \mathcal{O}_{n_1 n_2} \mathcal{O}_{n_2 n_3} \mathcal{O}_{n_3 n_1} \\
&= \sum \text{Diagram 1} + \sum \text{Diagram 2} \\
&\quad + \sum \text{Diagram 3} + \sum \text{Diagram 4} + \sum \text{Diagram 5}
\end{aligned}
\tag{5.220}$$

By using the ETH ansatz, we find

$$\begin{aligned} \mathbb{E}_R \left[\langle \mathcal{O}(t_1) \mathcal{O}(t_2) \mathcal{O}(t_3) \rangle_{\text{mc}}^{E, \Delta E} \right] &= \mathcal{K}_3(t_1, t_2, t_3) \\ &+ \mathcal{K}_2(t_1, t_2) \mathcal{K}_1(t_3) + \mathcal{K}_2(t_1, t_3) \mathcal{K}_1(t_2) + \mathcal{K}_2(t_2, t_3) \mathcal{K}_1(t_1) \\ &+ \mathcal{K}_1(t_1) \mathcal{K}_1(t_2) \mathcal{K}_1(t_3) \end{aligned} \quad (5.221)$$

- Example $p = 4$: We find that

$$\langle \mathcal{O}(t_1) \mathcal{O}(t_2) \mathcal{O}(t_3) \mathcal{O}(t_4) \rangle_{\text{mc}}^{E, \Delta E} \quad (5.222)$$

$$\begin{aligned} &= \sum n_4 \text{ (circle with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom)} + \sum n_4 \text{ (circle with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, double arc from } n_1 \text{ to } n_2) + \sum n_4 \text{ (circle with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, double arc from } n_1 \text{ to } n_3) \\ &+ \sum n_4 \text{ (circle with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, double arc from } n_2 \text{ to } n_3) + \sum n_4 \text{ (circle with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, double arc from } n_1 \text{ to } n_2 \text{ and } n_1 \text{ to } n_3) + \sum n_4 \text{ (circle with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, double arc from } n_2 \text{ to } n_3 \text{ and } n_1 \text{ to } n_3) \\ &+ \sum n_4 \text{ (circle with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, double arc from } n_1 \text{ to } n_2 \text{ and } n_2 \text{ to } n_3) + \sum n_4 \text{ (circle with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, double arc from } n_1 \text{ to } n_3 \text{ and } n_1 \text{ to } n_2) + \sum n_4 \text{ (circle with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, double arc from } n_1 \text{ to } n_2 \text{ and } n_2 \text{ to } n_3 \text{ and } n_1 \text{ to } n_3) \\ &+ \sum n_4 \text{ (circle with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, double arc from } n_1 \text{ to } n_2 \text{ and } n_1 \text{ to } n_3 \text{ and } n_2 \text{ to } n_3) + \sum n_4 \text{ (circle with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, double arc from } n_1 \text{ to } n_2 \text{ and } n_1 \text{ to } n_3 \text{ and } n_2 \text{ to } n_3 \text{ and } n_1 \text{ to } n_2 \text{ and } n_1 \text{ to } n_3) \\ &+ \sum n_4 \text{ (diamond shape with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, } n_4 \text{ at left)} \end{aligned} \quad (5.223)$$

$$+ \sum n_4 \text{ (diamond shape with } n_1 \text{ at top, } n_2 \text{ at right, } n_3 \text{ at bottom, } n_4 \text{ at left)} \quad (5.224)$$

The crossing terms are suppressed:

$$\frac{1}{D_{E, \Delta E}} \sum_{n, m} e^{i\vec{\omega} \cdot \vec{t}} \mathbb{E}_R [\mathcal{O}_{nm} \mathcal{O}_{mn} \mathcal{O}_{nm} \mathcal{O}_{mn}] = O\left(\frac{1}{D_{E, \Delta E}}\right). \quad (5.225)$$

Then, we see

$$\begin{aligned}
& \mathbb{E}_R \langle \mathcal{O}(t_1) \mathcal{O}(t_2) \mathcal{O}(t_3) \mathcal{O}(t_4) \rangle_{\text{mc}}^{E, \Delta E} \\
&= \mathcal{K}_4(t_1, t_2, t_3, t_4) \\
&+ \mathcal{K}_1(t_1) \mathcal{K}_3(t_2, t_3, t_4) + \mathcal{K}_1(t_2) \mathcal{K}_3(t_1, t_3, t_4) + \mathcal{K}_1(t_3) \mathcal{K}_3(t_1, t_2, t_4) + \mathcal{K}_1(t_4) \mathcal{K}_3(t_1, t_2, t_3) \\
&+ \mathcal{K}_2(t_2 - t_3, t_4 - t_1) \mathcal{K}_1(t_1 - t_2) \mathcal{K}_1(t_3 - t_4) + \mathcal{K}_2(t_1 - t_2, t_3 - t_4) \mathcal{K}_1(t_2 - t_3) \mathcal{K}_1(t_4 - t_1) \\
&+ \dots
\end{aligned} \tag{5.226}$$

We see that the expression of the correlation functions with full ETH is the same as the free cumulant expression of the moments in Definition (5.7.1). Thus we see:

Theorem

Theorem 13. *When we apply the full ETH ansatz (5.212), (5.213), the correlation functions have expressions in terms of the ETH simple loops $\mathcal{K}_q$. These expressions are the same as the expressions of the free moments in terms of free cumulants (5.7.1). Thus, we see*

$$\kappa_n[\mathcal{O}(t_1), \dots, \mathcal{O}(t_n)] = \mathcal{K}_n(\vec{t}) + O\left(\frac{1}{D_{E, \Delta E}}\right). \tag{5.227}$$

■ Asymptotic freeness

Here we discuss the physical consequence of the results above. To this end, we define the notion of *asymptotic freeness* [166] :

Definition

Definition 21. Suppose we have two $D \times D$ matrices A, B . Then we say A, B are asymptotically free if their mixed cumulants vanish for all $n \in \mathbb{N}$,

$$\lim_{D \rightarrow \infty} \kappa_{2n}[A, B, \dots, A, B] = 0. \tag{5.228}$$

We say A, B are asymptotically p -free if their mixed cumulants vanish for all $1 \leq n \leq p$,

$$\lim_{D \rightarrow \infty} \kappa_{2n}[A, B, \dots, A, B] = 0. \tag{5.229}$$

Examples are given by random matrix theory. Suppose we take two operators A, B and define $A^U := U^\dagger A U$ with $U \sim \text{Haar}$; then A^U, B are asymptotically free. If

we replace the Haar measure by a unitary p -design, then we obtain asymptotically p -free behavior.

The essence of chaotic dynamics is asymptotic freeness at long times. We can see that for observables $\mathcal{O}_1, \mathcal{O}_2$ which satisfy ETH, the infinite-time average of their mixed free cumulants vanishes:

$$\lim_{D_{E,\Delta E} \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \kappa_{2n}[\mathcal{O}_1(t), \mathcal{O}_2, \dots, \mathcal{O}_1(t), \mathcal{O}_2] = 0. \quad (5.230)$$

In actual systems, asymptotic freeness holds already at finite times, so we expect

$$\kappa_{2n}[\mathcal{O}_1(t), \mathcal{O}_2, \dots, \mathcal{O}_1(t), \mathcal{O}_2] \approx O\left(\frac{1}{D_{E,\Delta E}}\right), \quad t \gg t_n^* \quad (5.231)$$

for some characteristic time scale t_n^* .

Long-time freeness means that mixed moments at late times can be computed using factorization. For example, out-of-time-order correlators (OTOCs) are

$$\langle \mathcal{O}(t) \mathcal{O}' \mathcal{O}(t) \mathcal{O}' \rangle_{\text{mc}}^{E,\Delta E} = \kappa_2[\mathcal{O}(t), \mathcal{O}(t)] \kappa_1[\mathcal{O}']^2 + \kappa_1[\mathcal{O}(t)]^2 \kappa_2[\mathcal{O}', \mathcal{O}'] + O\left(\frac{1}{D_{E,\Delta E}}\right). \quad (5.232)$$

■ Relation to our clustering property

In [1], as a sufficient condition for strong ETH, the following condition is discussed:

$$\prod_{i=1}^{2p} \lim_{T_i \rightarrow \infty} \int_0^{T_i} \frac{dt_i}{T_i} G^{(2p)}(t_1, t_2, \dots, t_{2p}) = O\left(\frac{1}{(D_{E,\Delta E})^q}\right) \quad (5.233)$$

where

$$G^{(k)}(t_1, t_2, \dots, t_k) = \text{Tr} \left[\left(\rho_{\text{mc}}^{(E,\Delta E)} \right)^{\frac{1}{k}} \hat{\mathcal{O}}_{\text{mc}}(t_1) \left(\rho_{\text{mc}}^{(E,\Delta E)} \right)^{\frac{1}{k}} \hat{\mathcal{O}}_{\text{mc}}(t_2) \cdots \left(\rho_{\text{mc}}^{(E,\Delta E)} \right)^{\frac{1}{k}} \hat{\mathcal{O}}_{\text{mc}}(t_k) \right] \quad (5.234)$$

with $\hat{\mathcal{O}}_{\text{mc}} := \mathcal{O} - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E}$. This correlation function is given by

$$\varphi(\mathcal{O}) = \frac{1}{D_{E,\Delta E}} \text{Tr}[\mathcal{O}], \quad \mathcal{O} \in \mathcal{B}(\mathcal{H}_{E,\Delta E}). \quad (5.235)$$

We take $\mathcal{O} \in \mathcal{B}(\mathcal{H})$ and project it onto the energy shell by $\Pi_{E,\Delta E} \mathcal{O} \Pi_{E,\Delta E}$. The condition (5.233) expresses asymptotic freeness at long times. However, it seems that the finite-size scaling behavior is different. We will see this clustering condition later.

5.8 Late time behavior of the spectral form factor and correlation functions

5.8.1 Random matrix theory and Spectral Form Factor

The random Hamiltonian gives toy model of quantum chaos [167]. the Hamiltonians follow the random matrix theory (RMT). The typical examples we use are the gaussian type RMT like GUE, GOE and the circular type RMT like the CUE. Here we focus on the Gaussian type. For the Gaussian type RMT which classified by the Dyson index $\mathfrak{b} = 1(\text{GUE}), 2(\text{GOE}), 4(\text{GSE})$ has the following probability distribution for the Hamiltonian with $D \times D$ matrix, where D is the dimension of Hilbert space,

$$\begin{aligned} dHP(H) &= \frac{1}{\mathcal{Z}_0} \exp\{-\text{Tr}[H^2]\} \\ &= \frac{1}{\mathcal{Z}} dE_n \prod_{n < m} |E_n - E_m|^{\mathfrak{b}} \prod_n e^{-E_n^2} \\ &= \frac{dE_n}{\mathcal{Z}} \exp\left\{-\sum_n (E_n)^2 + \mathfrak{b} \sum_{n < m} \log |E_n - E_m|\right\} \\ &=: dE_n P(E_1, E_2, \dots, E_D) = DE \end{aligned} \tag{5.236}$$

The fundamental quantities are the density of states

$$\rho(E) := \sum_n \delta(E - E_n). \tag{5.237}$$

Also we have the n -points correlation functions

$$\rho^{(n)}(E_1, E_2, \dots, E_n) := \int dE_{n+1} \dots dE_D P(E_1, E_2, \dots, E_D). \tag{5.238}$$

One important RMT results in large D is the following formulas: the n -points correlation functions are determined by the Kernel K as

$$\rho^{(n)}(E_1, E_2, \dots, E_n) = \frac{(D-n)!}{D!} \det\{K(E_n, E_m)\}_{i,j=1}^n \tag{5.239}$$

where the kernel K in the large D limit behaves as

$$K(E_n, E_m) = \begin{cases} \frac{D \sin(D(E_n - E_m))}{\pi D(E_n - E_m)} & \text{if } n \neq m, \\ \frac{D}{2\pi} \sqrt{4 - E_n^2} & \text{if } n = m. \end{cases} \tag{5.240}$$

The diagonal part is called the Wigner semi-circle law and the off-diagonal part is called the sine kernel formula.

The spectral form factor is given by the Fourier transformation of the correlation functions

$$\begin{aligned} \text{SFF}^{(k)}(T) &:= \sum_{n,m} \int \text{D}E \, e^{i(E_{n_1} + \dots + E_{n_k} - E_{m_1} - \dots - E_{m_k})T}, \quad k \in \mathbb{N} \\ &= \left\langle \left| \text{Tr}[e^{-iHT}] \right|^{2k} \right\rangle_H \end{aligned} \quad (5.241)$$

where $\langle f(H) \rangle_H := \int \text{d}H P(H) f(H)$ is an average over the Hamiltonians. The simplest example is the two point function ($k = 1$). This is determined by the two point correlation function as follows

$$\begin{aligned} \text{SFF}(T) &= \sum_{n,m} \int \text{D}E e^{i(E_n - E_m)T} \\ &= D + D(D-1) \int \text{d}E_1 \text{d}E_2 \rho^{(2)}(E_1, E_2) e^{i(E_1 - E_2)T}. \end{aligned} \quad (5.242)$$

By directly computing the determinant, we obtain the famous universal sine kernel formula,

$$\rho^{(2)}(E_1, E_2) = \frac{D^2}{D(D-1)} \bar{\rho}(E_1) \bar{\rho}(E_2) - \frac{D^2}{D(D-1)} \frac{\sin^2(D(E_1 - E_2))}{(D\pi(E_1 - E_2))^2} \quad (5.243)$$

where

$$\bar{\rho}(E) = \rho^{(1)}(E) = \frac{1}{2\pi} \sqrt{4 - E^2} \quad (5.244)$$

is a Wigner semi-circle. This formula allows us the following of the SFF [168] ,

$$\begin{aligned} \text{SFF}(T) &= \text{SFF}_{\text{dis}}(T) + \text{SFF}_{\text{con}}(T) \\ \text{SFF}_{\text{dis}}(T) &:= D^2 \int \text{d}E_1 \text{d}E_2 \bar{\rho}(\rho_1) \bar{\rho}(\rho_2) e^{i(E_1 - E_2)T} \\ &= \left| \left\langle \text{Tr}[e^{-iHT}] \right\rangle_H \right|^2 \\ &= D^2 r_1^2(T), \\ r_1(T) &:= \frac{J_1(2T)}{T}. \end{aligned} \quad (5.245)$$

That is the disconnected piece is determined by the product of the Wigner semi-

circle. The connected part is defined by the complement

$$\begin{aligned} \text{SFF}_{\text{con}}(T) &:= \text{SFF}(T) - \text{SFF}_{\text{dis}}(T) \\ &= D - D^2 \int dE_1 dE_2 \frac{\sin^2(D(E_1 - E_2))}{(D\pi(E_1 - E_2))^2} e^{i(E_1 - E_2)T}. \end{aligned} \quad (5.246)$$

After doing the integral with some moderate regularization, we see

$$\begin{aligned} \text{SFF}_{\text{con}}(T) &= D(1 - r_2(T)) \\ r_2(T) &= \begin{cases} 1 - \frac{T}{2D}, & \text{for } T < 2D \\ 0 & \text{for } T > 2D. \end{cases} \end{aligned} \quad (5.247)$$

Then the total spectral form factor is plotted in the Figure 18. Now see the following

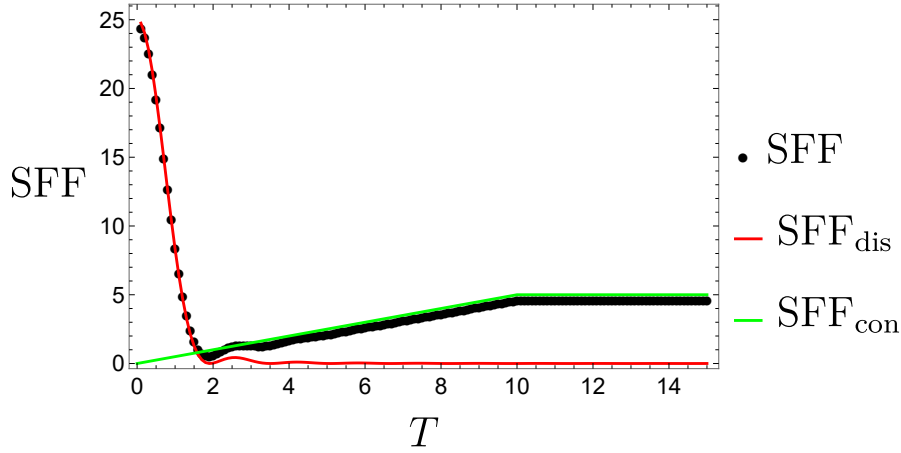


Figure 18: Behavior of spectral form factor: We plot with $D = 5$.

three parts of spectral form factor

1. Slope region: in early time, we see the spectral form factor shows decaying. To understand this, we see the another expression of SFF,

$$\text{SFF}(T) = \left\langle \left| \langle I | e^{-\frac{iHT}{2}(H_L + H_R)} | I \rangle \right|^2 \right\rangle_H, \quad |I\rangle = \sum_n |E_n\rangle |E_n\rangle. \quad (5.248)$$

Thus, the spectral form factor is a return amplitude from maximal entanglement state. For enough complicated Hamiltonian, we expect that the state non-trivially evolves and the amplitudes decreases. In this time scale the disconnected part $\text{SFF}_{\text{dis}} = \left(\left\langle \langle I | e^{-\frac{iHT}{2}(H_L + H_R)} | I \rangle \right\rangle_H \right)^2$ is essential contribute.

2. Ramp region: But around some time scale called dip time, the correlation between two amplitudes getting important. This is qualitatively regarded as

the connected part of $\rho^{(2)}(E_1, E_2)$ and thus correlation between two energy levels. For chaotic theories, because of the level repulsion, we see that the spectral form factor shows linear growth called ramp.

3. Plateau part: After ramp region, we see the spectral form factor reaches to stationary part called plateau.

5.8.2 Spectral form factor in large q -SYK models

We can also discuss the spectral form factor in large q SYK models [169]. For SYK models, spectral form factor can be expressed as a following path integral,

$$\begin{aligned} \text{SFF}(T, \beta) &= \left\langle \left| \text{Tr}[e^{-iHT}]^2 \right| \right\rangle_J \\ &= \int_{\text{P.B.C}} \mathcal{D}\psi_{\pm} \exp \left\{ i(S_{\text{SYK}}^{(+)}[\psi^{(+)}] - S_{\text{SYK}}^{(-)}[\psi^{(-)}]) \right\} \\ S_{\text{SYK}}^{(\pm)}[\psi^{(\pm)}] &= \int_C dt \left[\frac{i}{2} \sum_{j=1}^N \psi_{(\pm)}^j \partial_t \psi_{(\pm)}^j - \frac{(\pm i)^{\frac{q}{2}}}{q!} \sum_{j_1, j_2, \dots, j_q} J_{j_1 \dots j_q} \psi_{(\pm)}^{j_1} \cdots \psi_{(\pm)}^{j_q} \right]. \end{aligned} \quad (5.249)$$

The different sign $(\pm i)^{\frac{q}{2}}$ is coming from the convention related to the Hermite conjugation [36]. The periodic boundary condition (P.B.C) represents,

$$\psi_{(\pm)}^j(T) = -\psi_{(\pm)}^j(0). \quad (5.250)$$

We also obtain the bi-local fields representation,

$$\begin{aligned} \text{SFF}(T, \beta) &= \int_{\text{P.B.C}} \mathcal{D}G_{ab} \mathcal{D}\Sigma_{ab} e^{-NI_{\text{eff}}[G, \Sigma]} \\ I_{\text{eff}}[G, \Sigma] &= -\frac{1}{2} \log \det [\partial_t \delta(t - t') \delta_{ab} - \Sigma_{ab}(t, t')] \\ &\quad + \frac{1}{2} \int_C dt dt' \left[\Sigma_{ab}(t, t') G_{ab}(t, t') - \frac{J^2}{q} s_{ab} (G_{ab}(t, t'))^q \right] \\ s_{++} &= s_{--} = -1, \quad s_{+-} = s_{-+} = (-1)^{\frac{q}{2}} = i^q \end{aligned} \quad (5.251)$$

The saddle point equation is given by

$$\begin{aligned} \Sigma_{ab}(t, t') &= J^2 s_{ab} (G_{ab}(t, t'))^{q-1} + \Gamma_{ab}(t, t') \\ -iG_0^{-1} * G - \Sigma * G &= \mathbb{I} \\ -iG * G_0^{-1} - G * \Sigma &= \mathbb{I} \end{aligned} \quad (5.252)$$

where

$$(-iG_0^{-1})_{ab}(t, t') = \partial_t \delta(t - t') \delta_{ab} \quad (5.253)$$

. As an ansatz for the saddle point equations, we suppose the bi-local fields take a form of

$$G_{ab}(t, t') = G_{ab}(t - t'). \quad (5.254)$$

Because of the anti periodic boundary conditions, we have a Fourier transformation with Matsubara frequency $\omega_n = \frac{2\pi}{T}(n + \frac{1}{2})$, $n \in \mathbb{Z}$,

$$\begin{aligned} G_{ab}(\omega_n) &= \int_0^T dt e^{i\omega_n t} G_{ab}(t), \\ G_{ab}(t) &= \frac{1}{T} \sum_{n \in \mathbb{Z}} e^{-i\omega_n t} G_{ab}(\omega_n). \end{aligned} \quad (5.255)$$

Then, the second equation of the saddle point equations are in the form of

$$G_{ab}^{-1}(\omega_n) = -(i\omega_n \delta_{ab} + \Sigma_{ab}(\omega_n)). \quad (5.256)$$

Also we clarify the index symmetry of the bi-local fields. The bi-local field satisfies the anti-symmetry constraint

$$G_{ab}(t_1, t_2) = -G_{ba}(t_2, t_1) \quad (5.257)$$

which implies

$$G_{ab}(t) = -G_{ba}(-t), \quad G_{ab}(\omega_n) = -G_{ba}(-\omega_n). \quad (5.258)$$

Also from the anti-periodicity

$$G_{ab}(t) = -G_{ab}(T + t) = G_{ba}(-T - t) = G_{ba}(T - t). \quad (5.259)$$

In the last line, we use the anti-periodicity twice. To get some analytic result we consider large q limit of our SYK model. The recipe for large q limit is as follows [169]:

1. We take following scaling in large q limit: Fix $\mathcal{J}$ which is defined through $J^2 = \frac{2^{q-1}}{q} \mathcal{J}^2$.

2. Expand the bi-local fields G_{ab} , Σ_{ab} in $1/q$,

$$\begin{aligned} G_{ab}(t_1, t_2) &= G_{ab}^{(0)}(t_1, t_2) \exp \left\{ \frac{1}{q} g_{ab}(t_1, t_2) \right\} \\ &= G_{ab}^{(0)}(t_1, t_2) + \frac{1}{q} G_{ab}^{(1)}(t_1, t_2) + O\left(\frac{1}{q^2}\right), \\ G_{ab}^{(1)}(t_1, t_2) &= G_{ab}^{(0)}(t_1, t_2) g_{ab}(t_1, t_2) \\ \Sigma_{ab}(t_1, t_2) &= \Sigma_{ab}^{(0)}(t_1, t_2) + \frac{1}{q} \Sigma_{ab}^{(1)}(t_1, t_2) + O\left(\frac{1}{q^2}\right). \end{aligned} \quad (5.260)$$

The self energy is determined by the Kadanoff-Baym equation (5.252).

$$\Sigma_{ab}(t_1, t_2) = \frac{\mathcal{J}^2}{q} s_{ab} (2G_{ab}^{(0)})^{q-1} e^{g_{ab}(t_1, t_2)} + O\left(\frac{1}{q^2}\right). \quad (5.261)$$

This tells us that

$$\begin{aligned} \Sigma_{ab}^{(0)}(t_1, t_2) &= 0, \\ \Sigma_{ab}^{(1)}(t_1, t_2) &= \mathcal{J}^2 s_{ab} (2G_{ab}^{(0)}(t_1, t_2))^{q-1} e^{g_{ab}(t_1, t_2)}. \end{aligned} \quad (5.262)$$

3. Expand the Kadanoff-Baym equation (5.252) in $1/q$. The boundary conditions is moderately chosen in the situations.
4. Evaluate the on-shell action. We will derive the on-shell action later in detail. The action is now

$$\begin{aligned} I_{\text{eff}} &= -\frac{1}{2} \text{Tr} [\log (\partial_t \delta(t - t') \delta_{ab} - \Sigma_{ab}(t, t'))] \\ &+ \frac{1}{2} \int_0^T dt dt' \left[\Sigma_{ab}(t, t') G_{ab}(t, t') - \frac{\mathcal{J}^2}{2q^2} s_{ab} (2G_{ab}(t, t'))^q \right]. \end{aligned} \quad (5.263)$$

We do have different evaluation for this effective action depending on the order of time T .

■ Short Time Solution $T = O(1)$

Then, we start from expand the Kadanoff-Baym equation (5.252) in $1/q$.

Order $O(q^0)$

$$\begin{aligned} -iG_0^{-1} * G^{(0)} - \Sigma^{(0)} * G^{(0)} &= \mathbb{I} \\ -iG^{(0)} * G_0^{-1} - G^{(0)} * \Sigma^{(0)} &= \mathbb{I}. \end{aligned} \quad (5.264)$$

Since $\Sigma_{ab}^{(0)} = 0$ we conclude

$$\begin{aligned} -iG_0^{-1} * G^{(0)} &= \mathbb{I} \\ -iG^{(0)} * G_0^{-1} &= \mathbb{I}. \end{aligned} \quad (5.265)$$

This is equivalent to the Kadanoff-Baym equation for the free fermion $J \rightarrow 0$. By recalling the formula

$$(-iG_0^{-1} * A)_{ab}(t, t') = \partial_t A_{ab}(t, t'), \quad (5.266)$$

we obtain

$$\begin{aligned} \text{Diagonal } \partial_{t_1} G_{aa}^{(0)}(t_1, t_2) &= \frac{1}{2} \delta(t_1 - t_2) \\ \text{Off-diagonal } \partial_{t_1} G_{ab}^{(0)}(t_1, t_2) &= 0, \quad a \neq b. \end{aligned} \quad (5.267)$$

The generic solution is given by

$$\begin{aligned} \text{Diagonal } G_{aa}(t_1, t_2) &= \frac{1}{2} \text{sgn}(t_1 - t_2) + A_{aa} \\ \text{Off-diagonal } G_{ab}^{(0)}(t_1, t_2) &= B_{ab}, \quad a \neq b. \end{aligned} \quad (5.268)$$

Here A_{ab}, B_{ab} is some constants. From the Majorana fermion algebra $\{\psi_a^i, \psi_b^j\} = \delta^{ij} \delta_{ab}$ we see

$$A_{ab} = 0. \quad (5.269)$$

The off-diagonal components are determined by the boundary conditions of the path-integral we consider. If we focus on the replica symmetric saddle point we require

$$B_{++} = B_{--}, \quad B_{+-} = -B_{-+}. \quad (5.270)$$

So the zero-th order solution has one unfixed constant parameter B_{LR} . For latre use we set

$$B_{ab} = \frac{i}{2} b \epsilon_{ab}, \quad \epsilon_{+-} = -\epsilon_{-+} = 1. \quad (5.271)$$

Order $O(q^1)$

$$-iG_0^{-1} * G^{(1)} - \Sigma^{(1)} * G^{(0)} = 0. \quad (5.272)$$

By acting the $-iG_0^{-1}$ from right, we see

$$(-iG_0^{-1}) * G^{(1)} * (-iG_0^{-1}) = \Sigma^{(1)} * G^{(0)} * (-iG_0^{-1}). \quad (5.273)$$

By using the formula

$$\left((-iG_0^{-1}) * G^{(1)} * (-iG_0^{-1})\right)_{ab}(t, t') = -\partial_t \partial_{t'} A_{ab}(t, t') \quad (5.274)$$

and the order zero equation (5.265), the expression of $\Sigma^{(1)}$ (5.262), we obtain

$$\partial_{t_1} \partial_{t_2} G_{ab}^{(1)}(t_1, t_2) = -\mathcal{J}^2 s_{ab} \left(2G_{ab}^{(0)}(t_1, t_2)\right)^{q-1} \exp\{g_{ab}(t_1, t_2)\}. \quad (5.275)$$

Diagonal

The diagonal components are given by

$$\partial_t \partial_{t'} (\text{sgn}(t - t') g_{aa}(t, t')) = (2\mathcal{J}^2) \text{sgn}(t - t') e^{g_{aa}(t, t')}. \quad (5.276)$$

From the ansatz (5.254), we have

$$\partial_t^2 (\text{sgn}(t) g_{aa}(t)) = -(2\mathcal{J}^2) \text{sgn}(t) e^{g_{aa}(t)}. \quad (5.277)$$

The general solution is given by

$$e^{g_{aa}(t)} = \frac{c_a^2}{\mathcal{J}^2 \cosh^2(c_a |t| + d_a)} \quad (5.278)$$

where c_a, d_a are integration constants. These two are fixed by the boundary conditions. For simplicity, we focus on $\beta = 0$. In this case, the boundary conditions (5.259), we assign the boundary conditions,

$$g_{aa}(0) = 0, \quad g'_{aa}\left(\frac{T}{2}\right) = 0. \quad (5.279)$$

Then,, we find the solution

$$e^{g_{aa}(t)} = \left(\frac{\cosh\left(\frac{\tilde{c}_a}{2}\right)}{\cosh\left(\tilde{c}_a\left(\frac{1}{2} - \frac{t}{T}\right)\right)} \right)^2 \quad (5.280)$$

$$\tilde{c}_a = (\mathcal{J}T) \cosh \frac{\tilde{c}_a}{2}, \quad \tilde{c}_a > 0.$$

Notice that the second equation has two solutions. If $\tilde{c}_+ = \tilde{c}_-$, we have replica symmetric solution and otherwise it is replica symmetry is broken. If $\mathcal{J}T \geq \mathcal{J}T_* = 1.3254 \dots$, there is no real solution and we have complex solutions [169]. See Figure 19.

Off-Diagonal

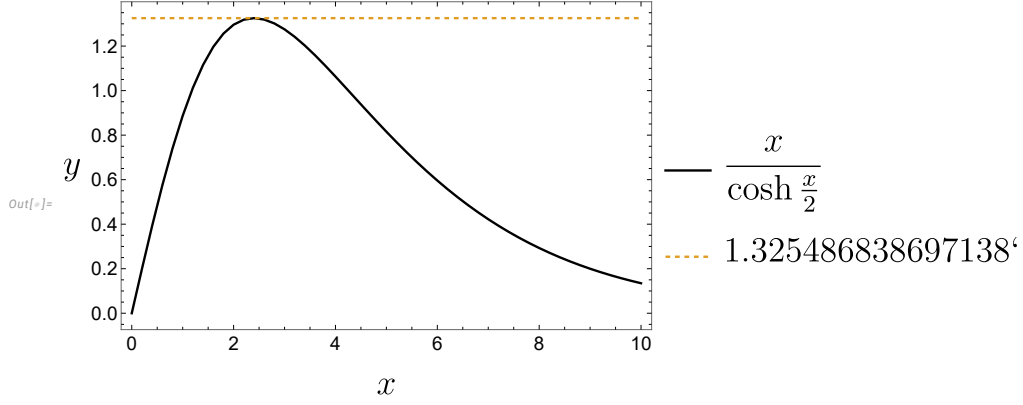


Figure 19: Plot of $y = \frac{x}{\cosh(\frac{x}{2})}$.

When $b \neq 0$, the off-diagonal component reduces to

$$\partial_t \partial_{t'} g_{ab}(t, t') = 2\mathcal{J}^2 b^{q-2} e^{g_{ab}(t, t')}. \quad (5.281)$$

The boundary conditions we assign are as follows: from the anti-periodicity (5.259), the off-diagonal components must satisfies

$$g_{ab}\left(\frac{T}{2}\right) = 0, \quad g_{ab}\left(-\frac{T}{2}\right) = 0. \quad (5.282)$$

Then we have the following solution

$$g_{+-}(t) = -g_{RL}(t) = c\left(t - \frac{T}{2}\right). \quad (5.283)$$

Evaluation of on-shell Action

Let's evaluate the on-shell action,

$$\begin{aligned} I_{\text{eff}}[G, \Sigma] &= -\frac{1}{2} \text{Tr} [\log [G_0^{-1} - \Sigma]] \\ &+ \frac{1}{2} \int_0^T dt dt' \left[\Sigma_{ab}(t, t') G_{ab}(t, t') - \frac{\mathcal{J}^2}{2q^2} s_{ab} (2G_{ab}(t, t'))^q \right]. \end{aligned} \quad (5.284)$$

We are considering the $1/q$ expansion of this on-shell action. Firstly, let us consider the determinant term. Notice that $\Sigma = O(1/q)$, we expand them up to $O(\Sigma^2) =$

$O(1/q^2)$.

$$\begin{aligned}
-\frac{1}{2}\text{Tr}[\log[G_0^{-1} - \Sigma]] &= -\frac{1}{2}\text{Tr}[\log G_0^{-1}] - \frac{1}{2}\text{Tr}[\log[\mathbb{I} - G_0 * \Sigma]] \\
&= -\frac{1}{2}\text{Tr}[\log G_0^{-1}] + \frac{1}{2}\text{Tr}[G_0 * \Sigma] + \frac{1}{4}\text{Tr}[G_0 * \Sigma * G_0 * \Sigma] + O\left(\frac{1}{q^3}\right).
\end{aligned} \tag{5.285}$$

We replace $\frac{1}{2}\text{Tr}[\log G_0^{-1}] = \log 2$ so that $\text{SFF}(0) = 2^{\frac{N}{2}}$. Next we are evaluating the first term in the second line.

$$\begin{aligned}
+\frac{1}{2}\int_0^T dt dt' \Sigma_{ab}(t, t') G_{ab}(t, t') &= -\frac{1}{2}\text{Tr}[\Sigma * G] \\
&= -\frac{1}{2}\text{Tr}[\Sigma * G^{(0)}] - \frac{1}{2q}\text{Tr}[\Sigma * G^{(1)}] + O\left(\frac{1}{q^3}\right).
\end{aligned} \tag{5.286}$$

So the term involving Σ is

$$(\text{Terms with } \Sigma) = \frac{1}{2}\text{Tr}[G_0 * \Sigma] - \frac{1}{2q}\text{Tr}[\Sigma * G^{(1)} * \Sigma * G^{(1)}] + O\left(\frac{1}{q^3}\right). \tag{5.287}$$

Then, we complete the integration over the Σ ,

$$\int \mathcal{D}\Sigma e^{\frac{N}{2q}\text{Tr}[G^{(1)} * \Sigma] - \frac{N}{4}\text{Tr}[\Sigma * G_0 * \Sigma * G_0]} \propto e^{+\frac{N}{4q^2}\text{Tr}[G^{(1)} * G_0^{-1} * G^{(1)} * G_0^{-1}] + O\left(\frac{1}{q^3}\right)} \tag{5.288}$$

In terms of $g_{ab}(t)$, this is rewritten as

$$\begin{aligned}
-\frac{1}{4q^2}\text{Tr}[G^{(1)} * G_0^{-1} * G^{(1)} * G_0^{-1}] &= -\frac{1}{4q^2}\int_0^T dt dt' \partial_t G_{ab}^{(1)}(t, t') \partial_{t'} G_{ba}^{(1)}(t', t) \\
&= \frac{1}{4q^2}\int_0^T dt dt' \partial_t G_{ab}^{(1)}(t, t') \partial_{t'} G_{ab}^{(1)}(t, t') \\
&= \frac{1}{16q^2}\int_0^T dt dt' \sum_{a=+,-} [\partial_t(\text{sgn}(t - t')g_{aa}(t, t')) \partial_{t'}(\text{sgn}(t - t')g_{aa}(t, t'))] \\
&\quad - \frac{b^2}{16q^2}\int_0^T dt dt' [\partial_t g_{+-}(t, t') \partial_{t'} g_{+-}(t, t') + \partial_t g_{-+}(t, t') \partial_{t'} g_{-+}(t, t')] \\
&= -\frac{T}{16q^2}\int_{-T}^T dt' \sum_{a=+,-} (\partial_t(\text{sgn}(t)g_{aa}(t)))^2
\end{aligned} \tag{5.289}$$

On the other hand, the second term in the second line is evaluated as

$$-\frac{\mathcal{J}^2}{4q^2} \int_0^T dt dt' s_{ab}(2G_{ab}(t, t'))^q = \frac{T\mathcal{J}^2}{4q^2} \int_{-T}^T dt \left[\sum_{a=+,-} e^{g_{aa}(t)} \right]. \quad (5.290)$$

In total, we have the action,

$$\begin{aligned} I_{\text{eff}} = & -\log 2 \\ & + \frac{T}{16q^2} \int_{-T}^T dt \sum_{a=+,-} [-(\partial_t(\text{sgn}(t)g_{aa}(t)))^2 + 4\mathcal{J}^2 e^{g_{aa}(t)}] \\ & - \frac{T}{16q^2} \int_{-T}^T dt [-b^2(\partial_t g_{+-}(t))^2 - b^2(\partial_t g_{-+}(t))^2 + 4\mathcal{J}^2 b^q (e^{g_{+-}(t)} + e^{g_{-+}(t)})]. \end{aligned} \quad (5.291)$$

Then, we substitute the solutions we found above. Firstly, let us consider the diagonal term. By using the saddle point equations and the periodic boundary conditions, we find that

$$\frac{T}{16q^2} \int_{-T}^T dt \sum_{a=+,-} [-(\partial_t(\text{sgn}(t)g_{aa}(t)))^2 + 4\mathcal{J}^2 e^{g_{aa}(t)}] = \frac{T}{q^2} \int_0^{\frac{T}{2}} dt \sum_a e^{g_{aa}(t)} \left(1 - \frac{1}{2} g_{aa}(t) \right). \quad (5.292)$$

Then, we substitute the solutions (5.280) and find

$$\frac{T}{q^2} \int_0^{\frac{T}{2}} dt \sum_a e^{g_{aa}(t)} \left(1 - \frac{1}{2} g_{aa}(t) \right) = \frac{1}{q^2} \sum_a \left[2\tilde{c}_a \tanh \frac{\tilde{c}_a}{2} - \frac{1}{2} (\tilde{c}_a)^2 \right]. \quad (5.293)$$

Actually, by computing the one-loop correction, we find that the spectral form factor shows the decays.

■ Large Time solution $T = O(q)$ or $O(q \log q)$

For large time period of $O(q)$, the $1/q$ expansion fail as discussed in [90]. So we need to use the other approximation. The idea is that we can approximate the Σ_{ab} to the delta function because they vary very quickly compared to G_{ab} . This is indeed true from the Kadanoff-Baym equation (5.252). Since the diagonal component of Σ_{ab} are odd-function, they are proportional to the derivative of the delta function $\Sigma_{aa} \propto \delta'(t)$. On the other hand, the off-diagonal component is related to the delta function itself. Thus, we put the ansatz,

$$\Sigma_{+-}(t) = -\Sigma_{-+}(t) = -i\nu\delta(t). \quad (5.294)$$

Even in the very large period $T \sim O(q)$ or $T \sim O(q \log q)$, we think that for shot time $t \sim O(1)$ the $1/q$ expansion works well and we consider the solutions for even

that time scales. But here we do not assign the periodic boundary conditions for these early time solutions. So we have the solution for the early time from the Liouville equation,

$$e^{g_{aa}(t)} = \frac{c_{aa}^2}{\mathcal{J}^2 \cosh^2(c_{aa}|t| + d_{aa})} \quad (5.295)$$

for diagonal solutions and

$$e^{g_{ab}^>(t)} = \frac{(c_{ab}^>)^2}{\mathcal{J}^2 b^{q-2} \cosh^2(c_{ab}^>t + d_{ab}^>)}, \quad t > 0 \quad (5.296)$$

for off-diagonal components. and obtain the solution of the off-diagonal component for early time.

$$e^{g_{ab}(t)} = \frac{(c_{ab})^2}{\mathcal{J}^2 b^{q-2} \cosh^2(c_{ab}|t| + d_{ab})} \quad (5.297)$$

With this solution, we can determine the exponent ν in (5.294). Here we consider the first possibility through while. Later we are coming back the other possibilities. The exponent is related by the early time ansatz via Kadanoff-Baym equation

$$\begin{aligned} \nu &= i \int_{-\infty}^{\infty} dt \Sigma_{+-}(t) \\ &= i \int_{-\infty}^{\infty} dt \frac{\mathcal{J}^2}{q} s_{+-} (2G_{+-}^{(0)})^{q-1} e^{g_{+-}(t)} \\ &= \frac{2b}{q} c_{ab}. \text{ for } \text{Re} b > 0. \end{aligned} \quad (5.298)$$

Here we use the Kadanoff-Baym equation (5.252) and early time solution (5.297) and suppose $q \equiv 0 \pmod{2}$.

Now we consider to solve the Kadanoff-Baym equation in the time scale of $t = O(q), O(q \log q)$. In this case, we use the ansatz (5.294). The diagonal components $\Sigma_{aa} \propto \delta'(t)$ just change the normalization of $\partial_t G_{ab}$ in $1/q$ and we ignore this contribution. Then, the Kadnoff-Baym equation (5.252) give four equations,

$$\begin{aligned} \partial_t G_{++}(t) + i\nu G_{-+}(t) &= 0, \\ \partial_t G_{+-}(t) + i\nu G_{--}(t) &= 0, \\ \partial_t G_{-+}(t) - i\nu G_{++}(t) &= 0, \\ \partial_t G_{--}(t) - i\nu G_{+-}(t) &= 0. \end{aligned} \quad (5.299)$$

These are easily solved and using the boundary conditions such that the G_{aa} are

symmetric around $t = \frac{T}{2}$, we see

$$\begin{aligned}
G_{++}(t) &= A \cosh \left(\nu \left(\frac{T}{2} - t \right) \right), \\
G_{-+}(t) &= -iA \sinh \left(\nu \left(\frac{T}{2} - t \right) \right), \\
G_{--}(t) &= B \cosh \left(\nu \left(\frac{T}{2} - t \right) \right), \\
G_{+-}(t) &= iA \sinh \left(\nu \left(\frac{T}{2} - t \right) \right).
\end{aligned} \tag{5.300}$$

where A, B are integration constants. From the boundary conditions (5.259), we can derive

$$G_{-+} \left(\frac{T}{2} - t \right) = -G_{-+} \left(t - \frac{T}{2} \right) = G_{+-} \left(y + \frac{T}{2} \right). \tag{5.301}$$

This means $A = B$. In summary, we have

$$\begin{aligned}
G_{++}(t) &= G_{--}(t) = A \cosh \left(\nu \left(\frac{T}{2} - t \right) \right), \\
G_{+-}(t) &= -G_{-+}(t) = iA \sinh \left(\nu \left(\frac{T}{2} - t \right) \right).
\end{aligned} \tag{5.302}$$

Next we are try to glue the two solutions, early time and late time solutions. This task is done by comparing the $t \rightarrow \infty$ of early time solutions (5.295) and (5.297),

$$\begin{aligned}
G_{aa}(t) &= \frac{1}{2} \left(1 + \frac{1}{q} g_{aa}(t) + O \left(\frac{1}{q^2} \right) \right) \\
&\approx \frac{1}{2} + \frac{1}{q} \log \frac{2c_{aa}}{\mathcal{J}} - \frac{d_{aa}}{q} - \frac{c_{aa}}{q} t, \\
G_{ab}(t) &= \frac{ib}{2} \epsilon_{ab} \left(1 + \frac{1}{q} g_{ab}(t) + O \left(\frac{1}{q^2} \right) \right) \\
&= \frac{ib}{2} \epsilon_{ab} + \frac{ib}{q} \epsilon_{ab} \log \frac{2c_{ab}}{\mathcal{J} b^{\frac{q-2}{2}}} - \frac{ib}{q} \epsilon_{ab} d_{ab} - \frac{ib}{q} \epsilon_{ab} c_{ab} t.
\end{aligned} \tag{5.303}$$

On the other hand, we consider the $t \rightarrow 0+$ of the late time solution (5.302),

$$\begin{aligned}
G_{aa}(t) &\approx A \cosh \left(\frac{\nu T}{2} \right) - A \nu t \sinh \left(\frac{\nu T}{2} \right), \\
G_{+-}(t) &= -G_{-+}(t) \approx iA \sinh \left(\frac{\nu T}{2} \right) - iA \nu t \cosh \left(\frac{\nu T}{2} \right).
\end{aligned} \tag{5.304}$$

For large T we have consistent solutions. From the leading part of constant term in (5.303) and (5.304), we observe

$$A \approx e^{-\frac{\nu T}{2}} \tag{5.305}$$

and from the linear t term,

$$\frac{c_{++}}{q} = \frac{c_{--}}{q} = \frac{c_{+-}}{q} = \frac{c_{-+}}{q} = \frac{\nu}{2}. \quad (5.306)$$

This is consistent with (5.298) when $b = 1$. From the $1/q$ order terms of the constant part, we see

$$\begin{aligned} \frac{1}{2} + \frac{1}{q} \log \frac{2c_{++}}{\mathcal{J}} - \frac{d_{++}}{q} &= A \cosh \left(\frac{\nu T}{2} \right), \\ \frac{1}{2} + \frac{1}{q} \log \frac{2c_{+-}}{\mathcal{J}} - \frac{d_{+-}}{q} &= A \sinh \left(\frac{\nu T}{2} \right). \end{aligned} \quad (5.307)$$

This means

$$d_{+-} = d_{++} + \sigma, \quad \sigma := qe^{-\nu T}. \quad (5.308)$$

This shows that $d_{+-} = d_{-+}$ are not in pure imaginary.

Next we evaluate the on-shell action. The action (5.263) has two parts. In this large time solution, we approximate the effective action such that we consider only contribution from late-time solutions (5.302). Firstly we evaluate the determinant term as follows.

$$\begin{aligned} -\frac{1}{2} \text{Tr} [\log (G_0^{-1} - \Sigma)] &= -\frac{1}{2} \text{Tr} [\log G_0^{-1}] - \frac{1}{2} \text{Tr} [\log (1 - G_0 * \Sigma)] \\ &= -\frac{1}{2} \text{Tr} [\log G_0^{-1}] + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \text{Tr} [(G_0 * \Sigma)^k]. \end{aligned} \quad (5.309)$$

Here every multiplication in $(G_0 * \Sigma)^k$ is taken for the $*$ product and matrix product on $+/-$ space. This trace is easy to work in the Fourier space. We can easily find that

$$(G_0^{-1})_{ab}(\omega_n) = \int_0^T dt e^{i\omega_n t} \partial_t \delta(t) \delta_{ab} = -i\omega_n \delta_{ab}, \Rightarrow (G_0)_{ab}(\omega_n) = \frac{1}{-i\omega_n} \delta_{ab}. \quad (5.310)$$

Also in the approximation (5.294)

$$\Sigma_{ab}(\omega_n) = -i\nu \epsilon_{ab}. \quad (5.311)$$

In the case that the time translation is kept, the Fourier transformation of $*$ product is just the product and thus

$$(G_0 * \Sigma)_{ab}(\omega_n) = \frac{\nu}{\omega_n} \epsilon_{ab}. \quad (5.312)$$

So for odd k the diagonal components are just zero and the traces are also zero. For

even $k = 2l$,

$$\text{Tr}[(G_0 * \Sigma)^{2l}] = 2 \sum_{n \in \mathbb{Z}} \left(-\frac{\nu^2}{\omega_n^2} \right)^l = 2(-1)^l \left(\frac{\nu T}{2\pi} \right)^{2l} \sum_{n \in \mathbb{Z}} \frac{1}{(n + \frac{1}{2})^{2l}}. \quad (5.313)$$

So in total

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \text{Tr}[(G_0 * \Sigma)^k] &= \frac{1}{2} \sum_{l=1}^{\infty} \frac{1}{2l} \text{Tr}[(G_0 * \Sigma)^{2l}] \\ &= \frac{1}{2} \sum_{l=1}^{\infty} \frac{1}{l} (-1)^l \left(\frac{\nu T}{2\pi} \right)^{2l} \sum_{n \in \mathbb{Z}} \frac{1}{(n + \frac{1}{2})^{2l}} \\ &= -\frac{1}{2} \sum_{n \in \mathbb{Z}} \log \left(1 + \left(\frac{\nu T}{2\pi} \frac{1}{n + \frac{1}{2}} \right)^2 \right) \\ &= -\log \cosh \left(\frac{\nu T}{2} \right). \end{aligned} \quad (5.314)$$

Here we use the relation $\sum_{l=1}^{\infty} \frac{(-1)^l}{l} z^{2l} = -\log(1 + z^2)$ and $\sum_{n \in \mathbb{Z}} \log \left(1 + \frac{x^2}{(n + \frac{1}{2})^2} \right) = 2 \log \cosh \pi x$ ¹¹ Also, we renormalize the free term

$$\frac{1}{2} \text{Tr}[\log G_0^{-1}] \rightarrow \log 2 \quad (5.316)$$

so that $\text{SFF}(0) = 2^{\frac{N}{2}}$. So in total,

$$-\frac{1}{2} \text{Tr}[\log G_0^{-1} - \Sigma] = -\log 2 - \log \cosh \frac{\nu T}{2} = -\frac{\nu T}{2} + O(e^{-\nu T}). \quad (5.317)$$

Next we evaluate the polynomial term,

$$(\text{Polynomial term}) = \frac{1}{2} \int_0^T dt dt' \left[\Sigma_{ab} G_{ab} - \frac{\mathcal{J}^2}{2q^2} s_{ab} (2G_{ab})^q \right]. \quad (5.318)$$

From the Kadanoff-Baym equation, $\Sigma_{ab} = \frac{\mathcal{J}^2}{q} s_{ab} (2G_{ab})^{q-1}$, we see

$$(\text{Polynomial term}) = \frac{T}{2} \left(1 - \frac{1}{q} \right) \int_{-T}^T dt (\Sigma_{ab}(t)) G_{ab}(t). \quad (5.319)$$

¹¹This is derived as follows: Let us define $f(x) := \sum_{n \in \mathbb{Z}} \log \left(1 + \frac{x^2}{(n + \frac{1}{2})^2} \right)$. It is easy to see $f(0) = 0$. Then,

$$f'(x) = \sum_{n \in \mathbb{Z}} \frac{2x}{x^2 + (n + \frac{1}{2})^2} = 2\pi \tanh \pi x. \quad (5.315)$$

We then integrate this relation and find equality.

By using the ansatz (5.294) and solutions (5.302) we see

$$\int_{-T}^T dt \Sigma_{ab}(t) G_{ab}(t) = 2\nu A \int_{-T}^T \delta(t) \sinh\left(\nu\left(\frac{T}{2} - t\right)\right) \approx \nu T(1 - O(e^{-\nu T})), \quad (5.320)$$

In total we see

$$I_{\text{eff}} = O(\nu T e^{-\nu T}). \quad (5.321)$$

Thus the effective action is almost zero in large T . We should notice that for given physical quantities q, T , we do not fix all of the parameters. Indeed, d_{++} is not fixed. Then, we need to integrate over d_{++} . Since the on-shell action is almost zero which is independent of choice of d_{++} . Also d_{++} corresponds to the time shift of the solution, we have $0 < d_{++} < T$,

$$\int_0^T d(d_{++}) e^{-I_{\text{eff}}} \propto T. \quad (5.322)$$

This shows that the spectral form factor have ramp.

■ Generic chaotic Hamiltonian

The above SYK discussion will be general in chaotic systems. Firstly, notice that the spectral form factor is given by the following Schwinger-Keldysh path integral,

$$\text{SFF}(T) = \overline{|\text{Tr}[e^{-iHT}]|^2} = \begin{array}{c} \bullet \longrightarrow \bullet \\ \bullet \longleftarrow \bullet \end{array} \quad (5.323)$$

Here we express the each unitary by the black line and the red points mean the identification. For the chaotic systems, we may think that the time evolution forget about the initial conditions which means that the Schwinger-Keldysh path integral is approximated by the connected path integral

$$\begin{aligned} \text{SFF}(T) &= \begin{array}{c} \bullet \longrightarrow \bullet \\ \bullet \longleftarrow \bullet \end{array} \\ &\approx \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \\ &= \text{Tr}[e^{-\beta H} e^{-iHT} e^{-\beta_{\text{aux}} H} e^{iHT}]. \end{aligned} \quad (5.324)$$

This connecting closed time contour breaks the doubled time translation symmetry and gives the NG boson which is corresponding to the relative time shift [36, 170]. Since $\text{Tr}[e^{-\beta H} e^{-iHT} e^{-\beta_{\text{aux}} H} e^{iHT}] = \text{Tr}[e^{-2\beta_{\text{aux}} H}]$ with some parameter β_{aux} . Then We have the zero Lorentzian action this NG mode gives the zero mode integral. The range of the integral will be the original periodicity T and this gives the ramp. However, we do not have nice explanation for plateau from this picture and this is interesting future problems.

5.8.3 Two point function

By using the ETH ansatz and RMT, we can find that the thermal two point function obeys similar behavior to the spectral form factors [171, 172]. Here we describe naive discussion. Suppose we have the operator $\mathcal{O}$ following to the ETH ansatz (5.182). For simplicity we assume that $\mathcal{O}(E) = 0$. Then from the property of pseudo random matrix R_{nm} we see

$$\begin{aligned}\mathbb{E}_R[\mathcal{O}_{nm}] &= 0, \\ \mathbb{E}_R[|\mathcal{O}_{nm}|^2] &= e^{-S_{\text{mc}}(\bar{E})} |F_{\mathcal{O}}(\bar{E}, \omega)|^2.\end{aligned}\tag{5.325}$$

Then, the variance of the operator $\mathcal{O}$ in microcanonical ensemble is given by

$$\begin{aligned}\mathbb{E}_R[(\Delta\mathcal{O})^2] &= \frac{1}{D_{E,\Delta E}} \sum_{|E_n\rangle, |E_m\rangle \in \mathcal{H}_{E,\Delta E}} \mathbb{E}_R[|\langle E_n | \mathcal{O} | E_m \rangle|^2] \\ &= \frac{1}{D_{E,\Delta E}} \sum_{|E_n\rangle, |E_m\rangle \in \mathcal{H}_{E,\Delta E}} e^{-S_{\text{mc}}(\bar{E})} |F_{\mathcal{O}}(\bar{E}, \omega)|^2 \\ &\approx e^{-S_{\text{mc}}(E)} |F_{\mathcal{O}}(E, 0)|^2\end{aligned}\tag{5.326}$$

where we approximate $\bar{E} \approx E$ and $\omega \approx \Delta E \ll E$ and we assume the support of $F_{\mathcal{O}}(E, \omega)$ lies around $\omega \approx 0$ ¹² Then the microcanonical two point function is given by

$$\begin{aligned}\mathbb{E}_R\left[\text{Tr}\left[\left(\rho_{\text{mc}}^{(E,\Delta E)}\right)^{\frac{1}{2}} \mathcal{O}(t) \left(\rho_{\text{mc}}^{(E,\Delta E)}\right)^{\frac{1}{2}} \mathcal{O}\right]\right] &= \frac{1}{D_{E,\Delta E}} \sum_{|E_n\rangle, |E_m\rangle \in \mathcal{H}_{E,\Delta E}} e^{i(E_n - E_m)t} e^{-S_{\text{mc}}(\bar{E})} |F_{\mathcal{O}}(\bar{E}, \omega)|^2 \\ &\approx (\Delta\mathcal{O})^2 \text{SFF}(t)\end{aligned}\tag{5.327}$$

where $\text{SFF}(t)$ is defined on the microcanonical window. We thus see that the two point functions are also have structure of slope, ramp, plateau like structure for chaotic Hamiltonian.

6 A Sufficient Condition for the Double Exponential Scaling

In this section, we present one possible strategy to derive the double exponential behavior (5.60), (5.63) from a set of assumptions that are expected to hold in highly chaotic systems, as discussed in [1]. Our main tools will be concentration inequalities [173, 174] together with a notion of quantum mixing, which we will introduce later.

¹²Typically, $F_{\mathcal{O}}(E, \omega)$ exponentially decays in ω [149].

6.1 Concentration Inequality and Sub-Gaussian Properties

Concentration inequalities are probabilistic bounds of the form

$$\text{Prob}[|Z - \mathbb{E}[Z]| > \epsilon] < f(\epsilon), \quad (6.1)$$

where $f(\epsilon)$ is some small function of ϵ . Here Z denotes a random variable and $\mathbb{E}[Z]$ its expectation value. Classical examples of such inequalities include Chebyshev's inequality and Markov's inequality

$$\begin{aligned} \text{Prob}[|Z - \mathbb{E}[Z]| > \epsilon] &< \frac{\mathbb{E}[Z - \mathbb{E}[Z]]^2}{\epsilon^2}, \\ \text{Prob}[|Z - \mathbb{E}[Z]| > \epsilon] &< \frac{\mathbb{V}[Z]}{\epsilon^2} \end{aligned} \quad (6.2)$$

6.1.1 Sub-Gaussian Random Variables: A Brief Review

We begin by recalling a commonly used class of probability distributions known as sub-Gaussian distributions. These distributions are characterized by concentration bounds exhibiting Gaussian-type decay in the tails. More precisely, we adopt the following definition.

Definition

Definition 22. A random variable Z is called σ -sub-Gaussian if it satisfies

$$\text{Prob}[|Z - \mathbb{E}[Z]| > \epsilon] < 2e^{-\frac{\epsilon^2}{2\sigma^2}}. \quad (6.3)$$

A wide variety of familiar random variables, including Gaussian, Bernoulli, and uniformly bounded variables, fall into the class of sub-Gaussian random variables with a finite and typically moderate value of σ . There exist several equivalent ways to characterize the condition (6.3). For simplicity of presentation, we assume throughout that $\mathbb{E}[Z] = 0$. This assumption is made solely for convenience and can be removed without affecting the generality of the discussion. One useful collection of equivalent formulations is summarized below.

Theorem

Theorem 14. Let Z be a centered σ -sub-Gaussian random variable. Then the condition (6.3) is equivalent to each of the following statements:

1. The tail probabilities of Z obey

$$\text{Prob}[|Z| > \epsilon] \leq 2e^{-\frac{\epsilon^2}{C_1^2}} \quad \text{for all } \epsilon > 0. \quad (6.4)$$

2. The moments of Z satisfy the bound

$$\|Z\|_{L^q} := (\mathbb{E}[|Z|^q])^{\frac{1}{q}} \leq C_2 \sqrt{q} \quad \text{for any } q \geq 1. \quad (6.5)$$

3. The exponential moment of Z^2 is controlled as

$$\mathbb{E}[\exp\{\eta^2 Z^2\}] \leq \exp\{C_3^2 \eta^2\} \quad \text{for all } \eta \text{ with } |\eta| < \frac{1}{C_3}. \quad (6.6)$$

4. The exponential moment of Z^2 is finite at a specific point, namely

$$\mathbb{E}\left[\exp\left\{\frac{Z^2}{C_4^2}\right\}\right] \leq 2. \quad (6.7)$$

5. For every $\eta \in \mathbb{R}$, the moment generating function of Z satisfies

$$\mathbb{E}[e^{\eta Z}] \leq \exp\left\{\frac{\eta^2 \sigma^2}{2}\right\}. \quad (6.8)$$

Here the constants C_1, C_2, C_3 , and C_4 depend on σ^2 only up to universal numerical factors.

The equivalence of these conditions can be established through the following sequence of implications: (6.8) $\Rightarrow$ (6.4) $\Rightarrow$ (6.5) $\Rightarrow$ (6.6) $\Rightarrow$ (6.7) $\Rightarrow$ (6.8). We briefly outline the main steps.

- (6.8) $\Rightarrow$ (6.4): this follows from a standard Chernoff-type bound. In particular, one may choose $C_1 = \sqrt{2}\sigma$.
- (6.4) $\Rightarrow$ (6.5): a direct calculation yields

$$\begin{aligned} \mathbb{E}[|Z|^q] &= \int_0^\infty d\epsilon \text{Prob}[|Z|^q > \epsilon] \\ &= \int_0^\infty dx q x^{q-1} \text{Prob}[|Z| > x] \\ &\leq 2q \int_0^\infty dx x^{q-1} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \quad \because (6.4) \\ &= 2^{\frac{q}{2}} q \sigma^q \Gamma\left(\frac{q}{2}\right). \end{aligned} \quad (6.9)$$

Consequently,

$$\|Z\|_{L^q} \leq \left(q\Gamma\left(\frac{q}{2}\right)\right)^{\frac{1}{q}} \sqrt{2}\sigma \leq 3\sigma\sqrt{q}. \quad (6.10)$$

Hence, taking $C_2 = 3\sigma$ establishes (6.5).

- (6.5) $\Rightarrow$ (6.6): we expand the exponential as

$$\mathbb{E}[\exp\{\eta^2 Z^2\}] = \mathbb{E}\left[1 + \sum_{q=1}^{\infty} \frac{(\eta^2 Z^2)^q}{q!}\right] = 1 + \sum_{q=1}^{\infty} \frac{\eta^{2q} \mathbb{E}[Z^{2q}]}{q!}. \quad (6.11)$$

Applying (6.5), we find

$$\mathbb{E}[\exp\{\eta^2 Z^2\}] \leq 1 + \sum_{q=1}^{\infty} \eta^{2q} 2^{q+1} \sigma^{2q} = \frac{1 + 2\eta^2 \sigma^2}{1 - 2\eta^2 \sigma^2}, \quad \text{for } |\eta| < \frac{1}{\sqrt{2}\sigma}. \quad (6.12)$$

For any $a > 2$, there exists a positive number $x_*(a)$ such that for $0 < x < x_*(a)$,

$$\frac{1+x}{1-x} < e^{ax}, \quad ae^{ax_*(a)} = \frac{1}{(1-x_*(a))^2}. \quad (6.13)$$

It then follows that

$$\mathbb{E}[\exp\{\eta^2 Z^2\}] \leq \exp\{2a\eta^2 \sigma^2\}, \quad \text{for } |\eta| < \frac{x_*(a)}{2\sigma}, \quad (6.14)$$

so we may choose $C_3 = \sqrt{2a}\sigma$.

- (6.6) $\Rightarrow$ (6.7): by setting $\eta = \frac{1}{C_4}$ with $C_4 = \frac{C_3}{\log 2}$, the bound (6.7) immediately follows.
- (6.7) $\Rightarrow$ (6.4):

$$\text{Prob}[|Z| > \epsilon] = \text{Prob}\left[e^{\frac{Z^2}{C_4^2}} > e^{\frac{\epsilon^2}{C_4^2}}\right] \leq e^{-\frac{\epsilon^2}{C_4^2}} \mathbb{E}\left[e^{\frac{Z^2}{C_4^2}}\right] \leq 2e^{-\frac{\epsilon^2}{C_4^2}}, \quad (6.15)$$

which allows us to take $C_1 = C_4$.

- (6.6) $\Rightarrow$ (6.8): using the elementary inequality $e^x \leq x + e^x$ for all $x \in \mathbb{R}$, we obtain

$$\mathbb{E}[e^{\eta Z}] \leq \mathbb{E}[\eta Z + e^{\eta^2 Z^2}] < e^{C_3^2 \eta^2}, \quad \text{for } |\eta| < \frac{1}{C_3}. \quad (6.16)$$

Furthermore, since $2\eta x \leq \eta^2 + x^2$ for any η, x , it follows that

$$\mathbb{E}[e^{\eta Z}] \leq e^{\frac{\eta^2}{2}} \mathbb{E}\left[e^{\frac{Z^2}{2}}\right] < e^{\frac{\eta^2}{2}} e^{\frac{1}{C_3^2}} < e^{C_3^2 \eta^2}. \quad (6.17)$$

6.1.2 Generalization of Sub-Gaussian Variables

Later we will want to allow mild departures from the strictly sub-Gaussian large-deviation profile. A convenient toy model is a bound of the schematic form “Gaussian + quartic,” namely

$$\mathbb{E}[e^{\eta Z}] \leq \exp \left\{ \frac{\eta^2 \sigma^2}{2} - \sum_{k=2} \frac{t'_k}{4^k} \eta^{2k} \right\} \quad (6.18)$$

for η in a certain range, or alternatively an explicit upper bound directly on the tail probability:

$$\text{Prob}[|Z| > \epsilon] \leq 2 \exp \left\{ -\frac{\epsilon^2}{2\sigma^2} - \sum_{k=1}^M \frac{g_{2k}}{2^k} \epsilon^{2k} \right\}. \quad (6.19)$$

When the coupling $g_{2k} > 0$, this quadratic (quartic in ϵ) correction remains below the purely Gaussian estimate. In particular, Z is still sub-Gaussian in the sense that

$$\text{Prob}[|Z| > \epsilon] \leq 2 \exp \left\{ -\frac{\epsilon^2}{2\sigma^2} \right\}. \quad (6.20)$$

Moreover, to ensure that the relevant probability integrals are well-defined, we indeed require $g_{2M} > 0$, and in what follows we restrict attention to this case. We consider following theorems which we do not have complete proof:

Conjecture 1: Let Z be a random variable with zero mean $\mathbb{E}[Z] = 0$. Assume that its moments satisfy the following upper bound for any $1 \leq q \leq L < +\infty$,

$$\mathbb{E}[Z^{2q}] \leq 2^{q+1} \sigma^{2q} q! \sum_{\{m_k\}: m \leq m_{\max}} \frac{\Gamma(q+m)}{\Gamma(q)} \prod_{k=2}^M \frac{(-g_{2k} \sigma^{2k})^{m_k}}{m_k!} \quad (6.21)$$

for some finite (sufficiently small) $m_{\max}$ and $g > 0$. Here we denote $m = \sum_{k=2}^M k m_k$. For any $q \geq L+1$ we only assume the weaker bound

$$\mathbb{E}[Z^{2q}] \leq (\sigma')^{2q}. \quad (6.22)$$

In addition, suppose that L satisfies $L+1 > \frac{(\sigma')^2}{8\sigma^2}$. Then we can deduce the following bounds for the generating function and the tail probabilities:

1. For some $|\eta| < \frac{1}{2\sigma}$, we have

$$\begin{aligned} \mathbb{E}[e^{\eta^2 Z^2}] &\leq 2e^{4\eta^2 \sigma^2 - \sum_{k=2}^M 2^k t_k g_{2k} \eta^{2k} \sigma^{4k}} + F^u(\eta), \\ F^u(\eta) &= \exp \left(-(L+1) \log \left(\frac{L+1}{\eta^2 \sigma'^2 e} \right) \right). \end{aligned} \quad (6.23)$$

2. For any $\eta > \frac{1}{2\sigma}$, it holds that

$$\mathbb{E}[e^{\eta Z}] \leq 2e^{2\eta^2\sigma^2} \left(2 \exp \left\{ - \sum_{k=2}^M \frac{t_k}{4^k} g_{2k} \sigma^{2k} \right\} + F^u \left(\sqrt{\frac{1}{8\sigma^2}} \right) \right). \quad (6.24)$$

3. For $\epsilon > 2\sigma$, we obtain

$$\text{Prob}[|Z| > \epsilon] \leq 2 \exp \left\{ - \frac{\epsilon^2}{8\sigma^2} \right\} \left(4 \exp \left\{ - \sum_{k=2}^M \frac{t_k}{4^k} g_{2k} \sigma^{2k} \right\} + F^u \left(\sqrt{\frac{1}{8\sigma^2}} \right) \right). \quad (6.25)$$

■ On the Form of the bound on moment (6.21)

Let us briefly explain how the moment inequality (6.21) naturally arises. Assume we are given the tail estimate (6.19). Then the q -th moment can be controlled as

$$\begin{aligned} \mathbb{E}[Z^q] &= \int_0^\infty d\epsilon \text{Prob}[|Z|^q > \epsilon] \\ &= q \int_0^\infty dx x^{q-1} \text{Prob}[|Z| > x] \\ &= 2q \int_0^\infty dx x^{q-1} \text{Prob}[|Z| > x] \\ &\leq 2q \int_0^\infty dx x^{q-1} \exp \left(- \frac{x^2}{2\sigma^2} - \sum_{k=2}^M \frac{g_{2k}}{2^k} x^{2k} \right). \end{aligned} \quad (6.26)$$

We now view the integral perturbatively in g . Since the resulting series is asymptotic and has vanishing radius of convergence, we truncate the expansion at a finite order $m = m_{\max}$. From this perturbative perspective, the upper bound is approximated by

$$\begin{aligned} \mathbb{E}[Z^{2q}] &\leq 2 \int_0^\infty dx x^{2q-1} e^{-\frac{x^2}{2\sigma^2}} \sum_{\{m_k\}: m \leq m_{\max}} x^{2m} \prod_{k=2}^M \left(\frac{1}{m_k!} \left(-\frac{g_{2k}}{2^k} \right)^{m_k} \right) \\ &= 2^{q+1} q! \sigma^{2q} \sum_{m \leq m_{\max}} \frac{\Gamma(m+q)}{\Gamma(q)} \prod_{k=2}^M \left(\frac{(-g_{2k} \sigma^{2k})^{m_k}}{m_k!} \right). \end{aligned} \quad (6.27)$$

Strictly speaking, one should not interchange the order of integration and summation, because the corresponding series is merely asymptotic. Nonetheless, once we truncate at a finite $m_{\max}$, this procedure provides a natural rationale for adopting a moment bound with finite $m_{\max}$.

In principle, one can bypass the perturbative expansion by evaluating the integral

exactly and then comparing the resulting asymptotic series with the exact expression via Borel resummation. We defer such a more refined treatment to future work. Note also that the bounds on the generating function and on tail probabilities can always be relaxed to those of a purely sub-Gaussian variable; however, keeping track of the quartic correction yields sharper estimates. A further remark concerns the modification of the bound for $q < L$: in later applications we will encounter situations where the moment estimate deteriorates for large q when σ is small. Past a critical value $q = L$, our control becomes weaker and we retain only a rough estimate with a larger parameter σ' . Even in such circumstances, provided that L is sufficiently large, we still expect the tail behavior to remain essentially sub-Gaussian. This motivates a careful treatment of finite- q corrections.

We now proceed to the proof of the theorem. Some intermediate steps are not completely rigorous mathematically; however, we support those steps with numerical checks.

■ **Upper bound for $\mathbb{E}[e^{\eta^2 Z^2}]$**

We first establish an upper bound for $\mathbb{E}[e^{\eta^2 Z^2}]$ from the assumed moment condition, again adopting a perturbative viewpoint:

$$\begin{aligned} \mathbb{E}[e^{\eta^2 Z^2}] &= 1 + \sum_{q=1}^{\infty} \frac{\eta^{2q}}{q!} \mathbb{E}[Z^{2q}] \\ &\leq 1 + \sum_{q=1}^L \eta^{2q} 2^{q+1} \sigma^{2q} a_q + \sum_{q=L+1}^{\infty} \frac{(\eta^2 (\sigma')^2)^q}{q!}, \end{aligned} \quad (6.28)$$

where

$$a_q = \sum_{m \leq m_{\max}} \frac{\Gamma(m+q)}{\Gamma(q)} \prod_{k=2}^M \left(\frac{(-g_{2k} \sigma^{2k})^{m_k}}{m_k!} \right). \quad (6.29)$$

The first two contributions can be bounded by an infinite series:

$$1 + \sum_{q=1}^{\infty} \eta^{2q} 2^{q+1} \sigma^{2q} a_q \leq 2 \sum_{q=0}^{\infty} \eta^{2q} 2^{q+1} \sigma^{2q} a_q := F_1(\eta). \quad (6.30)$$

Evaluation of $F_1(\eta)$

We can rewrite $F_1(\eta)$ as

$$\begin{aligned} F_1(\eta) &= 2 \sum_{q=0}^{\infty} s^q a_q, \quad s = 2\eta^2 \sigma^2 \\ &= 2 \sum_{\{m_k\}: m \leq m_{\max}} \frac{\Gamma(1+m)}{(1-s)^{m+1}} \prod_{k=2}^M \left(\frac{(-g_{2k} \sigma^{2k})^{m_k}}{m_k!} \right). \end{aligned} \quad (6.31)$$

If we keep only finitely many terms, then it is legitimate to exchange the order of summation.

Suppose we can show that, for sufficiently small $m_{\max}$, for s in a small interval $0 < s < s_* \sim \frac{1}{2}$, and for small $a_k > 0$ and some positive constants t_k

$$\sum_{\{m_k\}: m \leq m_{\max}} \frac{\Gamma(1+m)}{(1-s)^{m+1}} s \prod_{k=2}^M \left(\frac{(-a_k)^{m_k}}{m_k!} \right) \leq \exp \left\{ 2s - \sum_{k=2}^M t_k a_k s^k \right\} \quad (6.32)$$

where $a_k = g_{2k} \sigma^{2k}$. Then for $|\eta| < \frac{1}{2\sigma}$, with sufficiently small $m_{\max}$ and a_k , we obtain

$$\mathbb{E} \left[e^{\eta^2 Z^2} \right] \leq 2 \exp \left\{ 4\eta^2 \sigma^2 - \sum_{k=2}^M 2^k t_k g_{2k} \eta^{2k} \sigma^{4k} \right\}. \quad (6.33)$$

At present we do not have a fully rigorous derivation of (6.32), but we do have numerical evidence supporting it. Details are deferred to Appendix D. We choose $t_k = 6/2^k$. We see that we have $M \leq m_{\max} \leq 1/a_k$ as usual perturbation theories. Even though the numerical test is difficult for very large M but we expect the inequality is valid for $M = e^{O(S_{\text{mc}})}$. This point is highly non-trivial and still open questions.

Evaluation of the Third Term

Let us denote the third term in (6.28) by $F_2(\eta)$:

$$\begin{aligned} F_2(\eta) &= \sum_{q=L+1}^{\infty} \frac{1}{q!} (\eta^2 (\sigma')^2)^q \\ &= e^{\eta^2 (\sigma')^2} \frac{\gamma(L+1, \eta^2 (\sigma')^2)}{\Gamma(L+1)}, \end{aligned} \quad (6.34)$$

where $\gamma(a, x)$ is the lower incomplete gamma function,

$$\gamma(a, x) = \int_0^x d\epsilon \, \epsilon^{a-1} e^{-\epsilon}. \quad (6.35)$$

To obtain a simpler bound, we use the following identity [175],

$$e^x \frac{\gamma(a, x)}{\Gamma(a)} = \frac{x^a}{\Gamma(a)} \sum_{m=0}^{\infty} \frac{x^m}{(a)_{m+1}} \quad (6.36)$$

for $a > x$. Together with the estimates

$$\frac{1}{\Gamma(a)} \leq \left(\frac{a}{e} \right)^{-(a-1)} \quad (6.37)$$

and

$$(a)_{m+1} = a(a+1) \cdots (a+m) \geq a^{m+1}, \quad (6.38)$$

we arrive at the simpler bound

$$e^x \frac{\gamma(a, x)}{\Gamma(a)} = \frac{x^a}{\Gamma(a)} \sum_{m=0}^{\infty} \frac{x^m}{(a)_{m+1}} \leq \frac{1}{a-x} e^{a \log x - (a-1) \log \frac{a}{e}} < e^{-a \log \frac{a}{xe}} \quad (6.39)$$

for $x < a$. Consequently, we obtain the bound

$$F_2(\eta) \leq e^{-(L+1) \log \left(\frac{L+1}{\eta^2 \sigma'^2 e} \right)} =: F^u(\eta) \quad (6.40)$$

valid whenever $\eta^2 \sigma'^2 < L + 1$.

■ Inequality for the Generating Function

Next we consider the generating function $\mathbb{E}[e^{\eta Z}]$. Assume $|\eta| > \frac{1}{2\sigma}$. Using the inequality

$$\eta z \leq \frac{1}{2} \left(\frac{1}{4\sigma^2} z^2 + 4\sigma^2 \eta^2 \right), \quad (6.41)$$

valid for all $\eta, z \in \mathbb{R}$, we obtain

$$\begin{aligned} \mathbb{E}[e^{\eta Z}] &\leq e^{2\sigma^2 \eta^2} \mathbb{E}\left[e^{\frac{1}{8\sigma^2} Z^2}\right] \\ &\leq e^{2\sigma^2 \eta^2} \left(2 \exp \left\{ \frac{1}{2} - \sum_{k=2}^M \frac{t_k}{4^k} g_{2k} \sigma^{2k} \right\} + F^u \left(\sqrt{\frac{1}{8\sigma^2}} \right) \right) \\ &\leq L_1(\eta) + L_2(\eta), \end{aligned} \quad (6.42)$$

where we defined

$$\begin{aligned} L_1(\eta) &= 4 \exp \left\{ (2\sigma^2 \eta^2 + \sum_{k=2}^M \frac{t_k}{4^k} g_{2k} \sigma^{2k}) \right\}, \\ L_2(\eta) &= \exp \{ (2\sigma^2 \eta^2) \} F^u \left(\sqrt{\frac{1}{8\sigma^2}} \right). \end{aligned} \quad (6.43)$$

In the second line we used (6.33) together with $\eta^2 \geq \frac{1}{4\sigma^4}$. The first term can also be written as

$$e^{2\sigma^2 \eta^2} 2 \exp \left\{ \frac{1}{2} - \sum_{k=2}^M \frac{t_k}{4^k} g_{2k} \sigma^{2k} \right\} \leq 2 \exp \left\{ (4\sigma^2 \eta^2 - \sum_{k=2}^M \frac{t_k}{4^k} g_{2k} \sigma^{2k}) \right\}. \quad (6.44)$$

On the other hand, the second term becomes

$$e^{2\eta^2\sigma^2} F^u \left(\sqrt{\frac{1}{8\sigma^2}} \right) = \exp(2\eta^2\sigma^2) \exp \left[-(L+1) \log \frac{8(L+1)\sigma^2}{e\sigma'^2} \right]. \quad (6.45)$$

■ Inequality for the Probability

Finally, we derive a tail estimate. By Markov's inequality, for any $\eta > 0$ we have

$$\begin{aligned} \text{Prob}[|Z| > \epsilon] &= 2\text{Prob}[Z > \epsilon] = 2\text{Prob}[e^{\eta Z} > e^{\eta\epsilon}] \\ &\leq 2e^{-\eta\epsilon} \mathbb{E}[e^{\eta Z}]. \end{aligned} \quad (6.46)$$

Using (6.42) for $\eta > \frac{1}{2\sigma}$, we obtain

$$\begin{aligned} \text{Prob}[|Z| > \epsilon] &= 2 \inf_{\eta > \frac{1}{2\sigma}} e^{-\eta\epsilon} \mathbb{E}[e^{\eta Z}] \\ &= 2 \inf_{\eta > \frac{1}{2\sigma}} e^{-\eta\epsilon} (L_1(\eta) + L_2(\eta)) \\ &\leq 2 \inf_{\eta > \frac{1}{2\sigma}} e^{-\eta\epsilon} L_1(\eta) + 2 \inf_{\eta > \frac{1}{2\sigma}} e^{-\eta\epsilon} L_2(\eta) \\ &\leq 8 \exp \left(- \sup_{\eta > \frac{1}{2\sigma}} (\epsilon\eta - 2\eta^2\sigma^2) - \sum_{k=2}^M \frac{t_k}{4^k} g_{2k}\sigma^{2k} \right) \\ &\quad + 2 \exp \left(- \sup_{\eta > \frac{1}{2\sigma}} (\epsilon\eta - 2\eta^2\sigma^2) \right) F^u \left(\sqrt{\frac{1}{8\sigma^2}} \right). \end{aligned} \quad (6.47)$$

For $\epsilon > 2\sigma$, the supremum is achieved at $\eta = \frac{\epsilon}{4\sigma^2} > \frac{1}{2\sigma}$, and therefore

$$\text{Prob}[|Z| > \epsilon] \leq 2 \exp \left(- \frac{\epsilon^2}{8\sigma^2} \right) \left(4 \exp \left(- \sum_{k=2}^M \frac{t_k}{4^k} g_{2k}\sigma^{2k} \right) + F^u \left(\sqrt{\frac{1}{8\sigma^2}} \right) \right) \quad (6.48)$$

In later applications, the second term $F^u \left(\sqrt{\frac{1}{8\sigma^2}} \right)$ will be doubly exponentially suppressed in the microcanonical entropy. Hence, for practical purposes, the tail behaves as an almost sub-Gaussian one.

6.2 Almost Sub-Gaussianity of $\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$ and $\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}]$

One natural way to establish the double exponential behavior (5.60), (5.63) is to show that the random variables $\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$ and $\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}]$ are (almost) sub-Gaussian. In this subsection we discuss some analytic arguments in that direction. A numerical investigation for specific models was performed in [140].

To proceed, we rewrite the p -th moments in terms of thermal Wightman correlation functions. We regard the matrix elements $\langle E_n | \mathcal{O} | E_n \rangle$ and $\langle E_n | \mathcal{O} | E_m \rangle$

as random variables with respect to the uniform measure over $\mathcal{E}_{E,\Delta E}$ and $\mathcal{E}'_{E,\Delta E}$, respectively.¹³

We begin with the diagonal part $\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$. For the diagonal elements, the expectation value of a single matrix element with respect to the uniform measure, denoted $\mathbb{E}_{\mathcal{E}_{E,\Delta E}}[\langle E_n | \mathcal{O} | E_n \rangle]$, simply reproduces the microcanonical expectation value,

$$\mathbb{E}_{\mathcal{E}_{E,\Delta E}}[\langle E_n | \mathcal{O} | E_n \rangle] = \frac{1}{D_{E,\Delta E}} \sum_{|E_n\rangle \in \mathcal{H}_{E,\Delta E}} \langle E_n | \mathcal{O} | E_n \rangle = \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E}. \quad (6.49)$$

To consider a centered random variable, we thus introduce the operator with its one-point function subtracted,

$$\hat{\mathcal{O}}_{\text{mc}} := \mathcal{O} - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E}, \quad (6.50)$$

and define the microcanonical Wightman function

$$G_{\text{mc},E}^{(q)}(t_1, t_2, \dots, t_q) = \text{Tr} \left[(\rho_{\text{mc}}^{(E,\Delta E)})^{\frac{1}{q}} \hat{\mathcal{O}}_{\text{mc}}(t_1) (\rho_{\text{mc}}^{(E,\Delta E)})^{\frac{1}{q}} \hat{\mathcal{O}}_{\text{mc}}(t_2) \cdots (\rho_{\text{mc}}^{(E,\Delta E)})^{\frac{1}{q}} \hat{\mathcal{O}}_{\text{mc}}(t_q) \right]. \quad (6.51)$$

Then, similarly to the expectation value, the q -th moment can be expressed as

$$\begin{aligned} & \mathbb{E}_{\mathcal{E}_{E,\Delta E}} \left[\left(\langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_n \rangle \right)^q \right] \\ &= \frac{1}{D_{E,\Delta E}} \sum_{|E_n\rangle \in \mathcal{H}_{E,\Delta E}} \langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_n \rangle^q \\ &= \frac{1}{D_{E,\Delta E}} \sum_{|E_{n_1}\rangle, \dots, |E_{n_q}\rangle \in \mathcal{H}_{E,\Delta E}} \langle E_{n_1} | \hat{\mathcal{O}}_{\text{mc}} | E_{n_2} \rangle \langle E_{n_2} | \hat{\mathcal{O}}_{\text{mc}} | E_{n_3} \rangle \cdots \langle E_{n_q} | \hat{\mathcal{O}}_{\text{mc}} | E_{n_1} \rangle \\ & \quad \times \delta_{n_1 n_2} \cdots \delta_{n_q n_1}. \end{aligned} \quad (6.52)$$

Assuming a non-degenerate spectrum, we can use the identity

$$\delta_{nm} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt e^{i(E_n - E_m)t}. \quad (6.53)$$

Applying this relation and rewriting the phase $i(E_n - E_m)t$ as a real-time evolution

¹³We emphasize that we do not introduce any physical randomness. We consider a single Hamiltonian with a fixed spectrum and fixed operators. The “random variable” terminology is purely an artificial construction associated with the uniform measure that we impose.

of operators, we arrive at

$$\begin{aligned}
& \mathbb{E}_{\mathcal{E}_{E,\Delta E}} \left[\left(\langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_n \rangle \right)^q \right] \\
&= \frac{1}{D_{E,\Delta E}} \sum_{|E_{n_1}\rangle, \dots, |E_{n_q}\rangle \in \mathcal{H}_{E,\Delta E}} \prod_{i=1}^q \left(\lim_{T_i \rightarrow \infty} \frac{1}{T_i} \int_0^{T_i} dt_i \right) \\
&\quad \times \langle E_{n_1} | \hat{\mathcal{O}}_{\text{mc}}(t_1) | E_{n_2} \rangle \langle E_{n_2} | \hat{\mathcal{O}}_{\text{mc}}(t_2) | E_{n_3} \rangle \cdots \langle E_{n_q} | \hat{\mathcal{O}}_{\text{mc}}(t_q) | E_{n_1} \rangle \quad (6.54) \\
&= \prod_{i=1}^q \left(\lim_{T_i \rightarrow \infty} \frac{1}{T_i} \int_0^{T_i} dt_i \right) \text{Tr} \left[\left(\rho_{\text{mc}}^{(E,\Delta E)} \right)^{\frac{1}{q}} \hat{\mathcal{O}}_{\text{mc}}(t_1) \cdots \left(\rho_{\text{mc}}^{(E,\Delta E)} \right)^{\frac{1}{q}} \hat{\mathcal{O}}_{\text{mc}}(t_q) \right] \\
&=: \overline{G_{\text{mc}}^{(q)}(t_1, t_2, \dots, t_q)}^T,
\end{aligned}$$

where the long-time average is taken with respect to all time variables. Thus, the q -th moment is given by the long-time average of the q -point microcanonical Wightman function $G_{\text{mc}}^{(q)}(t_1, t_2, \dots, t_q)$. For every q , the quantity $\mathbb{E}_{\mathcal{E}_{E,\Delta E}} \left[\left(\langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_n \rangle \right)^{2q} \right]$ is manifestly positive, and therefore $\overline{G_{\text{mc}}^{(2q)}(t_1, \dots, t_{2q})}^T$ is also positive ¹⁴.

In a chaotic system, it is natural to expect a clustering behavior of the $2p$ -th moments of the form

$$\overline{G_{\text{mc}}^{(2q)}(t_1, t_2, \dots, t_q)}^T \leq C \cdot q! \left(a \cdot \overline{G_{\text{mc}}^{(2)}(t_1, t_2)}^T \right)^q \left(1 + O \left(\left(\overline{G_{\text{mc},E}^{(2)}}^T \right)^n \right) \right), \quad n > 0 \quad (6.55)$$

for q up to some $q \leq L$, where C and a are numerical constants independent of $S_{\text{mc}}(E)$. The parameter L is expected to grow exponentially with the entropy, and the finite- L correction produces only negligible deviations from an ideal sub-Gaussian behavior. This clustering property is a key assumption in our analysis and is expected to hold in simple holographic theories, as we will see in the next section. Note that this assumption is not exclusive to chaotic systems: for example, free theories also satisfy it. The distinction between chaotic and integrable systems will appear in additional conditions discussed in the next subsection. Under this clustering assumption, we obtain an almost sub-Gaussian property for $\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$ with variance parameter $\sigma^2 = \overline{G_{\text{mc},E}^{(2)}}^T$.

In many situations, due to ensemble equivalence (to be discussed in the next subsection), it is technically easier to work with canonical rather than microcanonical correlators. In particular, the microcanonical condition (6.57) can often be replaced by an analogous condition in the canonical ensemble. In the canonical case, we

¹⁴Notice that the original Wightman functions $G_{\text{mc},E}^{(2q)}(t_1, \dots, t_{2q})$ without long time averages are complex valued. The OTOCs $G_{\text{mc},E}^{(2q)}(t, 0, \dots, t, 0)$ are real valued.

consider the canonical Wightman functions

$$G_{\text{can},\beta}^{(2q)}(t_1, t_2, \dots, t_{2q}) = \text{Tr} \left[\left(\rho_{\text{can}}^{(\beta)} \right)^{\frac{1}{2q}} \hat{\mathcal{O}}_{\text{can}}(t_1) \left(\rho_{\text{can}}^{(\beta)} \right)^{\frac{1}{2q}} \hat{\mathcal{O}}_{\text{can}}(t_2) \cdots \left(\rho_{\text{can}}^{(\beta)} \right)^{\frac{1}{2q}} \hat{\mathcal{O}}_{\text{can}}(t_{2q}) \right], \quad (6.56)$$

where $\hat{\mathcal{O}}_{\text{can}}$ is the operator with its canonical one-point function subtracted, $\hat{\mathcal{O}}_{\text{can}} = \mathcal{O} - \langle \mathcal{O} \rangle_{\text{can}}^\beta$. We then expect the following clustering property for the long-time average of the canonical Wightman functions:

Condition 1 (Clustering property for the Wightman functions). *For a chaotic system at sufficiently high energy, we expect the microcanonical Wightman functions to satisfy*

$$\overline{G_{\text{mc}}^{(2q)}(t_1, t_2, \dots, t_q)}^T \leq C \cdot q! \left(a \cdot \overline{G_{\text{mc}}^{(2)}(t_1, t_2)}^T \right)^q \left(1 + O \left(\left(\overline{G_{\text{mc},E}^{(2)}}^T \right)^n \right) \right), \quad n > 0. \quad (6.57)$$

Moreover, when ensemble equivalence holds, we also expect the canonical Wightman functions to satisfy

$$\overline{G_{\text{can},\beta}^{(2q)}}^T \leq C' \cdot q! \left(a' \cdot \overline{G_{\text{can},\beta}^{(2)}}^T \right)^q \left(1 + O \left(\left(\overline{G_{\text{can},\beta}^{(2)}}^T \right)^n \right) \right) \quad (6.58)$$

for q up to some $q < L$, where C' , a' and $n > 0$ are numerical constants and L is again expected to be exponentially large in the entropy.

Another property we expect for chaotic systems is a quantum mixing behavior of the thermal correlators. We say that the connected canonical Wightman function exhibits quantum mixing if

$$G_{\text{can},\beta}^{(2)}(t_1, t_2) \rightarrow 0 \text{ as } |t_1 - t_2| \rightarrow \infty \quad (6.59)$$

in the thermodynamic limit. For example, this is expected for light operators beyond the Hawking–Page transition [176–178]. We will later illustrate this point using a holographic example. For large but finite time difference $t_1 - t_2$, we anticipate that after some decay time t_{dec} ,

$$G_{\text{can},\beta}^{(2)}(t_1, t_2) \rightarrow \frac{1}{(D_{E,\Delta E})^\gamma} g(t_1, t_2) \text{ as } |t_1 - t_2| \gg t_{\text{dec}}, \quad (6.60)$$

with a function $g(t_1, t_2) = O(1)$ and some positive constant γ . Summarizing:

Condition 2. (*Quantum Mixing Condition*)

For large but finite microcanonical entropy $S_{\text{mc}}(E)$, we assume that the canonical Wightman function satisfies

$$\overline{G_{\text{can},\beta}^{(2)}}(t_1, t_2)^T = O(1) \|\mathcal{O}\|_\infty^2 e^{-\gamma S_{\text{mc}}(E)} = e^{-O(S_{\text{mc}}(E))}. \quad (6.61)$$

This behavior is in fact expected in holographic theories with black hole duals. Combining the assumptions (6.58), (6.61) with Conjecture or Theorem 14 when we can neglect subleading term, we obtain a double exponential scaling for the diagonal matrix elements:

Theorem

Theorem 15. By assuming (6.58) and (6.61) for the operator $\mathcal{O}$ and setting $\epsilon = e^{-\eta \gamma S_{\text{mc}}(E)}$ with $\eta < \frac{1}{2}$, we find that $\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$ exhibits the double-exponential behavior (5.60), namely

$$\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}] \leq 4e^{-\# \frac{\epsilon^2}{G_{\text{can},\beta}^{(2)}} + O\left(\left(\overline{G_{\text{can},\beta}^{(2)}}\right)^T\right)^n} = 4e^{-\# e^{O(S_{\text{mc}}(E))}}, \quad \# > 0. \quad (6.62)$$

The proof follows by substituting $\sigma^2 = \overline{G_{\text{can},\beta}^{(2)}}^T$ into Conjecture 1 or Theorem 14 when we dismiss the subleading part and choosing an appropriate value of g . The correction term $\left(\overline{G_{\text{can},\beta}^{(2)}}^T\right)^n$ will be analyzed in explicit examples in Section 8.2.2.

6.2.1 Ensemble Equivalence

We now discuss the ensemble equivalence used above. In many situations, microcanonical correlators are technically harder to analyze directly, so one often works instead with the canonical ensemble. In particular, the canonical real-time Wightman functions can be obtained from Euclidean time-ordered correlators. In holographic theories, the canonical ensemble is naturally associated with an AdS black hole geometry with imaginary time periodicity, and Euclidean time-ordered correlators can be computed straightforwardly via the standard AdS/CFT dictionary.

In the large- $D_{E,\Delta E}$ limit, we generally expect ensemble equivalence between microcanonical and canonical ensembles for suitable observables [63, 179–182]. More precisely, we expect to be able to bound the microcanonical moments $\mathbb{E}_{\mathcal{E}_{E,\Delta E}} \left[\langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_n \rangle^{2q} \right]$ (equivalently, the long-time averages of microcanonical Wightman functions) in

terms of canonical ones under suitable assumptions. Concretely, we assume

$$\mathbb{E}_{\mathcal{E}_{E,\Delta E}} \left[\langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_n \rangle^{2q} \right] = \overline{G_{\text{mc},E}^{(2q)}}^T \leq \frac{e^{\beta(E+\Delta E)} Z(\beta) \overline{G_{\text{can},\beta}^{(2q)}}^T}{D_{E,\Delta E}} \quad (6.63)$$

for some inverse temperature β . Furthermore, we expect that there exists a positive numerical constant and some temperature $\beta(E)$ such that

$$\frac{e^{\beta(E)(E+\Delta E)} Z(\beta(E))}{D_{E,\Delta E}} \leq O(S_{\text{mc}}(E)). \quad (6.64)$$

Here $\beta(E)$ is determined by the standard thermodynamic relation $\frac{\partial S_{\text{mc}}(E)}{\partial E} = \beta(E)$. This inequality can be understood heuristically from the thermodynamic limit:

$$Z(\beta) \sim e^{-\beta F(\beta)} \sim e^{-\beta(E - S_{\text{th}}(E))} \sim e^{-\beta E} D_{E,\Delta E}, \quad (6.65)$$

where $\sim$ indicates equality up to multiplicative factors of order $e^{O(S_{\text{mc}}(E))}$. Combining the clustering property (6.58) with the quantum mixing condition (6.61), we infer that the long-time averages of the canonical Wightman functions satisfy

$$\overline{G_{\text{can},\beta}^{(2q)}}^T < e^{-O(S_{\text{mc}}(E))} \quad (6.66)$$

for the corresponding inverse temperature. Then, using (6.63) and (6.64), we can bound the microcanonical correlators with the same scaling behavior.

Finally, we comment on when we can justify (6.63). For the variance ($p = 1$), one can show that (6.63) is always satisfied. In particular, if the canonical correlation function obeys the quantum mixing property, then the microcanonical variance also decays:

$$\overline{G_{\text{mc},E}^{(2)}}^T < O(S_{\text{mc}}(E)) \overline{G_{\text{can},\beta}^{(2)}}^T = e^{-O(S_{\text{mc}}(E))}. \quad (6.67)$$

For higher moments $p > 1$, the situation is more subtle. We do not provide a general proof here, but instead highlight a couple of relevant examples for later models. One example is when the uniform distribution on $\mathcal{E}_{\text{eq}}$ is symmetric about the average $\mathbb{E}[\langle E_n | \mathcal{O} | E_n \rangle] = \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E}$; then all odd moments vanish and (6.63) follows. Another case is when the long-time averages of the microcanonical Wightman correlators are themselves exponentially suppressed. Further details of these inequalities (6.63) and (6.64) are given in [1].

6.2.2 Off-Diagonal Elements

We now turn to the off-diagonal matrix elements. The expectation value of $|\langle E_n | \mathcal{O} | E_m \rangle|^2$ with respect to the uniform measure on $\mathcal{E}'_{E,\Delta E}$ is

$$\mathbb{E}_{\mathcal{E}'_{E,\Delta E}} [|\langle E_n | \mathcal{O} | E_m \rangle|^2] = \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \sum_{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E,\Delta E}} |\langle E_n | \mathcal{O} | E_m \rangle|^2 \quad (6.68)$$

$$= \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \sum_{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E,\Delta E}} \left| \langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_m \rangle \right|^2 \quad (6.69)$$

$$= \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \sum_{\substack{(|E_n\rangle, |E_m\rangle) \\ \in \mathcal{E}_{E,\Delta E} \times \mathcal{E}_{E,\Delta E}}} \left| \langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_m \rangle \right|^2 \quad (6.70)$$

$$- \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E}} \left| \langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_n \rangle \right|^2 \quad (6.71)$$

$$= \frac{1}{D_{E,\Delta E} - 1} \left(\text{Tr} \left[(\rho_{\text{mc}}^{(E,\Delta E)})^{\frac{1}{2}} \hat{\mathcal{O}}_{\text{mc}} (\rho_{\text{mc}}^{(E,\Delta E)})^{\frac{1}{2}} \hat{\mathcal{O}}_{\text{mc}} \right] - \overline{G_{\text{mc}}^{(2)}}^T \right). \quad (6.72)$$

Using the quantum mixing property, the right-hand side can be bounded as

$$\frac{1}{D_{E,\Delta E} - 1} \left(\text{Tr} \left[(\rho_{\text{mc}}^{(E,\Delta E)})^{\frac{1}{2}} \hat{\mathcal{O}}_{\text{mc}} (\rho_{\text{mc}}^{(E,\Delta E)})^{\frac{1}{2}} \hat{\mathcal{O}}_{\text{mc}} \right] - \overline{G_{\text{mc}}^{(2)}}^T \right) \leq \frac{1}{D_{E,\Delta E}} \left(\|\hat{\mathcal{O}}_{\text{mc}}\|_{\infty}^{\text{TFD}} \right)^2, \quad (6.73)$$

where we have introduced an operator norm on the thermofield-double Hilbert space,

$$\|\mathcal{O}\|_{\infty}^{\text{TFD}} := \sup_{|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}} \sqrt{\frac{{}_{LR}\langle \Psi | \mathcal{O}_L \otimes \mathcal{O}_R | \Psi \rangle_{LR}}{{}_{LR}\langle \Psi | \Psi \rangle_{LR}}}. \quad (6.74)$$

This norm is UV-finite even if $\mathcal{O}$ is a genuinely local operator at a single point. For light operators we expect this norm to be independent of the Hilbert space dimension, leading to

$$\mathbb{E}_{\mathcal{E}'_{E,\Delta E}} [|\langle E_n | \mathcal{O} | E_m \rangle|^2] = O(1) e^{-S_{\text{mc}}(E)}. \quad (6.75)$$

For higher moments, we again rewrite them in terms of thermal Wightman cor-

relators:

$$\begin{aligned}
& \mathbb{E}_{\mathcal{E}'_{E,\Delta E}} [|\langle E_n | \mathcal{O} | E_m \rangle|^{2q}] \\
&= \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \sum_{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E,\Delta E}} |\langle E_n | \mathcal{O} | E_m \rangle|^{2q} \\
&= \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \sum_{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E,\Delta E}} \left| \langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_m \rangle \right|^{2q} \\
&= \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \sum_{\substack{(|E_n\rangle, |E_m\rangle) \\ \in \mathcal{E}_{E,\Delta E} \times \mathcal{E}_{E,\Delta E}}} \left| \langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_m \rangle \right|^{2q} \\
&= A_1 - A_2,
\end{aligned} \tag{6.76}$$

where we define

$$\begin{aligned}
A_1 &= \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \sum_{\substack{(|E_n\rangle, |E_m\rangle) \\ \in \mathcal{E}_{E,\Delta E} \times \mathcal{E}_{E,\Delta E}}} \left| \langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_m \rangle \right|^{2q}, \\
A_2 &= \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \sum_{|E_n\rangle \in \mathcal{E}_{E,\Delta E}} \left| \langle E_n | \hat{\mathcal{O}}_{\text{mc}} | E_n \rangle \right|^{2q}.
\end{aligned} \tag{6.77}$$

Since both A_1 and A_2 are non-negative, we have

$$\mathbb{E}_{\mathcal{E}'_{E,\Delta E}} [|\langle E_n | \mathcal{O} | E_m \rangle|^{2q}] < A_1 + A_2. \tag{6.78}$$

The second term A_2 is equal to the long-time average of the $2q$ -point microcanonical Wightman function and can be bounded from above by inverse powers of the energy shell dimension using the quantum mixing property (6.61) and the clustering property (6.58):

$$A_2 \leq \frac{1}{D_{E,\Delta E} - 1} \overline{G_{(\text{mc})}^{(2q)}}^T \leq q! \cdot O(1) e^{-(q\gamma+1)S_{\text{mc}}(E)}. \tag{6.79}$$

What is new for the off-diagonal elements is the contribution A_1 . By applying

the same manipulations as in the diagonal case, we can rewrite A_1 as

$$\begin{aligned}
A_1 &= \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \prod_{i=1}^q \lim_{\substack{T_i \rightarrow \infty \\ S_i \rightarrow \infty}} \frac{1}{T_i} \frac{1}{S_i} \int_0^{T_i} dt_i \int_0^{S_i} ds_i \\
&\quad \times \sum_{\substack{(|E_{n_i}\rangle, |E_{m_i}\rangle) \\ \in (\mathcal{E}_{E,\Delta E} \times \mathcal{E}_{E,\Delta E})^q}} \langle E_{n_i} | \hat{\mathcal{O}}_{\text{mc}} | E_{m_i} \rangle \langle E_{m_i} | \hat{\mathcal{O}}_{\text{mc}} | E_{n_i} \rangle e^{i(E_{n_i} - E_{n_{i+1}})t_i} e^{i(E_{m_i} - E_{m_{i+1}})s_i} \\
&= \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \prod_{i=1}^q \lim_{\substack{T_i \rightarrow \infty \\ S_i \rightarrow \infty \\ \mathfrak{T}_{2q} \rightarrow \infty}} \frac{1}{T_i} \frac{1}{S_i} \frac{1}{\mathfrak{T}_{2q}} \prod_{k=1}^{2q} \int_0^{\mathfrak{T}_k} dt_k G_{\text{mc}}^{(2q)}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{2q}) \\
&= \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} \overline{G_{\text{mc}}^{(2q)}}^T,
\end{aligned} \tag{6.80}$$

where the times $\mathbf{t}_k$ are determined by the recursion relations

$$\begin{aligned}
-\mathbf{t}_{2k-1} + \mathbf{t}_{2k} &= s_{k-1} - s_k, \\
-\mathbf{t}_{2k} + \mathbf{t}_{2k+1} &= -t_k + t_{k+1},
\end{aligned} \tag{6.81}$$

and the intervals $\mathfrak{T}_k$ are defined analogously. Using the clustering property (6.58) together with the quantum mixing property (6.61), we obtain

$$A_1 < \frac{C}{D_{E,\Delta E}(D_{E,\Delta E} - 1)} q! \cdot \left(\overline{G_{\text{can},\beta}^{(2)}}^T \right)^q = O(1) q! e^{-(\gamma q + 2)S_{\text{mc}}(E)}. \tag{6.82}$$

This shows that the off-diagonal elements also exhibit a double exponential suppression consistent with (5.63).

Theorem

Theorem 16. *By assuming (6.58) and (6.61) for the operator $\mathcal{O}$ and setting $\epsilon = e^{-\eta \gamma S_{\text{mc}}(E)}$ with $\eta < \frac{1}{2}$, we find that $\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}]$ also displays double-exponential behavior (5.63), namely*

$$\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}] \leq 4e^{-\#'\frac{\epsilon^2}{G_{\text{can},\beta}^{(2)}}T + O\left(\left(\overline{G_{\text{can},\beta}^{(2)}}^T\right)^n\right)} = 4e^{-\#'\epsilon^{O(S_{\text{mc}}(E))}}, \quad \# > 0. \tag{6.83}$$

In explicit models, one can incorporate the deviations from exact sub-Gaussianity into the above argument without difficulty.

7 Non-equilibrium evolution and information problems in AdS/CFT correspondence

7.1 Thermal correlators and information problems in AdS/CFT

7.1.1 Decay of two point functions

Here we discuss the real time evolution of thermal correlator in AdS/CFTs. Firstly we notice that two copies of the holographic CFT in the thermofield double state is approximately described by gravity on the extended AdS Schwartzschild spacetime if the temperature is enough high [24]. Some comments are in order.

- The boost symmetry in AdS Schwartzschild spacetime is corresponding to the symmetry generated by $H_{\text{TFD}} = H_L - H_R$. Indeed, the TFD state is invariant under this time translation.
- In the semiclassical approximation, besides giving a geometry, we need to specify the state of the all other fields on this spacetime. From the discussion in Section 2.3, we know that the QFT on blackhole is also in thermal bulk QFTs. Thus we can think that the bulk QFTs also in TFD states which lies in matter Hilbert space. Since the TFD states are vacuum state for TFD Hamiltonian H_{TFD} , we consider the excitation from this vacuum states. This structure of the excitation is same as one of zero temperature vacuum. Thus we consider weak coupling QFT on AdS blacholes.

Below we see the ways of computing the thermal correlator in different manner and learn how they looks universal.

■ Via conformal transformation [176]

Let us consider Schwartzschild BH in AdS_3 space.

$$ds^2 = \frac{-4dU_+dU_-}{(1 + U_+U_-)^2} + \frac{(1 - U_+U_-)^2}{(1 + U_+U_-)^2}d\phi^2 \quad (7.1)$$

In this coordinate, the horizons are at $U_+ = 0$ and $U_- = 0$. The singularity is at $U_+U_- = 1$. There are two conformal boundaries at $U_+U_- = -1$. There are a boost-isometry $U_+ \rightarrow e^\lambda U_+$, $U_- \rightarrow e^{-\lambda} U_-$ and reflection symmetry $U_+ \Leftrightarrow U_-$. We can glue the Euclidean BH at $t = 0$, which type of geometries are called the Hartle Hawking state. The boundary manifold of the Euclidean BH is $S^1_\beta \times S^1$. For gluing it we cut the Eulclidean BH at $\tau = 0$ and the boundary manifold is $I_{\beta/2} \times S^1$. For AdS_3 case, we can obtain the blackhole by coordinate transformation and compactification. Recall the transformation map from the Poincare to the BTZ (see

Appendix A.1),

$$w_{\pm} = \sqrt{\frac{r^2 - r_{\pm}^2}{r^2 - r_-^2}} e^{2\pi T_{\pm} u_{\pm}}, \quad (7.2)$$

$$z = \sqrt{\frac{r^2 - r_{\pm}^2}{r^2 - r_-^2}} e^{\pi T_{+} u_{+} + \pi T_{-} u_{-}} \quad (7.3)$$

,where

$$T_{\pm} = \frac{r_{+} \mp r_{-}}{2\pi L_{\text{AdS}}} \quad (7.4)$$

$$u_{\pm} = \phi \pm \frac{t}{L_{\text{AdS}}} \quad (7.5)$$

If we are interested only in the boundary correlation functions it is enough to consider the region $r \gg r_{\pm}$ and the expression became simple

$$w_{\pm} = e^{2\pi T_{\pm} u_{\pm}} \quad (7.6)$$

$$z = \sqrt{\frac{r^2 - r_{\pm}^2}{r^2}} e^{\pi T_{+} u_{+} + \pi T_{-} u_{-}} \quad (7.7)$$

The source field transform as the primary field with conformal dimension $(h, \bar{h}) = (h_{+}, h_{-})$ and thus

$$\Phi \rightarrow r^{-2h_{-}} \Phi_0(u_{+}, u_{-}) \quad (7.8)$$

From the expression of the bulk to boundary propagator in the Poincare coordinate and the map above we obtain naive bulk to boundary propagator in the BTZ

$$K_{\text{BTZ}}(r, u_{+}, u_{-}; u'_{+}, u'_{-}) = c' \frac{\left(\frac{r_{+}^2 - r_{-}^2}{r^2}\right)^{h_{+}} e^{-2\pi h_{+}[T_{+}\Delta u_{+} + T_{-}\Delta u_{-}]} }{\left(\frac{r_{+}^2 - r_{-}^2}{r^2} + (1 - e^{-2\pi T_{+}\Delta u_{+}})(1 - e^{-2\pi T_{-}\Delta u_{-}})\right)^{2h_{+}}} \quad (7.9)$$

,where $\Delta u_{\pm} = u_{\pm} - u'_{\pm}$. Since we obtained the BTZ blackhole as the identification of the boundary coordinates, we consider the mirror images for the 2π shift of the angle ϕ . Then the true bulk to boundary propagator in the BTZ is

$$K_{\text{BTZ}}(r, u_{+}, u_{-}; u'_{+}, u'_{-}) = c' \sum_{n \in \mathbb{Z}} \frac{\left(\frac{r_{+}^2 - r_{-}^2}{r^2}\right)^{h_{+}} e^{-2\pi h_{+}[T_{+}\Delta u_{+} + T_{-}\Delta u_{-} + (T_{+} + T_{-})2\pi n]} }{\left(\frac{r_{+}^2 - r_{-}^2}{r^2} + (1 - e^{-2\pi T_{+}(\Delta u_{+} + 2\pi n)})(1 - e^{-2\pi T_{-}(\Delta u_{-} + 2\pi n)})\right)^{2h_{+}}} \quad (7.10)$$

And in $r \rightarrow \infty$ limit

$$\Phi(r, u_+, u_-) \propto r^{-2h_+} \int du'_+ du'_- \sum_{n \in \mathbb{Z}} \frac{e^{-2\pi h_+ [T_+ \Delta u_+ + T_- \Delta u_- + (T_+ + T_-) 2\pi n]}}{((1 - e^{-2\pi T_+ (\Delta u_+ + 2\pi n)})(1 - e^{-2\pi T_- (\Delta u_- + 2\pi n)}))^{2h_+}} \Phi_0(u'_+, u'_-) \quad (7.11)$$

Computing the on-shell action and finally

$$\begin{aligned} \langle \mathcal{O}(u_+, u_-) \mathcal{O}(u'_+, u'_-) \rangle_{\text{can}}^\beta &\propto \sum_{n \in \mathbb{Z}} \frac{e^{-2\pi h_+ [T_+ \Delta u_+ + T_- \Delta u_- + (T_+ + T_-) 2\pi n]}}{((1 - e^{-2\pi T_+ (\Delta u_+ + 2\pi n)})(1 - e^{-2\pi T_- (\Delta u_- + 2\pi n)}))^{2h_+}} \\ &\propto \sum_{n \in \mathbb{Z}} \frac{e^{-2\pi h_+ [T_+ \Delta u_+ + T_- \Delta u_- + (T_+ + T_-) 2\pi n]}}{((1 - e^{-2\pi T_+ (\Delta u_+ + 2\pi n)})(1 - e^{-2\pi T_- (\Delta u_- + 2\pi n)}))^{2h_+}} \end{aligned} \quad (7.12)$$

This result matches with the conformal transformation from the Poincare vacuum two point function. From QFT point of view this is just the Bogoliubov transformation from the Poincare(Minkowski) modes to the BTZ modes (Rindler). Indeed the BTZ two point function satisfies the KMS condition

$$(\Delta t, \Delta \phi) \rightarrow (\Delta t + i\beta_H, \Delta \phi + i\Omega_H \beta_H) \quad (7.13)$$

,where

$$T_H = \frac{r_+^2 - r_-^2}{2\pi L_{\text{AdS}}^2}, \quad \Omega_H = \frac{r_-}{L_{\text{AdS}} r_+} \quad (7.14)$$

They are the hawking temperature and the angular momentum of the outer horizon of the BTZ BH. To see the real time evolution in simple manner, let us consider the case $T_+ = T_- = T_H$ which is the case $r_- = 0$. In this case

$$\langle \mathcal{O}(t, \phi) \mathcal{O}(t', \phi') \rangle_{\text{can}}^\beta \propto \sum_{n \in \mathbb{Z}} \frac{1}{\left(\cosh \left(\frac{2\pi \Delta t}{\beta_H} \right) - \cosh \left(\frac{2\pi (\Delta \phi + 2\pi n)}{\beta_H} \right) \right)^{2\Delta}}. \quad (7.15)$$

Thus the thermal two point function shows exponential decays $\langle \mathcal{O}(t, \phi) \mathcal{O}(t', \phi') \rangle_{\text{can}}^\beta \approx e^{-\frac{2\pi \Delta}{\beta_H} \Delta t}$. Also, by using the KMS relation,

$$G_{\text{can}, \beta}^{(2)}(t_1, t_2) \propto \sum_{n \in \mathbb{Z}} \frac{1}{\left(\cosh \left(\frac{2\pi \Delta t}{\beta_H} \right) + \cosh \left(\frac{2\pi (\Delta \phi + 2\pi n)}{\beta_H} \right) \right)^{2\Delta}} \quad (7.16)$$

and again see the exponential decays.

■ Via solving the wave equations: quasi normal modes [183–185]

Of corse, we can obtain the thermal correlator with the GKPW dictionary (3.15). The decaying behavior is related to complex poles of the retarded Green function $G(\omega)$ in frequency space. The complex poles are called quasi normal modes (QNM).

Indeed, by using the residue theorems

$$G(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} G(\omega) = \sum_{\omega_{\text{QNM}}} e^{-i\text{Re}[\omega_{\text{QNM}}]t - \text{Im}[\omega_{\text{QNM}}]t} \text{Res}_{\omega=\omega_{\text{QNM}}} [G(\omega)]. \quad (7.17)$$

In AdS/CFT correspondence, quasi normal mode is a mode with in-going boundary condition on the horizon and normalizable condition on the asymptotic boundary. To see this, let us consider the scalar quasi normal mode in the BTZ blackhole. Scalar perturbations are described by the wave equation,

$$(\square_G - m^2)\Phi = 0. \quad (7.18)$$

We use the one useful coordinate for BTZ blackhole

$$ds^2 = \frac{(1-z)^2}{4z^2} dz^2 - \frac{z}{1-z} (dx^+)^2 + \frac{1}{(1-z)^2} (dx^-)^2. \quad (7.19)$$

Note that this z is different from Poincare coordinate. We use the ansatz

$$\Phi = e^{-i(k_+ x^+ + k_- x^-)} f_k(z). \quad (7.20)$$

With these ansatz, we end up with hypergeometric equation

$$z(1-z) \frac{d^2 R}{dz^2} + (1-z) \frac{dR}{dz} + \left[\frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{m^2}{4(1-z)} \right] R = 0. \quad (7.21)$$

The solution with in-going boundary condition at the horizon is given by

$$f_k(z) = z^\alpha (1-z)^\beta {}_2F_1(a, b, c; z), \quad (7.22)$$

where

$$\alpha = -\frac{ik_+}{2}, \quad \beta = \frac{1 - \sqrt{1 + m^2}}{2} \quad (7.23)$$

$$a = \frac{k_+ - k_-}{2i} + \beta, \quad b = \frac{k_+ + k_-}{2i} + \beta, \quad c = 1 + 2\alpha. \quad (7.24)$$

Indeed we can see the asymptotic behavior around the near-horizon is in-going: The expansion around the horizon $z = 0$ is

$$e^{-ik_+ x^+ - ik_- x^-} f_k(z) \sim e^{-ik_+ x^+ - ik_- x^-} z^\alpha \quad (7.25)$$

$$= e^{-ik_+ x^+ - ik_- x^- - \frac{ik_+}{2} \log z} \quad (7.26)$$

$$= e^{-ik_+ (x^+ - \frac{1}{2} \log z) - ik_- x^-}. \quad (7.27)$$

For out-out going wave we should take $\alpha = -\frac{ik_+}{2}$. To discuss the behavior around the conformal boundary, we use the identity

$$\begin{aligned}
f_k(z) &= z^\alpha (1-z)^\beta {}_2F_1(a, b, c; z) \\
&= z^\alpha (1-z)^\beta (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c-a, c-b, c-a-b; 1-z) \\
&\quad + z^\alpha (1-z)^\beta \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(c-a, c-b, a+b-c+1; 1-z).
\end{aligned} \tag{7.28}$$

Since $\beta + c - a - b = \frac{1}{2}(1 + \sqrt{1+m^2})$ and $\beta = \frac{1}{2}(1 - \sqrt{1+m^2})$, the first term corresponds to the normalizable mode and the second term corresponds to the non-normalizable mode. If we demand the vanishing field value at $z = 1$, we obtain the discrete quasi-normal mode from the pole of $\Gamma(c-a)\Gamma(c-b)$. That is

$$\begin{aligned}
c-a = -n \quad \text{or} \quad c-b = -n, \quad n = 0, 1, 2, \dots \\
\frac{i}{2}(k_+ \pm k_-) = n + \frac{1}{2}(1 + \sqrt{1+m^2}).
\end{aligned} \tag{7.29}$$

We can read off the quasi-normal mode by using the identities

$$\begin{aligned}
(k_+ + k_-)(r_+ - r_-) &= \omega - k, \\
(k_+ - k_-)(r_+ + r_-) &= \omega + k,
\end{aligned} \tag{7.30}$$

then we have

$$\begin{aligned}
\omega_L &= l - 2\pi i T_L (2n + \Delta), \\
\omega_R &= -l - 2\pi i T_R (2n + \Delta),
\end{aligned} \tag{7.31}$$

with $l \in \mathbb{Z}$ and $n = 0, 1, 2, \dots$.

At first, let us assume that we can use the same procedure as the Euclidean AdS/CFT.

The on-shell action can be evaluated by the boundary intgeral,

$$I[\phi_0] = \left[\int \frac{d^d k}{(2\pi)^d} \phi_0(-k) \mathcal{F}(k, z) \phi_0(k) \right]_{z=\epsilon}^{z=z_H} \tag{7.32}$$

,where

$$\mathcal{F}(k, z) = K \sqrt{-G} G^{zz} f_{-k}(z) \partial_z f_k(z). \tag{7.33}$$

By taking the second derivative of the on-shell action, we see

$$G(k) = -[\mathcal{F}(k, z) + \mathcal{F}(-k, z)]_{z=\epsilon}^{z=z_H} \tag{7.34}$$

One can see that this quantity is real and hence cannot be a candidate to the retarded Green function which is generally real. By noticing $f_k(z)^* = f_{-k}(z)$, we can read off

$$\text{Im}\{\mathcal{F}(k, z)\} = \frac{K}{2i} \sqrt{-G} G^{zz} [f_{-k}^*(z) \partial_z f_k(z) - f_k(z) \partial_z f_k^*(z)] \quad (7.35)$$

The correct prescription is conjectured as

$$G^{R/A}(k) = -2 \lim_{z \rightarrow \epsilon} \mathcal{F}(k, z). \quad (7.36)$$

The prescription for computing the retarded/advanced Green function is formulated as follows:

1. Find a solution to the wave equation with following properties:
 - The solution equals to 1 at the boundary $z = \epsilon$;
 - For timelike momenta, the solution has an asymptotic expression corresponding to the in-going/ out-going wave at the horizon. For space like momenta, the solution is regular at the horizon.
2. The retarded Green function is then given by $G = -2\mathcal{F}_{\partial M}$. Only contribution from the boundary has to be taken. Here we only consider the contribution just from the asymptotic boundary.

Let us discuss why this works. To get an intuition we consider the expansion around the asymptotic boundary $z = \epsilon$. Unnormalized solution has an expansion

$$f_k(z) \sim A(k)(1-z)^{\beta_-}(1+\dots) + B(k)(1-z)^{\beta_+}(1+\dots) \quad (7.37)$$

,where

$$\beta_+ = \frac{1 + \sqrt{1+m^2}}{2} = \frac{\Delta}{2}, \quad \beta_- = \frac{1 - \sqrt{1+m^2}}{2} = \frac{d - \Delta}{2}. \quad (7.38)$$

Thus $A(k)$ corresponds to the non-normalizable mode and $B(k)$ corresponds to normalizable mode. Similar to the usual AdS/CFT dictionary, $B(k)$ corresponds to the sourced one-point function and $A(k)$ is source. If we consider the normalized f_k , we see

$$f_k(z) \sim (1-z)^{\frac{d-\Delta}{2}}(1+\dots) + \frac{B(k)}{A(k)}(1-z)^{\frac{\Delta}{2}}(1+\dots) \quad (7.39)$$

From this we can finally calculate the flux factor

$$\mathcal{F}(k, \epsilon) = d \frac{B(k)}{A(k)}. \quad (7.40)$$

Similar to the usual AdS/CFT dictionary, the two point function is given by the

derivative of the sourced one-point function

$$G \sim \frac{B(k)}{A(k)} \sim d\mathcal{F}(k, \epsilon). \quad (7.41)$$

When $d = 2$ case, the normalized wave is

$$f_k(z) = \frac{z^\alpha(1-z)^\beta {}_2F_1(a, b, c; z)}{\epsilon^\alpha(1-\epsilon)^\beta {}_2F_1(a, b, c; \epsilon)}. \quad (7.42)$$

By evaluating $\mathcal{F}$ we see the retarded/ and obtain the same behavior as (7.15).

■ Geodesics approximation

There is another way of obtaining correlation functions. We can compute the boundary correlation function from bulk Green function called bulk-to-bulk correlation function $G_{BB}(x_1, x_2)$

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \lim_{r \rightarrow \infty} e^{2\Delta r} G_{BB}(r, x_1; r, x_2) \quad (7.43)$$

where the bulk-to-bulk correlation function $G_{BB}(x_1, x_2)$ satisfying [26, 186]

$$\begin{aligned} (\square_G - m^2) G_{BB}(x_1^A; x_2^A) &= \frac{1}{\sqrt{G}} \delta^{d+1}(x_1^A - x_2^A), \\ m^2 &= \Delta(\Delta - d). \end{aligned} \quad (7.44)$$

This is based on the one another dictionary of AdS/CFT correspondence called HKLL dictionary

$$\mathcal{O}(x) = \lim_{r \rightarrow \infty} e^{\Delta r} \Phi(x^A) \quad (7.45)$$

where $\Phi(x^A)$ is bulk fields. It is known that HKLL dictionary is equivalent to GKPW dictionary [65, 187]. For vacuum AdS_{d+1} , it is known that [26, 188]

$$G_{BB}(x_1, x_2) = G_{BB}(\xi) \propto \xi^\Delta \cdot {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}, \Delta - \frac{d}{2} + 1; \xi^2\right) \quad (7.46)$$

where ξ is defined through the geodesic distance between two points x_1^A and x_2^A

$$\xi = \frac{1}{\cosh(\text{Dist}(x_1; x_2))}. \quad (7.47)$$

Note that when the distance of two points are infinite $\xi \rightarrow 0$. From this expression we can get simple formula for correlation function with light but large conformal dimension ¹⁵ The key things is the integration expression of the hypergeometric function

¹⁵This mean that we take $\Delta \rightarrow \infty$ after large N limit.

$${}_2F_1(a, b, c; \xi^2) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 ds s^{b-1} (1-s)^{b-c-1} (1-\xi^2 s)^{-a} \quad (7.48)$$

for $\text{Re}(c) > \text{Re}(b) > 0$, $|\arg(1-\xi^2)| < \pi$. In our set up, we have

$$G_{\text{BB}}(\xi) \propto \xi^\Delta \int_0^1 ds s^{\frac{\Delta-1}{2}} (1-s)^{\frac{\Delta-d-1}{2}} (1-\xi^2 s)^{-\frac{\Delta}{2}} \quad (7.49)$$

We consider the saddle point approximation in $\Delta \rightarrow \infty$. Let's focus on the integral in the form,

$$I(\xi) = \int_0^1 ds s^{-\frac{1}{2}} (1-s)^{-\frac{d+1}{2}} e^{\frac{\Delta}{2} v(s)} \quad (7.50)$$

$$v(s) = \log s + \log(1-s) - \log(1-\xi^2 s)$$

We have two points such that $v'(s_*) = 0$ as

$$s_*^{(1)} = \frac{1 - \sqrt{1-\xi^2}}{\xi^2}, \quad s_*^{(2)} = \frac{1 + \sqrt{1-\xi^2}}{\xi^2}. \quad (7.51)$$

Note that $0 < s_*^{(1)} < 1$, $s_*^{(2)} > 1$. Thus, we consider the saddle-point approximation for the first point. Since $v''(s_*^{(1)}) < 0$, we do not deform the integral contour from $s: 0 \rightarrow 1$, we obtain

$$I(\xi) \approx \xi^{-\Delta} \sqrt{\frac{2\pi}{\Delta}} \left(\frac{1 + \sqrt{1-\xi^2}}{\sqrt{1-\xi^2}} \right)^{\frac{d}{2}} e^{-\Delta \text{Dist}(x_1, x_2)}. \quad (7.52)$$

By combining these approximations, we obtain

$$G_{\text{BB}}(x_1^A, x_2^A) \propto e^{-\Delta(\text{Dist}(x_1, x_2))} \left(\frac{\Delta}{2\pi} \frac{1 + \sqrt{1-\xi^2}}{\sqrt{1-\xi^2}} \right)^{\frac{d}{2}}. \quad (7.53)$$

From this we can compute the boundary correlator. Since the distance of two points set in conformal boundary are infinite and we need to renormalize. Actually the factor $e^{2r\Delta}$ play a role of renormalization and we find

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle \propto e^{-\Delta \text{Dist}_{\text{ren}}(x_1^A, x_2^A)} \quad (7.54)$$

where we denote $\text{Dist}_{\text{ren}}(x_1, x_2)$ is renormalized geodesic length. For simple illustra-

tion, for Poincare coordinate we have

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle \propto \frac{1}{(\eta_{ab}(x_1^a - x_2^a)(x_1^b - x_2^b))^{2\Delta}}. \quad (7.55)$$

Here we use the formula in Appendix A.1. Also as we compute in the next section 7.2, we can compute the geodesic length in AdS₃ blackholes and we can compute the geodesics length between two points in the maximal extended geometry in same time slice is linear in t . See Figure 20. And we find

$$G_{\text{can},\beta}^{(2)} \propto e^{-\frac{2\pi\Delta}{\beta}t}. \quad (7.56)$$

Now several comments are in order:

- The geodesic approximation method here is actually considered to be applicable for more general geometry and higher dimension. In [189, 190]. In their papers they consider worldline formalism for bulk QFTs and find that for large mass we have the saddle point approximation

$$\langle \mathcal{O}(P) \mathcal{O}(Q) \rangle \approx \sum_{\text{geodesics connecting } P \text{ and } Q} e^{-m \text{Dist}_{\text{ren}}(P:Q)}. \quad (7.57)$$

The worldline formalism is very general formulation fo QFTs and works for any spacetime dimension.

- Then we can give approximate formulas for thermal correlator from higher dimension AdS blackholes [191]. We can find that the geodesic function increase up some time where the effect of singularity is important.
- The decay of correlation functions are related to the increasing the length of the ER bridge. See 20.
- The discussion can be more accurately in JT gravity including the quantum effects [192].
- One may think the geodesic approximation is very similar to the holographic entanglement entropy formulas in three dimensional bulk since in this case RT surface is given by geodesics length. Especially, we know that the entanglement entropy is computed from the twist operator with conformal dimension $\Delta_n = \frac{c}{12}(n - \frac{1}{n})$ with $n \rightarrow 1$. Even though the Δ_n is proportional to c but $n \rightarrow 1$ limit makes the dimension is small and looks like light operator. However there is distinction. For correlation function possibly there is no need for homologous conditions. And this distinction will be important for late time behavior. The

entanglement entropy saturate the time scale of $O(N_{\text{dof}})$ (we will see this fact in next subsection 7.2) but the correlation functions can continue to decay up to $e^{O(S_{\text{mc}}(E))}$ [193].

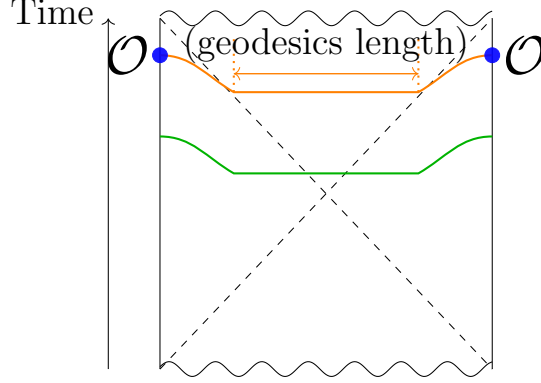


Figure 20: The thermal correlation functions are computed in geodesic length between two side. The length is proportional to the length of ER bridge and increase.

7.1.2 Information problems in AdS/CFT

The information problems are more accurately formulated in AdS/CFT correspondence [24]. The key point is the behavior of the thermal two point function in late time (See Figure 21).

Above we saw that the holographic thermal two point function decays from semi-classical computations when the bulk back ground is given by the blackhole. If we naively believe the semi-classical computation and decay behavior for any time, we see the correlation function finally reach to arbitrarily small value and small for any late time. However when we consider boundary CFTs live on spatially compact manifold like cylinder and large but finite N , this behavior never happens. In this setting we expect that the theory has discrete energy spectrum even though they are almost continuous. In any discrete systems there are (quantum) Poincare recurrence theorem: Consider a quantum system that is unitary and has a discrete energy spectrum. For any initial state and any $\epsilon > 0$, there exists a $t_{\text{rec}} > 0$ such that the following holds:

$$||\psi(t_{\text{rec}} > 0)\rangle - |\psi(0)\rangle|| > 1 - \epsilon. \quad (7.58)$$

The essence of the theorem is the spectrum expansion

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t} |E_n\rangle. \quad (7.59)$$

Then each $e^{-iE_n t}$ is a kind of clock with period $1/E_n$. The recurrence time is estimated when all of the hands on the clock go around and align. For example some harmonic oscillators like system where $E_n - E_m = \omega(n - m)$, the recurrence time is short $T = \omega$. However, the non-intangible system they are very long $e^{D_{\text{eff}}}$ but still finite. Also the two point function can be have spectrum decomposition

$$G_{\text{can}}^{(2)}(t) = \sum_n e^{-\beta E_n} e^{i(E_n - E_m)t} |\langle E_n | \mathcal{O} | E_m \rangle|^2 \quad (7.60)$$

and should shows recurrence. So at least we need not to believe the semi-classical picture for any late time. Maldacena discussed that actually the semi-classical picture fail down in more early time. Simple estimation is following: The correlation function decays in early time but for the time scale when the correlation functions from other saddle points can compatible to semi-classical correlation function *i.e.*,

$$G_{\text{BH}}^{(2)} = e^{-O(1/G_N)} = \frac{1}{\text{Poly}(D_{E,\Delta E})}. \quad (7.61)$$

¹⁶ However the large N limit discussed so far only pick up one saddle and we expect

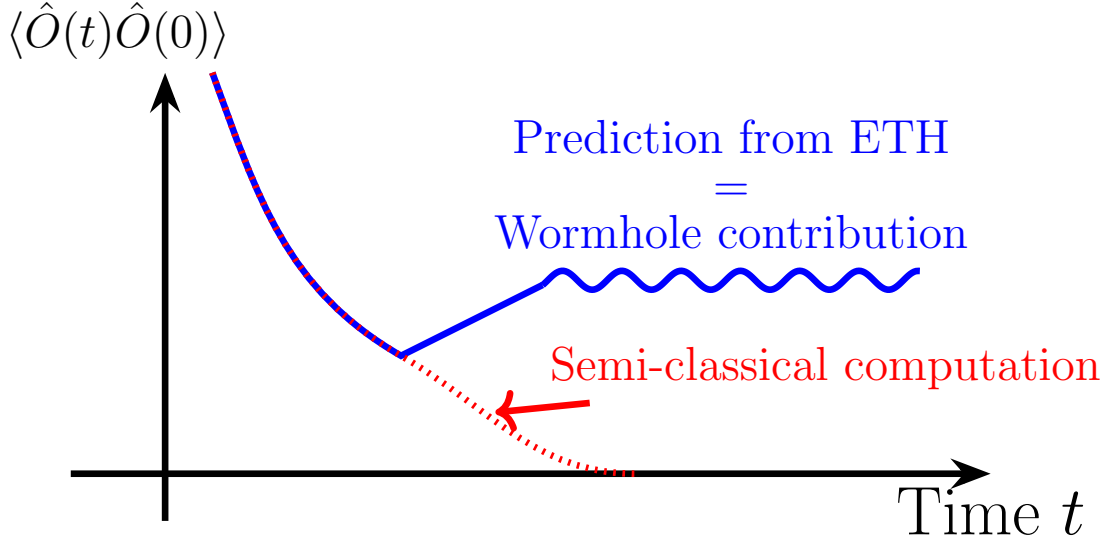


Figure 21: Time dependence of the two-point function $\langle \hat{O}(t) \hat{O}(0) \rangle$. The blue curve represents the contribution predicted by ETH and the associated wormhole saddle, while the red dotted curve shows the semi-classical computation. These two lines do not agree in time $t = O(D_{E,\Delta E})$. This is information problems.

¹⁶ Actually because of the blackhole singularity we have another time scale we loose some control of semi-classical picture. Also the late time the correlation becomes polynomial decay in time.

the true correlation function will be

$$G_{\text{can}}^{(2)}(t) = \frac{\sum_{G_* \in \text{saddle}} G_{G_*}(t) e^{-I_{\text{grav}}[G_*]} + (\text{off-shell contribution})}{\sum_{G_* \in \text{saddle}} e^{-I_{\text{grav}}[G_*]} + (\text{off-shell contribution})} \quad (7.62)$$

where $G_{G_*}(t)$ is correlation functions obtained by the bulk-to-bulk correlators on saddle point G_* . Then we may seek other saddles or off-shell contributions. One possibilities for the non-decaying behavior of the late time two point function is coming from the gravitational path integral with wormholes for gravitational Schwinger-Keldysh [38]. To see this see the Figure 22. The role of wormholes is reducing the length of ER bridge.

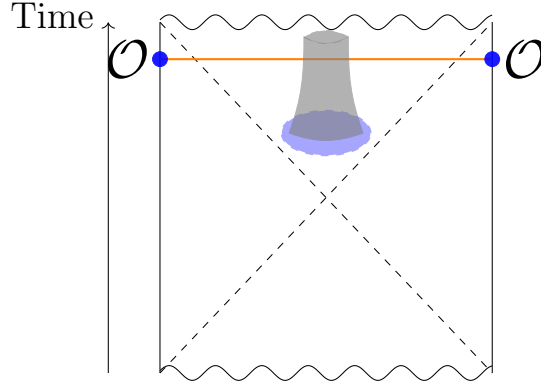


Figure 22: Baby universe emission reduces the length of ER bridge

For two dimensional JT gravity case, the order of the late time behavior is expected $O(e^{-S_0})$ where S_0 is ground state microcanonical entropy. Even though we do not have concrete discussion for the higher dimensional for the actual gravitational late time behavior of the correlation function, we expect the similar story. In there dimensional case, there are non-wormhole gravitational saddle points which are given by $SL(2, \mathbb{Z})$ of the non-rotating blackhole and orbifolds. Let us consider BTZ black hole, which has the modular parameter $\tau = i\tau_2 = \frac{i\beta}{2\pi}$. Then, the other saddle points are given by the $SL(2, \mathbb{Z})$ transformation, $\tau' = \frac{a\tau+b}{c\tau+d}$. The late-time behavior of two-point functions and the regularized action with modular parameter τ' are given by [194],

$$(G_{\text{can}, \beta}^{(2)})_{\tau'} \sim \exp \left\{ -2\Delta \frac{|a|\tau_2}{|a\tau + b|^2} t \right\} \quad (7.63)$$

The contribution from $a = 0$ corresponds to thermal AdS_3 , where the two-point function shows periodic function in time. Except for the thermal AdS contribution, we find decaying behavior. If we can exchange the long-time average and the

geometry sum, we find that

$$\overline{G_{\text{can},\beta}^{(2)}}^T = e^{-I_{\text{BH}} + I_{\text{ThermalAdS}}} \overline{G_{\text{ThermalAdS}}^{(2)}}^T = O(1) \|\mathcal{O}\|_\infty^2 e^{-\frac{1}{2} \left(1 - \left(\frac{S_{\text{mc}}(E)}{2\pi E}\right)^2\right) S_{\text{mc}}(E)}. \quad (7.64)$$

However, there is two unsatisfactory points

1. If we consider microcanonical correlation functions, the thermal AdS geometry is not included among the saddle points, since it is obtained from the vacuum AdS geometry and has energy below the black-hole threshold.
2. The coefficient $\frac{1}{2} \left(1 - \left(\frac{S_{\text{mc}}(E)}{2\pi E}\right)^2\right)$ is not equal to $1/2$, contrary to the expectation from ETH [195].

Thus, it seems that the wormhole is the best candidate for explaining late-time correlation functions. A concrete discussion in higher dimensions, however, has not yet been fully established.¹⁷

Moreover, in Section 5.8, we saw that the two-point function is related to the spectral form factor, as shown in (5.327). As discussed in [36], the connected part of the spectral form factor is obtained from the so-called double-cone wormhole. The idea is as follows. In the ramp phase, the Schwinger–Keldysh contour for the spectral form factor can be approximated by a closed time contour, which is nothing but a path integral over the thermofield double (TFD) state. In the context of the AdS/CFT correspondence, this corresponds to a black hole geometry. By respecting the periodicity in real time, one wraps the black hole along the real-time direction, leading to the double-cone geometry. After performing a suitable complex deformation to avoid singular behavior near the cone, one finally obtains the wormhole geometry shown in Fig. 23.

The remind task for solving the information problems is discussing the correct gravitational non-perturbative objects and boundary behavior which will be related to the ETH. Importantly, ETH is a concept free from dimension and specific structure of systems like supersymmetry. In up-coming section, we see the direct relationship between the gravitational path integral and ETH.

7.1.3 Aside: Entanglement pattern in double cone wormholes

Before going to the non-equilibrium phenomena in AdS/CFT, we give a short comments on double cone wormholes. As a spirit of ER=EPR states that the wormhole

¹⁷Recently, as an analogue of SYK models, random averages of two-dimensional CFTs have been considered. In these models, it has been confirmed that the averaged spectral form factor reproduces the known wormhole contribution, similar to the double-cone geometry [196]. See also recent developments [197, 198].

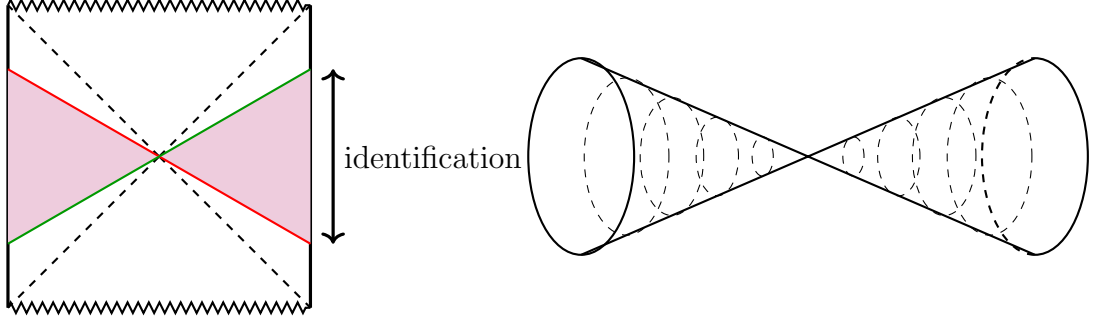


Figure 23: The picture of double cone wormholes. We do the compactification of the blackhole in real time.

are related to entanglement, we try to consider an entanglement pattern created by double cone wormholes.

Actually this problems is same as the problem of UV regularization in QFTs. We saw in Section 2, we see that the brickwall on stretched horizon play a role of UV regularization. Actually, the double cone also [6]. This is because complex deformation smooths out the conical singularity and yields the double cone geometry. Now with this regularization let us compute Rényi entropies of free scalar on double cone wormhole. We employ the orbifold method, in which the trace $\text{Tr}[\rho_A^n]$ is obtained from partition functions on $\mathbb{C}/\mathbb{Z}_n \times \mathbb{R}^{d-2}$. Here we take subsystem A to be one time slice of right Rindler wedge. For free theories, these partition functions can be evaluated exactly by the heat kernel technique. The relevant heat kernel obeys a diffusion-type equation and, for flat space, takes a Gaussian form.

$$\log Z[M] = \int_0^\infty \frac{dt}{2t} \int_M d^d x \sqrt{G} K(t, x, x) \quad (7.65)$$

$$K_{\mathbb{R}^d}(x, x'; t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{r^2}{4t} - tm^2}, \quad r = |x - x'|, \quad (7.66)$$

$$K_{\mathbb{C}/\mathbb{Z}_n \times \mathbb{R}^8}(t; x; r', \theta', x'_i) = \frac{1}{n} \sum_{k=0}^{n-1} K_{\mathbb{C} \times \mathbb{R}^{d-2}} \left(t; x; r', \theta' - \frac{2\pi k}{n}, x'_i \right). \quad (7.67)$$

On the orbifold, the kernel can be written as an image sum over angular shifts by $2\pi/N$. The Rényi entropy then reduces to an integral over the heat kernel along a contour in the complex t -plane

$$\begin{aligned} (1-n)S_A^{(n)} &= \frac{2\pi V_{d-2}}{n} \sum_{k=1}^{n-1} \int_\Gamma \frac{dt}{2t} \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_\gamma dr \sqrt{r^2} e^{-tm^2 - \frac{r^2}{t} \sin^2 \frac{\pi k}{n}} \\ &= \frac{\pi V_{d-2}}{(4\pi)^{\frac{d}{2}} n} \sum_{k=1}^{n-1} \frac{1}{\sin^2 \left(\frac{k\pi}{n} \right)} \int_\Gamma dt \cdot t^{-\frac{d}{2}} e^{-tm^2 + \frac{\epsilon^2}{t} \sin^2 \left(\frac{k\pi}{n} \right)} \end{aligned} \quad (7.68)$$

The usual real contour leads to divergences, and the correct contour must be chosen

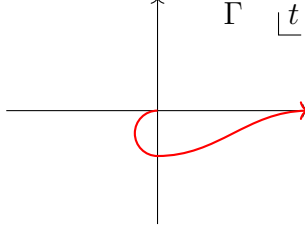


Figure 24: Deformed contour for double cone regularization

so that the heat kernel integral reproduces the known Green function. This selects a contour that winds around the origin in the lower half-plane, producing the desired regularized expression.

The resulting Rényi entropies can be written in closed form in terms of modified Bessel functions. For even spacetime dimension, one finds the expected divergent area-law contributions together with universal logarithmic terms proportional to $\log(m\epsilon)$, which agree with known results and are independent of the regularization scheme

$$\begin{aligned}
S_A^{(n)} &= a_{d-2} \frac{V_{d-2}}{\epsilon^{d-2}} + \dots \\
&+ \frac{n+1}{3n} \frac{2\pi}{(4\pi)^{\frac{d}{2}}} \frac{(-1)^{\frac{d}{2}}}{(\frac{d}{2}-1)!} \cdot V_{d-2} m^{d-2} \log\left(\frac{1}{m\epsilon}\right) \\
&+ a_0 \left(\psi\left(\frac{d}{2}-1\right) - \gamma + \frac{i\pi}{2} \right) \cdot V_{d-2} m^{d-2} + \dots
\end{aligned} \tag{7.69}$$

Interestingly, the final Rényi entropies acquire complex values. This originates from the fact that the double cone deformation renders the modular Hamiltonian non-Hermitian: the generator of Rindler time translations is modified by an imaginary contribution proportional to the Hamiltonian [36, 199]. Solving the field equation on the complex double cone shows that this modification leads to quasi-normal modes with complex frequencies. As a consequence, the formal expression $\text{Tr } e^{-\beta K_{\text{modified}}}$ becomes complex, which in turn produces complex Rényi entropies. This non-hermicity is related to quasi normal mode of double cone wormholes. Interestingly this complex entanglement spectrum is new class of von Neumann algebras.

So far we do not have any gravitational contributions. Now we take into account. Similar to the brick wall case, however, we may think that the some series of UV divergence will be renormalized to the Newton constant G_N . And this procedure should be independent of UV regularizations. Thus when we have gravity, the generalized entropies computed by two regularization should match. In this sense, we cannot distinguish two spacetime is connected (in complex manner) or not. These discussion is highly naive and we need to do more serious computations.

7.2 Quantum quench in CFTs and its holographic duals

One method for preparing a non equilibrium initial state is a *quantum quench* [78, 200]. Let us consider the following quench protocol. We prepare a Hamiltonian

$$H_{\text{deformed}} = H_{\text{CFT}} + \int d^d x g(t) \mathcal{O}(x), \quad (7.70)$$

where the time-dependent coupling $g(t)$ is the step function

$$g(t) = \begin{cases} g & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases} \quad (7.71)$$

As the initial condition, we take an energy eigenstate $|\psi_0\rangle$ of H_{CFT} . For $t < 0$, the dynamics are governed solely by H_0 , and the initial state remains unchanged. At $t = 0$, the perturbation is turned on; since the energy eigenstates of H_{CFT} are no longer eigenstates of H_{deformed} , the system undergoes nontrivial time evolution.

We assume that the post-quench Hamiltonian H_{CFT} possesses conformal symmetry and that the perturbation is introduced by an operator $\mathcal{O}$ of conformal dimension Δ in the conformal field theory. This quench protocol is characterized by a single length scale

$$\beta \propto g^{1/(d-\Delta)}. \quad (7.72)$$

Suppose that our deformation is a relevant deformation. Then the path integral in $t < 0$ prepares the ground states of the gapped systems [201]. Calabrese and Cardy assumed that, as long as we focus on distances larger than β , the initial gapped state can be approximated by a boundary state of undeformed conformal Hamiltonian:

$$|\psi_0\rangle = \mathcal{N} e^{-\frac{\beta}{4} H} |\text{B}\rangle, \quad (7.73)$$

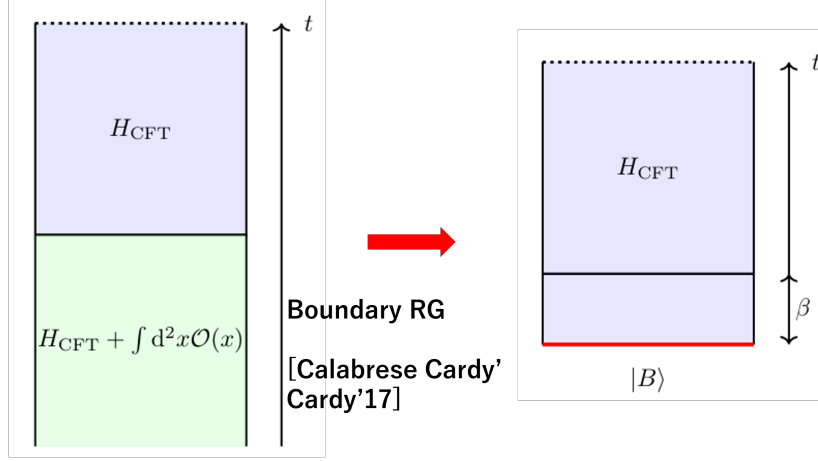
where $\mathcal{N}$ is a normalization constant. And we can determine which boundary states prepared by the optimization [202]. In precisely, we consider a variational states

$$|\{\mathcal{N}_a\}, \{\beta_a\}\rangle := \sum_a \mathcal{N}_a e^{-\frac{\beta_a}{4} H_{\text{CFT}}} |\text{B}_a\rangle \quad (7.74)$$

where a is the index of the physical boundary states,

$$|\text{B}_a\rangle = \sum_h \frac{S_a^h}{\sqrt{S_0^h}} ||h\rangle\rangle, \quad (7.75)$$

with Ishibashi states $||h\rangle\rangle$ and modular S matrix. The variational parameters are



$$|\psi(t)\rangle = e^{-iH_{\text{CFT}}t} |0_{\text{gapped}}\rangle \quad |\psi(t)\rangle = e^{-iH_{\text{CFT}}t} e^{-\frac{\beta}{4}H_{\text{CFT}}} |B\rangle$$

Figure 25: Path-integral representation of the quantum quench.

chosen by minimizing

$$\lim_{N_{\text{dof}} \rightarrow \infty} \frac{1}{N_{\text{dof}}} \frac{\langle \{\mathcal{N}_a\}, \{\beta_a\} | H_{\text{deformed}} | \{\mathcal{N}_a\}, \{\beta_a\} \rangle}{\langle \{\mathcal{N}_a\}, \{\beta_a\} | \{\mathcal{N}_a\}, \{\beta_a\} \rangle}. \quad (7.76)$$

The degree of freedom depends on the thermodynamic limit we consider. It is discussed that this optimization procedure works well in unitary minimal models. Thus, for $t > 0$, the quantum state is given by

$$|\psi(t)\rangle = \mathcal{N} e^{-itH_{\text{CFT}} - \frac{\beta}{4}H_{\text{CFT}}} |B\rangle. \quad (7.77)$$

We now discuss the time evolution of the entanglement entropy in this quantum quench protocol [201]. We focus on a two-dimensional conformal field theory defined on the real line. We take subsystem A to be the interval $[-\frac{l}{2}, \frac{l}{2}]$.

The quantity $\text{Tr}_A[\rho_A^n]$ is obtained from n replicas of this path integral. Using twist operators, we can evaluate this quantity:

$$\text{Tr}_A[\rho_A^n] = \langle B | e^{-\frac{\beta}{4}H_{\text{CFT}}} \bar{\sigma}(t, -\frac{l}{2}) \sigma(t, \frac{l}{2}) e^{-\frac{\beta}{4}H_{\text{CFT}}} |B\rangle. \quad (7.78)$$

where $\sigma(t, x)$ are twist fields with conformal dimension $\Delta_n = \frac{c}{12}(n - \frac{1}{n})$. After Wick rotation,

$$\text{Tr}_A[\rho_A^n] = \langle B | e^{-\frac{\beta}{4}H_{\text{CFT}}} \bar{\sigma}(z_1) \sigma(z_2) e^{-\frac{\beta}{4}H_{\text{CFT}}} |B\rangle, \quad (7.79)$$

where the complex coordinate is defined as $z = it_E + x$, and the twist fields are inserted at $z_1 = it_E - \frac{l}{2}$, $z_2 = it_E + \frac{l}{2}$.

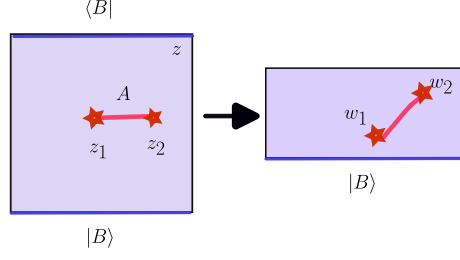


Figure 26: Conformal mapping from the strip to the upper half-plane.

To evaluate the twist-operator correlator, we map the strip on which z resides to the upper half-plane via

$$z = \frac{\beta}{2\pi} \log w. \quad (7.80)$$

Using the doubling trick, the two-point function on the upper half-plane becomes a four-point function on the complex plane:

$$\text{Tr}_A[\rho_A^n] = \left(\frac{dw_1}{dz_1} \right)^{\Delta_n} \left(\frac{dw_2}{dz_2} \right)^{\Delta_n} \langle \bar{\sigma}(w_1) \sigma(w_2) \bar{\sigma}(\bar{w}_1) \sigma(\bar{w}_2) \rangle. \quad (7.81)$$

Using the Ward–Takahashi identity, this four-point function can be evaluated when $\beta \ll t, l$:

$$\langle \bar{\sigma}(w_1) \sigma(w_2) \bar{\sigma}(\bar{w}_1) \sigma(\bar{w}_2) \rangle \approx c_n \left(\frac{|w_1 - \bar{w}_2| |w_2 - \bar{w}_1|}{|w_1 - \bar{w}_1| |w_2 - \bar{w}_2| |w_1 - w_2| |\bar{w}_1 - \bar{w}_2|} \right)^{2\Delta_n}, \quad (7.82)$$

where c_n is a constant independent of t and l [77]. Substituting $t_E = it + \frac{\beta}{4}$, one finds

$$\text{Tr}_A \rho_A^n = c'_n \left[\left(\frac{\beta\pi}{4} \right)^2 \frac{\cosh\left(\frac{2\pi l}{\beta}\right) + 2 \cosh\left(\frac{2\pi t}{\beta}\right)}{\cosh^2\left(\frac{\pi l}{\beta}\right) \cosh^2\left(\frac{2\pi t}{\beta}\right)} \right]^{2\Delta_n}. \quad (7.83)$$

In the limit $\beta \ll t, l$, the entanglement entropy becomes

$$S_A(t) = \begin{cases} \frac{c}{3} \frac{2\pi t}{\beta}, & t < \frac{l}{2}, \\ \frac{c}{3} \frac{\pi l}{\beta}, & t > \frac{l}{2}. \end{cases} \quad (7.84)$$

■ Gravity Duals

What is the gravitational dual process corresponding to the relaxation of a non-equilibrium state prepared by a quantum quench? If a holographic conformal field theory possessing a gravity dual satisfies thermalization conditions such as the ETH, then the late-time steady state is expected to be close to an equilibrium ensemble such as the microcanonical or canonical distribution. Recalling that static black hole

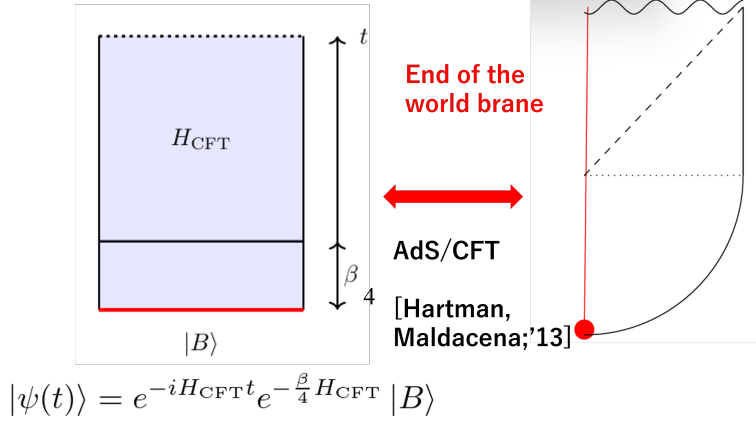


Figure 27: Gravity dual of $|\text{TFD}(t)\rangle$ and its $\mathbb{Z}_2$ quotient, corresponding to the time evolution of a pure state.

geometries correspond to such equilibrium distributions, one expects the relaxation process to correspond, on the gravity side, to the gravitational collapse of an object with energy that eventually forms a black hole.

With this in mind, in this section we re-derive on the gravity side the discussion of the quantum quench in previous sections. Although there exist approaches based on the Vaidya geometry as the gravitational dual of a quantum quench [151, 203], here we follow the argument given by Hartman and Maldacena [204], which employs the eternal black hole. To construct the gravity dual of the path-integral state $|\psi(t)\rangle = \mathcal{N} e^{-\frac{\beta}{4}H_{\text{CFT}} - itH_{\text{CFT}}} |B\rangle$, recall that the gravity dual of the TFD state is the maximally extended eternal black hole.¹⁸ By performing a $\mathbb{Z}_2$ identification between the left and right sides of this eternal black hole, one can construct the gravity dual of the time evolution of a boundary-state initial condition [24]; see Fig. 27.

This eternal black hole possesses a Killing vector generating opposite time translations on the left and right boundaries, but it is not invariant under time translations that shift both sides in the same direction. Thus, such time evolution is interpreted as a nontrivial spacetime evolution. This corresponds to using the Hamiltonian $H = H_L + H_R$ instead of $H = H_L - H_R$ to generate nontrivial time evolution of the TFD state. Indeed,

$$|\text{TFD}(t)\rangle := e^{-i(H_L + H_R)t} |\text{TFD}\rangle = \sum_n e^{-2itE_n + \frac{E_n}{2}\beta} |E_n\rangle_L |E_n\rangle_R \quad (7.85)$$

is different from the original $|\text{TFD}\rangle$ which is consistent with non-trivial time evolution of $|\psi(t)\rangle$.

¹⁸In this section, the same symbol κ is used both for the regularization parameter of the boundary state and for a time-like coordinate inside the black hole. The former is set to $\kappa = \pi/2$.

Motivated by this, let us consider a more general spacetime of the form

$$ds^2 = -g(\rho)^2 dt^2 + h(\rho)^2 (dx_1^2 + \cdots + dx_{d-1}^2) + d\rho^2. \quad (7.86)$$

For simplicity, we set the black hole temperature to $\beta = \frac{1}{2\pi}$. We assume that g and h behave near the horizon ($\rho \sim 0$) as

$$g(\rho) = \rho + o(\rho^3), \quad h(\rho) = C + o(\rho^2), \quad (7.87)$$

and asymptotically grow exponentially toward the boundary $\rho \rightarrow \infty$. In the case of the TFD state, the relevant metric is that of an AdS_{d+1} black hole:

$$h(\rho) = \frac{2}{d} \left(\coth \frac{d\rho}{2} \right)^{\frac{2}{d}}, \quad g(\rho) = h(\rho) \tanh \frac{d\rho}{2}. \quad (7.88)$$

Because the geometry is not static in the time coordinate of interest, the HRT prescription must be used to compute the entanglement entropy. In this situation, HRT surfaces may pass inside the black hole. To go path through the blackhole interior by the Rindler type coordinate (7.86), we need to use the complex $\rho \in \mathbb{C}$. In the original reference [204], they consider this method. Here we discuss another way computations. By using the global coordinates, we can escape the complex coordinate. To demonstrate this we consider a concrete examples, we consider Kruskal coordinates in AdS_3 . First of all, let us write the standard Rindler coordinate

$$ds^2 = -r_H^2 \sinh^2 \rho dt^2 + d\rho^2 + r_H^2 \cosh^2 \rho d\theta^2. \quad (7.89)$$

This coordinate and the Janus coordinates are related by the transformations

$$\begin{aligned} \sinh y &= \sinh \rho \cosh r_H t, \\ \cosh y \sin \tau &= \sinh \rho \sinh r_H t, \\ \cosh y \cos \tau &= \cosh \rho. \end{aligned} \quad (7.90)$$

and obtain the metric

$$ds^2 = dy^2 + \cosh^2 y (-d\tau^2 + r_H^2 \cos^2 \tau d\theta^2) \quad (7.91)$$

See the Penrose diagram 28. Then we consider the geodesics connecting two point $P_1 = (\tau_\infty, y_\infty)$ and $P_2 = (\tau_\infty, -y_\infty)$. See Fig.35 for intuition. The asymptotic values

τ_∞ and y_∞ are related to the boundary values $\frac{e^{\rho_\infty}}{2} \approx \frac{2}{\epsilon}$, $\tau_\infty \approx t_b$ by

$$\begin{aligned} f(y_\infty) &\approx \frac{1}{4} e^{2\rho_\infty} \cosh^2 r_H t_b \\ \tan \tau_\infty &\approx \sinh r_H t_b \end{aligned} \quad (7.92)$$

where t_b is the time at which the subsystem is defined. Then, we consider the geodesics which connects two asymptotic boundaries. The geodesics is along $\theta = \text{const.}$. Thus, we consider the curve $\tau = \tau(y)$. As discussed in Appendix A.1, since the BTZ blackhole is locally equivalent to the vacuum AdS spacetime we can easily compute the geodesic length. In this case,

$$\text{Dist}(P_1 : P_2) = 2L_{\text{AdS}} y_\infty = 2L_{\text{AdS}} \log \left(\frac{1}{\epsilon} \cosh \left(\frac{2\pi t}{\beta} \right) \right) \quad (7.93)$$

which means the linear growth of entanglement

$$S_A = \frac{c}{3} \log \left(\frac{1}{\epsilon} \cosh \left(\frac{2\pi t}{\beta} \right) \right) \underset{t \gg \beta}{=} \frac{c}{3} \frac{2\pi t}{\beta} + \dots \quad (7.94)$$

and we see the linear growth of entanglement entropy.

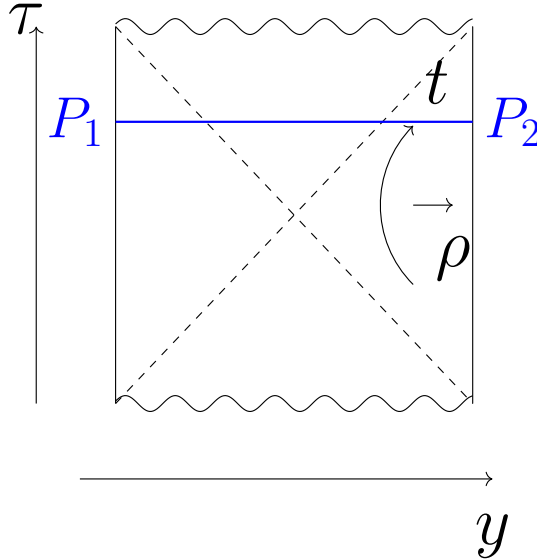


Figure 28: Picture of setting in quantum quench.

8 ETH in Holographic Theories

In this section, we try to present concrete examples that satisfy the conditions (6.58) and (6.61), which, together with some other natural assumptions, lead to the double exponential scaling (5.60). We find if we assume some mathematical inequalities

which we do not have complete proof, we have double exponential scalings for holographic theories in $1/N$. We focus on simple toy models of holographic theories that are dual to QFTs on AdS black holes and to setups with wormholes. Throughout, we regard the thermodynamic limit as the large N limit and take $\mathcal{A}$ to be the set of light primary operators. Rather than working with specific top-down constructions that originate from superstring theory, we consider bottom-up holographic models. The structure of the discussion is as follows:

1. We first analyze a bulk free theory on AdS black holes and on a certain class of wormhole geometries that control the late-time behavior of two-point functions.
2. Next, we incorporate $1/N$ corrections.
3. Finally, we briefly comment on the role of backreaction and on possible wormhole contributions. This part is heuristic, and we will not present a fully controlled computation.

These considerations together suggest that

If there exists wormholes which contribute to two point functions with order $O(1/\text{Poly}(D_{E,\Delta E}))$ and any other larger contribution to higher order correlation functions, the boundary theories show strong ETH.

Notice that our derivation does not depend on

- theory randomness. we can use for single specific Hamiltonians
- spacetime dimension

8.1 Bulk free theories

8.1.1 Clustering Property

A very simple realization of the clustering property for long-time-averaged thermal Wightman correlators (6.58) arises in generalized free field theories exhibiting exact large N factorization. Prior to invoking large N factorization itself, it is convenient to rewrite $G_{\text{can},\beta}^{(2q)}$ in the conventional language of thermal correlators:

$$G_{\text{can},\beta}^{(2q)}(t_1, t_2, \dots, t_{2q}) = \text{Tr} \left[\rho_{\text{can}}^{(\beta)} \hat{\mathcal{O}}_{\text{can}}(\tau_1) \hat{\mathcal{O}}_{\text{can}}(\tau_2) \cdots \hat{\mathcal{O}}_{\text{can}}(\tau_{2q}) \right], \quad (8.1)$$

with

$$\tau_k = t_k - i a_k \beta, \quad a_k = \frac{2q - k}{2q}, \quad k = 1, \dots, 2q. \quad (8.2)$$

Following the standard discussion of Wick rotation [205], such real-time Wightman correlators are obtained by analytic continuation from Euclidean time-ordered correlators,

$$G_{E,\beta}^{(2q)}(t_{E1}, t_{E2}, \dots, t_{E2q}) = \text{Tr} \left[\rho_{\text{can}}^{(\beta)} \mathcal{T}_E \left(\hat{\mathcal{O}}_{\text{can}}(t_{E1}) \hat{\mathcal{O}}_{\text{can}}(t_{E2}) \cdots \hat{\mathcal{O}}_{\text{can}}(t_{E2q}) \right) \right], \quad (8.3)$$

via the continuation $t_{Ek} = \frac{2q-k}{2q}\beta + it_k$.

Let us now assume that Wick contraction is valid:

$$\begin{aligned} G_{\text{can},\beta}^{(2q)}(t_1, t_2, \dots, t_{2q}) &= \text{Tr} \left[\rho_{\text{can}}^{(\beta)} \hat{\mathcal{O}}_{\text{can}}(\tau_1) \hat{\mathcal{O}}_{\text{can}}(\tau_2) \cdots \hat{\mathcal{O}}_{\text{can}}(\tau_{2q}) \right] \\ &= \sum_{\text{Pairs}} \prod_{(i,j) \in \text{Pairs}} \text{Tr} \left[\rho_{\text{can}}^{(\beta)} \hat{\mathcal{O}}_{\text{can}}(\tau_i) \hat{\mathcal{O}}_{\text{can}}(\tau_j) \right], \end{aligned} \quad (8.4)$$

where Pairs denotes the collection of all possible pairings appearing in Wick contractions, and the sum runs over every such pairing. We take N to be large but finite, so that exact factorization is not guaranteed in a strict sense, in contrast to the idealized $N \rightarrow \infty$ limit. Nevertheless, as a toy model, we temporarily assume that exact large N factorization holds even at large but finite N . Corrections of order $1/N$ will be incorporated later.

Under this idealized exact factorization, the clustering of long-time-averaged thermal correlators follows immediately:

$$\begin{aligned} \overline{G_{\text{can},\beta}^{(2q)}(t_1, t_2, \dots, t_{2q})}^T &= \sum_{\text{Pairs}} \prod_{(i,j) \in \text{Pairs}} \overline{\text{Tr} \left[\rho_{\text{can}}^{(\beta)} \hat{\mathcal{O}}_{\text{can}}(\tau_i) \hat{\mathcal{O}}_{\text{can}}(\tau_j) \right]}^T \\ &= (2q-1)!! \times \left(\overline{G_{\text{can},\beta}^{(2)}(\tau_i, \tau_j)}^T \right)^q \\ &\leq q! \left(2 \overline{G_{\text{can},\beta}^{(2)}}^T \right)^q. \end{aligned} \quad (8.5)$$

This expression has precisely the form of (6.58), and hence implies sub-Gaussian behavior with variance parameter $\sigma^2 = \overline{G_{\text{can},\beta}^{(2)}}^T$.

8.1.2 Quantum Mixing

As explained in section 7.1, thermal two-point functions decay with time, and at sufficiently late times their magnitude becomes non-perturbatively small due to wormhole contributions. This behavior is exactly what is encoded in the quantum mixing condition (6.61).

A subtle issue, however, arises from conformal symmetry. Owing to conformal symmetry and its representation theory, the non-degeneracy and non-resonance con-

ditions (5.46) are strictly violated. For example, there exist many pairs of energy eigenstates whose energy difference is exactly one. There are three natural strategies to deal with this complication.

The first possibility is to relax the assumptions entering Theorem 5.4 and Theorem 5.4. A modified set of inequalities appropriate to this situation is presented in Appendix C. A second approach is to weakly deform the conformal field theory by adding irrelevant operators that slightly break conformal symmetry. Such irrelevant deformations are not expected to significantly modify the bulk interior geometry, in particular the black hole background, and therefore the qualitative structure of quantum mixing (as well as the clustering property) should persist. A third and conceptually clean option is to restrict attention to conformal primary operators and primary states. In conformal field theories, ETH can be formulated entirely in terms of primaries [128]¹⁹. To implement this idea, we introduce the set of primary energy eigenstates with energies (equivalently, conformal dimensions) concentrated around a given value E^p , namely those satisfying $E \leq E^p \leq E + \Delta E$. We denote this set by $\mathcal{E}_{E,\Delta E}^p$ and define a microcanonical ensemble restricted to primary states. Expectation values in this primary microcanonical ensemble are given by

$$\langle \mathcal{O} \rangle_{P,\text{mc}}^{E,\Delta E} = \frac{1}{D_{E,\Delta E}^P} \sum_{|E_m^p\rangle \in \mathcal{E}_{E,\Delta E}^p} \langle E_m^p | \mathcal{O} | E_m^p \rangle. \quad (8.6)$$

We then adopt an ETH ansatz tailored to primary operators:

$$\begin{aligned} \text{Diagonal} \quad & \left| \langle E_m^p | \mathcal{O}^p | E_m^p \rangle - \langle \mathcal{O} \rangle_{\text{mc}}^{E,\Delta E} \right| = e^{-O(S_{\text{mc}}(E))}, \\ \text{Off-diagonal} \quad & |\langle E_m^p | \mathcal{O}^p | E_l^p \rangle| = e^{-O(S_{\text{mc}}(E))}. \end{aligned} \quad (8.7)$$

for any primary states $|E_m^p\rangle, |E_l^p\rangle \in \mathcal{E}_{E,\Delta E}^p$ and any light primary operators $\mathcal{O}^p$. In complete analogy with the standard ETH, this ansatz implies

$$\left| \langle E_m^p | \mathcal{O} | E_m^p \rangle - \langle \mathcal{O} \rangle_{P,\text{mc}}^{E,\Delta E} \right| < \|\mathcal{O}\|_\infty \left(\epsilon + \sqrt{D_{E,\Delta E}^P \Delta_{P,\text{diag,ETH}}^\epsilon[\mathcal{O}]} \right), \quad (8.8)$$

where $D_{E,\Delta E}^P = |\mathcal{E}_{E,\Delta E}^p|$ and $\Delta_{P,\text{diag,ETH}}^\epsilon[\mathcal{O}]$ is defined as

$$\Delta_{P,\text{diag,ETH}}^\epsilon[\mathcal{O}] := \text{Prob}_{\mathcal{E}_{E,\Delta E}^p} \left[\left| \langle E_m^p | \mathcal{O} | E_m^p \rangle - \langle \mathcal{O} \rangle_{P,\text{mc}}^{E,\Delta E} \right| > \epsilon \right]. \quad (8.9)$$

An analogous inequality holds for off-diagonal matrix elements. One can further show that the second moment of the one-point function with respect to the uniform distribution over primary energy eigenstates is bounded from above by the corre-

¹⁹In that work, the thermodynamic limit is taken as the large volume limit.

sponding second moment taken over the full set of energy eigenstates. As a result, we obtain quantum mixing for primary microcanonical correlators:

$$\begin{aligned} \frac{1}{D_{E,\Delta E}^P} \sum_{|E_m^p\rangle \in \mathcal{E}_{E,\Delta E}^P} \left(\langle E_m^p | \left(\mathcal{O}^p - \langle \mathcal{O}^p \rangle_{P,\text{mc}}^{E,\Delta E} \right) | E_m^p \rangle \right)^2 &\leq \mathbb{E}_{\mathcal{E}_{E,\Delta E}} \left[\langle E_m | \hat{\mathcal{O}}_{\text{mc}} | E_m \rangle^2 \right] \\ &= \overline{G_{\text{mc},E}^{(2)}}^T = e^{-O(S_{\text{mc}}(E))}. \end{aligned} \quad (8.10)$$

For higher moments, if one could establish that microcanonical averages of one-point functions in the two ensembles are exponentially close, then the corresponding results for the full energy-eigenstate microcanonical ensemble would follow directly. At present, however, a fully satisfactory justification of this step is lacking, and we therefore refrain from pursuing this issue further here.

8.2 Adding $1/N$ Corrections

In the previous subsection, we restricted our attention to the generalized free part arising from large N factorization. As emphasized there, this description is exact only in the strict large N limit. In realistic holographic theories with large but finite N , one must include subleading contributions arising in the $1/N$ expansion. Since our goal is to understand how various quantities scale for large yet finite N , we now carefully incorporate these $1/N$ corrections. Physically, these corrections originate from weak interactions among the fields, which generate non-trivial connected correlation functions beyond pure products of two-point functions. In what follows, we analyze whether such interactions modify the assumption (6.58).

8.2.1 Corrections to the Two-Point Function

We begin by examining the $1/N$ corrections to the two-point function. Let us assume that the bulk effective field theory contains a Φ^4 interaction:

$$S_{\text{EFT}} = \int d^{d+1}x \sqrt{-G} \left[\nabla_M \Phi \nabla^M \Phi + m^2 \Phi^2 + \frac{g_4}{4!} \Phi^4 \right] \quad (8.11)$$

where $g_4 = 1/\text{Poly}(N)$. One could equally well include higher polynomial interactions Φ^n with $n = O(N^0)$. Moreover, we will ignore gravitational corrections here and focus solely on the Φ^4 theory in a fixed thermal background for illustration. Using standard QFT techniques, the two-point function can be expressed in terms of the unperturbed propagator and a series of 1PI diagrams. Even in three bulk dimensions, computing the exact quantum corrections on a black hole background is challenging. Nevertheless, we can estimate the late-time behavior of the full two-

point function perturbatively. The relevant corrections include:

- From diagrams of the sunset type, one finds at each order in perturbation theory [178]:

$$\begin{aligned} G_{\text{can},\beta}^{(2)}(t) &= \sum_{n=0}^{m_{\text{max}}} G_{\text{can},\beta}^{(2,n)}(t) \\ G_{\text{can},\beta}^{(2,n)}(t) &= \sum_{k=0}^n c_{n,k} t^k G_{\text{can},\beta}^{(2,0)}(t) \end{aligned} \tag{8.12}$$

where $G_{\text{can},\beta}^{(2,0)}(t)$ denotes the two-point function in the unperturbed theory. Thus, the full two-point function still exhibits the quantum mixing behavior.

- Loop diagrams generate UV divergences, as usual in quantum field theory. After renormalization, the conformal dimension of the operator receives anomalous contributions [206], which in turn induce corrections to the quasi-normal mode spectrum.

8.2.2 Perturbative Computation and Almost Sub-Gaussianity

We now turn to the Wick diagram expansion of the long-time-averaged thermal correlators in perturbation theory, focusing on the $1/N$ corrections.

Let us first consider $p = 2$, i.e. the four-point function. Suppose that the connected Euclidean thermal correlator can be written in terms of Witten diagrams as

$$\begin{aligned} G_{\text{E},\beta}^{(4)}(t_{\text{E}1}, t_{\text{E}2}, t_{\text{E}3}, t_{\text{E}4}) &= \prod_{k=1}^4 \int d^{d+1} X_{\text{E}k} K_{\text{Bb}}(x_{\text{E}1}, X_{\text{E}1}) K_{\text{Bb}}(x_{\text{E}2}, X_{\text{E}2}) \\ &\quad \times K_{\text{Bb}}(x_{\text{E}3}, X_{\text{E}3}) K_{\text{Bb}}(x_{\text{E}4}, X_{\text{E}4}) U(X_{\text{E}1}, X_{\text{E}2}, X_{\text{E}3}, X_{\text{E}4}). \end{aligned} \tag{8.13}$$

Here, $K_{\text{Bb}}(x_{\text{E}}, X_{\text{E}})$ denotes the bulk-to-boundary propagator, and U stands for the sum of all amputated Witten diagrams with four external legs. At large N , we have

a perturbative expansion of the form

$$\begin{aligned}
G_{E,\beta}^{(4)}(t_{E1}, t_{E2}, t_{E3}, t_{E4}) &= G_{\text{can},\beta}^{(2)}(t_{E1}, t_{E2}) G_{\text{can},\beta}^{(2)}(t_{E3}, t_{E4}) + (\text{permutation}) \\
&\quad + G_{\beta,\text{conn}}^{(4)}(t_{E1}, t_{E2}, t_{E3}, t_{E4}) \\
G_{\beta,\text{conn}}^{(4)}(t_{E1}, t_{E2}, t_{E3}, t_{E4}) &:= \prod_{k=1}^4 \int d^{d+1} X_{Ek} K_{\text{Bb}}(x_{E1}, X_{E1}) K_{\text{Bb}}(x_{E2}, X_{E2}) \\
&\quad \times K_{\text{Bb}}(x_{E3}, X_{E3}) K_{\text{Bb}}(x_{E4}, X_{E4}) \\
&\quad \times \Gamma_4(X_{E1}, X_{E2}, X_{E3}, X_{E4}).
\end{aligned} \tag{8.14}$$

Here, Γ_4 is the 1PI four-point vertex. See Figure 29 and Figure 30 for an illustration.

In the semi-classical limit, the bulk-to-boundary propagator has a structure analogous to that of the boundary two-point function:

$$K_{\text{Bb}}(x, X) = K_{\text{Bb}}^{\text{BH}}(x, X) e^{-I_{\text{BH}}} + e^{-\gamma S_{\text{mc}}(E)} K_{\hat{\mathcal{O}}}(X_E). \tag{8.15}$$

After Wick rotation, the boundary points always lie outside the black hole horizon, so the first term exhibits the same type of decay as the boundary correlators:

$$\overline{K_{\text{Bb}}^{\text{BH}}(x, X_E)}^T = 0. \tag{8.16}$$

Consequently, in analogy with the boundary correlators, we assume

$$\overline{K_{\text{Bb}}(x, X_E)}^T = e^{-\gamma S_{\text{mc}}(E)} K_{\hat{\mathcal{O}}}(X_E), \tag{8.17}$$

for some $O(1)$ function $K_{\hat{\mathcal{O}}}(X)$. Substituting these expressions into the four-point Witten diagram, we find

$$\begin{aligned}
\overline{G_{\beta,\text{conn}}^{(4)}}^T &= \left(\prod_{k=1}^4 \int d^{d+1} X_{Ek} \overline{K_{\text{Bb}}(x_k, X_k)}^T \right) \Gamma_4(X_1, X_2, X_3, X_4) \\
&= e^{-4\gamma S_{\text{mc}}(E)} \left(\prod_{k=1}^4 \int d^{d+1} x_k K_{\hat{\mathcal{O}}}(X_k) \right) \Gamma_4(X_1, X_2, X_3, X_4).
\end{aligned} \tag{8.18}$$

Thus, the connected contribution is of order $O\left(\left(\overline{G_{\text{can},\beta}^{(2)}}^T\right)^4\right)$ and is exponentially

suppressed relative to the generalized free part. Altogether, we obtain

$$\begin{aligned}\overline{G_{\text{can},\beta}^{(4)}}^T &= 3\left(\overline{G_{\text{can},\beta}^{(2)}}^T\right)^2 + \Gamma_4\left(\overline{G_{\text{can},\beta}^{(2)}}^T\right)^4, \\ \Gamma_4 &= \frac{1}{(G_{\hat{\mathcal{O}}})^4} \left(\prod_{k=1}^4 \int d^{d+1}x_k K_{\hat{\mathcal{O}}}(X_k) \right) \Gamma_4(X_1, X_2, X_3, X_4).\end{aligned}\tag{8.19}$$

General $2q$ point function

We now extend this discussion to the general $2q$ -point function $\overline{G_{\text{can},\beta}^{(2q)}}^T$ and its decomposition into exact two-point functions and connected four-point contributions (or four-point vertices). From the Wick contraction analysis, we obtain the following decomposition:

$$\overline{G_{\text{can},\beta}^{(2q)}}^T = \sum_{\{n_k\}: \sum_{k=1}^q kn_k=q} \left(\overline{G_{\text{can},\beta}^{(2)}}^T\right)^{n_1} \frac{(2q)!}{\prod_{k=1}^q (n_k)!((2k)!)^{n_k}} \prod_{k=2}^q \left(\Gamma_{2k}\left(\overline{G_{\text{can},\beta}^{(2)}}^T\right)^{2k}\right)^{n_k}.\tag{8.20}$$

where the formula is derived by the combinations. Γ_{2k} are the $2k$ -order 1PI diagrams obtained by similar manner to Γ_4 . For example, Γ_6 involves loop diagram with three $\Gamma_4(X_1, \dots, X_4)$. We assume Γ_{2k} are small in large N and Γ_{2k} are monotonically decrease in k . The power of $\overline{G_{\text{can},\beta}^{(2)}}^T$ is determined as follows: First we have n_1 pairs of external lines which are not connected in each other. Next we consider connect parts of left $(2q - 2n_1)$ lines in n_2 number of 4 vertex diagrams. We continue this procedure up to $2q$ vertex diagrams. Next, we compare the bound (6.21) with the

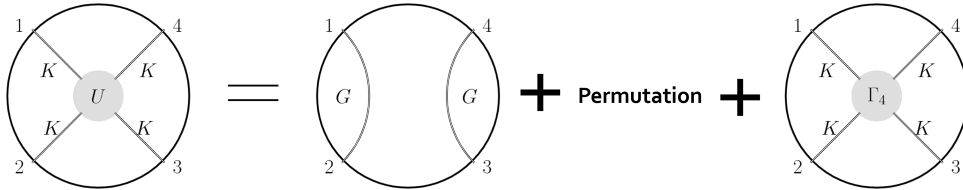


Figure 29: Schematic Witten diagram for the four-point function. Here K denotes the bulk-to-boundary propagator, G stands for the boundary correlators, and Γ_4 represents the sum of 1PI diagrams for the four-point vertex.

expansion (8.20). In doing so, it is natural to identify

$$\sigma^2 = \overline{G_{\text{can},\beta}^{(2)}}^T, \quad \Gamma_{2k} = -\epsilon_k g_{2k}, \quad \epsilon_k = \pm 1.\tag{8.21}$$

With this identification, we are led to the following mathematical conjecture :

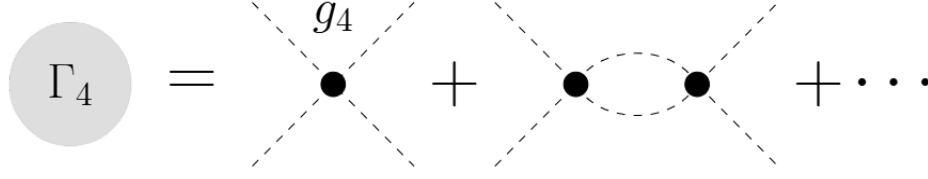


Figure 30: Illustration of the 1PI diagram Γ_4 appearing in the perturbative expansion in powers of g_4 .

Conjecture 2: Consider the two positive functions for vectors $x = (x_1, x_2, x_3, \dots, x_L)$,

$$G^{(2q)}(x) := \sum_{\{n_k\}: \sum_{k=1}^q kn_k = q} \frac{(2q)!}{\prod_{k=1}^q ((2k)!)^{n_k} (n_k)!} x_1^{n_1 - q} \prod_{k=2}^q (x_k)^{n_k} \quad (8.22)$$

$$E^{(2q)}(x) := 2^{q+1} q! \sum_{\{m_k\}: m \leq m_{\max}} \frac{\Gamma(m+q)}{\Gamma(q)} \prod_{k=2}^M \frac{1}{m_k!} (\epsilon_k x_k)^{m_k}.$$

We put $x_1 = \overline{G_{\text{can},\beta}^{(2)}}^T$ and $x_k = \Gamma_{2k} \left(\overline{G_{\text{can},\beta}^{(2)}}^T \right)^{2k}$. For sufficiently small x and for some positive integer $m_{\max}$, we have

$$G^{(2q)}(x) \leq E^{(2q)}(x). \quad (8.23)$$

In Appendix E, we provide numerical evidence supporting this proposition for the case $\epsilon_k = -1$. In our application, we have $\epsilon_k = -1$, which is extremely small and we take $x_k = y^k, k \geq 2$ with $y = \frac{\overline{G_{\text{can},\beta}^{(2)}}^T}{S}$. The numerics suggest that the inequality holds for $M = m_{\max} = \lfloor \frac{q}{2} \rfloor$, and up to $y \sim 1/q^c$ with $c \sim 1$. Since $y = \frac{1}{\text{Poly}(S_{\text{mc}})} e^{-O(S_{\text{mc}})}$, we may thus use the inequality for q up to $q = O\left((S_{\text{mc}}(E))^{\frac{1}{c}}\right) e^{\frac{2}{c}\gamma S_{\text{mc}}(E)}$. Notice that we do not prove the inequality for really large q, M and $m_{\max}$ and thus still open questions. Possibly we may wonder the correlation functions become large because of many diagram but we expect we can bounded them by taking large enough $m_{\max}$

²⁰. For arbitrarily large q , we can still employ a simple, albeit loose, bound:

$$\begin{aligned}\mathbb{E}_{\mathcal{E}_{E,\Delta E}} \left[\langle E_n | \hat{\mathcal{O}} | E_n \rangle^{2q} \right] &= \frac{1}{D_{E,\Delta E}} \sum_{|E_n\rangle \in \mathcal{H}_{E,\Delta E}} \langle E_n | \hat{\mathcal{O}} | E_n \rangle^{2q} \\ &\leq (\|\hat{\mathcal{O}}\|_\infty)^{2q} \\ &\leq (2\|\mathcal{O}\|_\infty)^{2q}.\end{aligned}\tag{8.24}$$

We now invoke Conjecture (D). Setting $\sigma^2 = \overline{G_{\text{can},\beta}^{(2)}}^T$, $g_{2k} = \Gamma_{2k}$, $M = m_{\text{max}} = \lfloor \frac{q}{2} \rfloor$, $\sigma' = 2\|\mathcal{O}\|_\infty$ and $L = O\left((S_{\text{mc}}(E))^{\frac{1}{c}}\right) e^{\frac{2}{c}\gamma S_{\text{mc}}(E)}$, we obtain the scaling

$$\frac{\sigma'^2}{(L+1)\sigma^2} \sim \frac{1}{S_{\text{mc}}(E)^{\frac{1}{c}}} e^{-\left(\frac{2}{c}-1\right)S_{\text{mc}}(E)} = e^{-O(S_{\text{mc}}(E))},\tag{8.25}$$

and by choosing $\epsilon = \sigma^\lambda$, $\lambda < 1$ we find

$$\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}] \leq e^{-\#e^{O(S_{\text{mc}}(E))}} \left(e^{-\#'e^{-O(S_{\text{mc}}(E))}} + e^{-\#''e^{O(S_{\text{mc}}(E))}} \right), \quad \#, \#', \#'' > 0.\tag{8.26}$$

Note that M is originally independent from q and possibly relate to L . However, for computational reason we set M depends on q . Therefore, even after including perturbative $1/N$ corrections, we still recover the double exponential scaling of $\Delta_{\text{diag,ETH}}^\epsilon[\mathcal{O}]$ if the inequality (8.23) is correct. An entirely analogous analysis applies to the off-diagonal quantity $\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}]$. In conclusion, under the assumptions of non-degeneracy and non-resonance (which might be further relaxed) and with assumptions that the conjectured inequalities are valid, this simple holographic toy model, dual to a weakly interacting scalar theory in AdS, continues to exhibit strong ETH.

8.2.3 Discussion: Including Back Reaction and wormhole

In the preceding analysis, we largely ignored gravitational back reaction. In realistic AdS/CFT setups at large but finite N , however, such back reaction effects must eventually be accounted for. In particular, when we consider very large $q = O(S_{\text{mc}}(E))$, products of many light operators should be treated as effectively heavy operators. In this regime, it becomes extremely difficult to analyze correlation functions, since we no longer have a simple effective field theory description on a fixed black hole background at temperature β . Nevertheless, it may be possible to gain some control over these very high-point correlators by changing the “vacuum” with respect to which we define the generalized free fields. Let us explain this idea

²⁰We cannot have Gaussian upper bound for $G^{(2q)}$.

more concretely. Suppose we have a weakly coupled QFT on an AdS black hole with temperature β . From the AdS/CFT perspective, this corresponds to a Fock space obtained via the GNS construction from the canonical ensemble with inverse temperature β [207]. Thus, the effective weakly coupled field theory description is intrinsically state-dependent. On the other hand, one can alternatively view the black hole as a highly excited state of the AdS vacuum. In that picture, we have another Fock space built on the AdS vacuum, and black holes arise from inserting a large number of operators on this vacuum.

We can adopt a conceptually similar viewpoint for the higher-point thermal correlators $G_{\text{can},\beta}^{(2q)}(t_1, t_2, t_3, \dots, t_{2q})$ at finite temperature. Let us consider the regime $t_1, t_2 \gg t_3, \dots, t_{2q}$ and split the operators into the pair at (t_1, t_2) and the remaining $(2q - 2)$ operators. As in the previous discussion, we treat these $(2q - 2)$ operators as creating a large excitation of the black hole, while the first two operators are interpreted as probes propagating on this excited geometry. Concretely, we assume that the following clustering behavior emerges:

$$G_{\text{can},\beta}^{(2q)}(t_1, t_2, t_3, \dots, t_{2q}) \approx G_{\text{Excited}}^{(2)}(t_1, t_2) G_{\text{can},\beta}^{(2q-2)}(t_3, \dots, t_{2q-2}), \quad (8.27)$$

where $G_{\text{Excited}}^{(2)}(t_1, t_2)$ is the two-point function evaluated in the excited black hole background. We now outline a concrete implementation of this picture. To model the large excitation, we consider the AdS Vaidya metric sourced by a fluid whose stress tensor scales as $O\left(\frac{1}{G_N}\right)$. The AdS Vaidya metric takes the form

$$ds^2 = -\left(\frac{r^2}{L_{\text{AdS}}^2} - M(v)\right)dv^2 + 2dvdr + r^2 d\phi^2. \quad (8.28)$$

The specific profile $M(v)$ is determined once the bulk fluid stress tensor is specified, but the detailed form will not be important here. We assume that by the time we insert the probe operators at late times, the geometry has relaxed to a new equilibrium black hole, so that $G_{\text{Excited}}^{(2)}(t_1, t_2)$ can be approximated by the two-point function in the new equilibrium state. Consequently, $G_{\text{Excited}}^{(2)}(t_1, t_2)$ again decays exponentially. Hence,

$$\begin{aligned} & \lim_{T_2 \rightarrow \infty} \lim_{T_1 \rightarrow \infty} \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} dt_1 dt_2 G_{\text{can},\beta}^{(2q)}(t_1, t_2, \dots, t_{2q}) \\ &= \overline{G_{\text{can},\beta'}^{(2)}}^T G_{\text{can},\beta}^{(2q-2)}(t_3, \dots, t_{2q-2}) \end{aligned} \quad (8.29)$$

where β' denotes the temperature of the final equilibrium black hole. Repeating the same reasoning for $G_{\text{can},\beta}^{(2q-2)}(t_3, \dots, t_{2q-2})$, we obtain the clustering property (6.58). We stress that this argument, which relies on heuristic features of AdS/CFT, is not

rigorous and remains to be established in a more systematic manner. We leave a detailed treatment of this mechanism to future work.

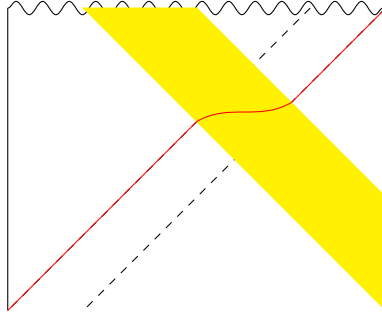


Figure 31: Schematic picture of an energy injection that distorts the black hole horizon. The yellow region represents a shock-wave-like fluid created by the large number of operator insertions. We evaluate the two-point function of probe operators inserted well above this region.

One might also hope to derive similar recursion relations for thermal correlators purely from symmetry considerations. Suppose, again, that we are interested in the long-time behavior in the regime where $t_1, t_2 \gg t_3, \dots, t_{2q}$. In frequency space, after Fourier transformation, the late-time regime is controlled primarily by the analytic structure near zero frequency, $\omega \sim 0$. The factorization pattern suggested by (8.29) then resembles soft theorems in quantum field theory, typically associated with massless gauge symmetries. In our context, one might expect recursion relations for thermal correlators to follow from a soft theorem associated with an emergent symmetry in the effective long-time theory. Possible candidates include the symmetries that arise in hydrodynamics and the asymptotic symmetries of black hole horizons, which have recently been discussed in [208]. Developing this perspective is a natural direction for future research.

Finally, we comment on wormhole contributions, which we have mostly neglected so far. If such wormholes are taken into account, then at sufficiently late times the correlation functions are dominated by wormhole-induced contributions, and the long-time average of the correlators may not simply follow the form (8.20). As emphasized in [38], however, the non-decaying part of the correlation function arises from worldlines that wind around the wormhole cycles, and these contributions scale as $O(e^{-3S_{\text{mc}}(E)})$, which is still compatible with (6.57). At even later times, a larger family of wormhole saddles contributes, and the correlator develops a plateau of order $O(e^{-2S_{\text{mc}}(E)})$. These qualitative expectations are rather naive, and a concrete analysis, especially in higher dimensions, is still lacking. Clarifying this behavior is an important open problem.

Part III

Open quantum systems and Gravity

9 Open Quantum Systems and Their Importance in Holography

9.1 Double Holography and Dissipation

So far, we have considered black holes in AdS spacetime and the information paradox related to these black holes. We saw that they are closely connected to the statistical mechanics of closed systems.

However, in Section 16, we discussed that the original information problem involves evaporating black holes which are asymptotically flat. There are some similarity and big differences between (large) AdS black holes and evaporating black holes:

1. They both have black hole horizons. Correspondingly, we have black hole thermodynamics²¹ and quasinormal modes. After some time scale, we lose information if we are stuck in the single black hole saddle.
2. Large AdS black holes have positive heat capacity, whereas evaporating black holes do not. The thermodynamic instability induces evaporation.
3. The main reason for these differences is the existence of the conformal boundary in asymptotic AdS. Hawking radiation basically occurs because of the existence of two distinct time coordinates arising from the Rindler structure of black holes. The Rindler structure is highly universal for any black hole. However, if we have a boundary, the Hawking radiation can be reflected and return to the horizon in finite time. Then, naively, there is local detailed balance around the horizon and the total configuration is stabilized. In asymptotically flat spacetime, there is no solid boundary and the radiation escapes to infinity, making the black hole unstable.

Suppose now we try to discuss the holography of evaporating black holes to tackle the original Hawking information problem. There are several possibilities:

1. There are small AdS black hole saddles in AdS spacetime. These black holes are small enough to completely evaporate before the Hawking radiation returns. In principle, we can discuss them within the usual AdS/CFT correspondence; however, they always appear as subdominant saddles compared to the large AdS black hole and thermal AdS. In this sense, they are difficult to treat.

²¹Possibly assuming that the evaporation process is very slow. In that case, we may treat the evaporating black hole at each moment as some Schwarzschild black hole with asymptotically flat structure.

2. We can consider holography for flat spacetimes directly. Recently there are many works under the names of celestial holography and Carrollian holography (See [209] and [210] for lecture). One difficulty in this direction is that they describe a 4d/2d duality and we need to consider four-dimensional black holes, which are very difficult to analyze by hand. The difficulty is mostly related to the non-trivial Weyl tensor, and the solutions to the perturbation wave equations are given by Heun functions, which are significantly more complicated than hypergeometric functions.
3. There are ways to obtain evaporating black holes by deforming AdS/CFT. We consider adding a flat spacetime bath to the AdS spacetime and gluing them. From the CFT perspective, this gluing procedure corresponds to introducing dissipation.

In this section, we focus on the third option. Over the last decade, there has been extensive discussion of how gluing AdS to a bath provides a possible solution to the information paradox. In [41, 211], a holography of gluing geometries is considered. They propose a duality between three theories:

1. d -dimensional AdS gravity coupled to a flat, non-gravitating bath
2. $(d+1)$ -dimensional AdS gravity with an EOW brane and parts of the conformal boundary,
3. d -dimensional holographic boundary CFTs.

This triality is called *double holography*. See the Figure 32. Using double holography, one can derive the Page curve.

To see this, consider the third picture: an AdS black hole glued to a flat bath. Initially we have distinct spacetimes; after some time $t = 0$, we perform a joint quench. We take a subsystem in the flat bath. To compute the entropy of this subsystem, we use the first and third pictures and apply holographic entanglement entropy for AdS/BCFT. After the Page time, we must consider the channel in which the RT surface ends on the EOW brane. In this configuration there is a large entanglement wedge which non-trivially includes some region of the EOW brane. This means, in the second picture, that operators in this region can be reconstructed. This region is called an *island*, and we must incorporate the island contribution to the entanglement entropy. This leads to the *island formula*

$$S_A = \min_I \text{Ext} \left[\frac{\text{Area}[\partial I]}{4G_N} + S_{\text{mat}}[A \cup I] \right]. \quad (9.1)$$

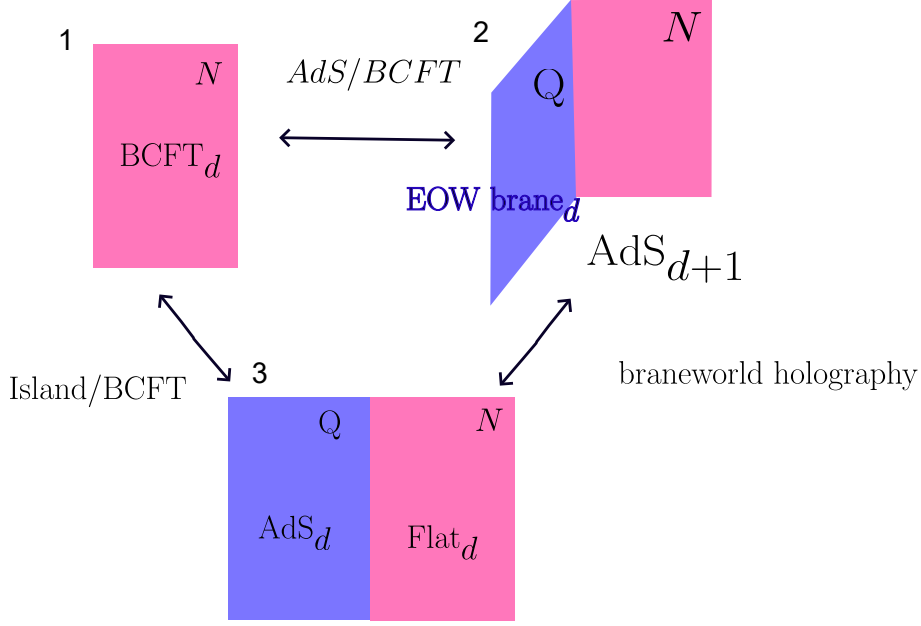


Figure 32: Sketch of the double holography. We have three pictures. 1) d dim. BCFT 2) $(d + 1)$ dim. AdS gravity with EOW brane 3) d dim. AdS brane gravity couples to flat bath.

Later, the island formula was derived purely within gravity, without double holography. In that derivation, one considers a wormhole connecting gravitational replica manifolds [211, 212], similar to double-cone wormholes.

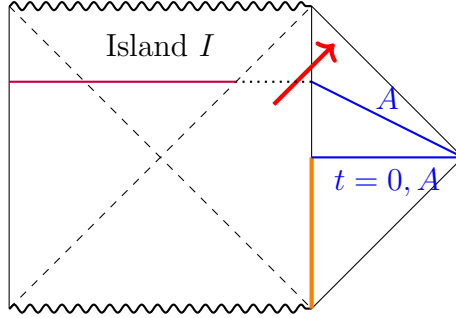


Figure 33: Set up of AdS coupled to bath. The Hawking radiation passes through the AdS boundary (red). The orange line expresses the hard wall which separates two regions in $t \leq 0$.

It seems that the island mechanism works well. However, there are the following subtleties:

1. When we couple an AdS system to a bath, the graviton becomes massive. This is essentially the same as the discussion in Section 4. However, in this case, we turn off dynamical gravity on one side, so there is no massless graviton. Thus, the island discussion above resolves the information paradox only for a

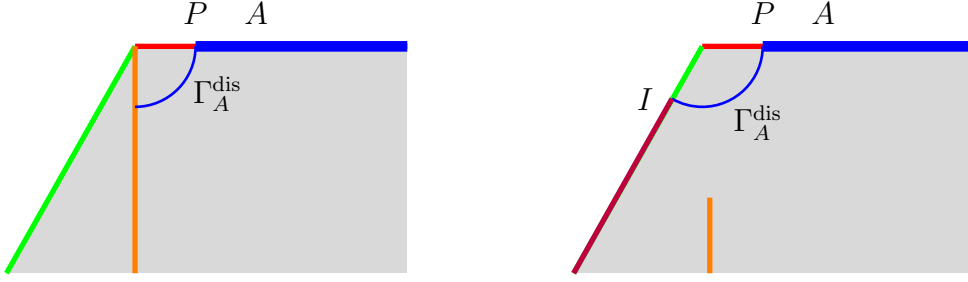


Figure 34: Holographic entanglement entropy in AdS/BCFT with gluing two spacetimes. the orange lines express the brane duals to the orange hard wall in Figure 33. The left figure shows the timeslice in $t = 0$. The right is one of after the page time.

massive graviton [213]. We may think that we can avoid the massive graviton by taking the flat limit, which corresponds to pushing the EOW brane to the conformal boundary. However, in this limit, from the AdS/BCFT perspective, the holographic entanglement entropy of the disconnected channel has a doubled UV divergence compared to the connected contribution, and is never favored. We need to understand whether the effect of mass is truly important in the gravitational path integral around the Page time. This is a highly dynamical problem that must be taken seriously.

2. Possibly we consider the entire flat space gravitational systems. However, because of the gauge constraint of the gravity, the definition of sub region is ambiguous.
3. Most computations of replica wormholes are based on Euclidean methods, with the number of Hawking quanta interpreted as time. To be precise, one should consider gravitational Schwinger–Keldysh computations. In that case, there is a new contribution called the bra-ket wormhole, which connects the bra and ket states in the gravitational path integral.
4. The detailed mechanism of double holography is not fully understood. In particular, we do not know what happens to the $(d - 1)$ -dimensional holographic CFT that is dual to the d -dimensional AdS region in the second picture. Naively, we deform the CFT with some dissipation. The holography of open systems is itself an interesting subject, since we may obtain theoretical predictions for strongly coupled open systems.
5. Islands in dS: Actually, it is known that the island formula (9.1) does not work in some set up involving de Sitter gravity. In [9], we consider spacetime dS_2 couple to flat bath similar to Figure 33 instead of AdS blackhole. By naively applying (9.1), we found there is no nice solution of optimization. In

concrete, we consider double holography with AdS_3 bulk with dS EOW brane and find that HRT surface for interval intersects and its non-extremal points. By considering the original treatment of one point function in AdS/BCFT we can obtain the correct non-extremal island.

In this thesis, we discuss parts of these questions based on our papers [2–5].

9.2 Open Quantum Systems

Before moving to the holographic discussion, we give a short review of open quantum systems [42]. Quantum open systems are systems in which we divide the degrees of freedom into a main system and a bath:

$$\begin{aligned} H_{\text{tot}} &= H_S + H_B + H_{\text{int}}, \\ \mathcal{H}_{\text{tot}} &= \mathcal{H}_S \otimes \mathcal{H}_B. \end{aligned} \tag{9.2}$$

We consider a total system evolving unitarily with e^{-iHt} ²².

The main focus in the literature is the time evolution of the reduced density matrix of the system, $\rho_S(t)$. In general, the time evolution is very complicated, but under some assumptions we can approximate it by a simple equation called the Lindblad equation or the GKSL equation:

$$\frac{\partial}{\partial t} \rho_S(t) = -i[H_S, \rho_S(t)] - \sum_k \left(L_k \rho_S L_k^\dagger - \frac{1}{2} \{L_k L_k^\dagger, \rho_S(t)\} \right). \tag{9.3}$$

This Lindblad equation also arises in quantum dynamics with continuous measurement. For example, as an effective description of continuous measurement, we can consider a stochastic Schrödinger equation

$$\begin{aligned} d(|\psi(t)\rangle) &= -iH |\psi(t)\rangle dt - \frac{1}{2} \sum_k (L_k L_k^\dagger - \langle L_k L_k^\dagger \rangle_{\psi(t)}) |\psi(t)\rangle dt \\ &\quad + \sum_k \left(\frac{L_k |\psi(t)\rangle}{\langle L_k^\dagger L_k \rangle_{\psi(t)}} - |\psi(t)\rangle \right) \cdot dN_k, \end{aligned} \tag{9.4}$$

where dN_k is a Poisson increment and thus is a classical random variable following

$$\mathbb{E}[dN_k] = \langle L_k^\dagger L_k \rangle_{\psi(t)} dt, \quad dN_k dN_l = \delta_{kl} dN_k. \tag{9.5}$$

Each realization $\{dN_k\}$ is called a trajectory and describes a specific set of measure-

²²Even if the systems we are interested in are not unitary and are given by some generic CPTP map, by adding auxiliary systems to the bath we always obtain unitary time evolution.

ment outcomes. We then consider the averaged density matrix

$$\rho(t) = \mathbb{E}[\lvert\psi(t)\rangle\langle\psi(t)\rvert]. \quad (9.6)$$

One can easily confirm that the averaged density matrix follows the Lindblad equation. In this sense, the stochastic equation contains much more information than the Lindblad equation, particularly the correlations between trajectories, which are important in measurement-induced phase transitions.

Now consider another expression of the Lindblad equation or stochastic Schrödinger equation. We can write

$$\frac{\partial}{\partial t}\rho(t) = -i[H_{\text{eff}}, \rho(t)] + \sum_k \gamma_k L_k \rho L_k^\dagger, \quad (9.7)$$

where $H_{\text{eff}} = H - \frac{i}{2} \sum_k L_k L_k^\dagger$. Then we can divide the dynamics into two parts:

1. **Non-Hermitian dynamics:** evolution under the non-Hermitian Hamiltonian H_{eff} .
2. **Jump term:** the last term is called the jump term. For example, if L_k is a projection onto some classical state k , then $L_k \rho L_k^\dagger$ gives a jump to that specific state. This is indeed what happens in the measurement case.

For the Lindblad equation, it is also useful to consider the doubled picture. We can always map the mixed state $\rho \in \mathcal{S}(\mathcal{H})$ to a pure state $|\rho\rangle \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$:

$$\frac{\partial}{\partial t} |\rho(t)\rangle = \left(-i(H_{\text{eff}})_+ \otimes I + iI_+ \otimes (H_{\text{eff}})_- + \sum_k (L_k)_+ \otimes (L_k)_- \right) |\rho(t)\rangle. \quad (9.8)$$

Similar to the in-out formalism, we can represent the amplitude by path integral as

$$\begin{aligned} \text{Tr}[\rho(t)] &= (I|\rho(t)) = \int \mathcal{D}X_\pm \exp(iS_{\text{Lindblad}}[X_+, X_-]), \\ S_{\text{Lindblad}}[X_+, X_-] &= S_0^{(+)}[X_+] - S_0^{(-)}[X_-] \\ &\quad + \int_{\mathcal{C}} dt \sum_k \left[L_k[X_+] L_k^\dagger[X_+] + L_k[X_-] L_k^\dagger[X_-] + L_k[X_+] L_k^\dagger[X_-] \right]. \end{aligned} \quad (9.9)$$

Here $S_0^{(\pm)}$ correspond to the actions of H . We see that the jump term is given by the interaction between the forward and backward paths. When we take L_k to be some single-trace operator $\mathcal{O}(x)$, the Lindblad equation is essentially a double-trace deformation [214].

To understand Lindblad dynamics, we consider non-Hermitian dynamics and jump (or measurement) dynamics individually. In the upcoming sections, we discuss their holography.

10 Non-hermitian holography and wormholes

As one of realizations of a traversable wormhole in AdS, we consider a Janus solution with the Janus deformation parameter taken to be imaginary valued (i.e. model A). This model has an advantage that we can clearly understand its CFT dual in terms of the transition matrix of two different thermofield double states.

10.1 Traversable wormholes from Janus solutions

Now we consider the Janus solution with AdS₂ black hole [115, 116, 215–217]

$$\begin{aligned} ds^2 &= dy^2 + f(y)ds_{\text{AdS}_2}^2, \\ f(y) &= \frac{1}{2}(1 + \sqrt{1 - 2\gamma^2} \cosh 2y), \\ ds_{\text{AdS}_2}^2 &= -d\tau^2 + r_H^2 \cos^2 \tau d\theta^2, \quad \left(-\frac{\pi}{2} \leq \tau \leq \frac{\pi}{2}\right). \end{aligned} \tag{10.1}$$

In previous section, in order to keep the metric and scalar real valued, we need to require $0 \leq \gamma^2 \leq \frac{1}{2}$. In this case, we have $\mu_0 > \frac{\pi}{2}$ and the left and right boundary are not causally connected, as depicted in the upper right picture in Fig.35.

Now we allow the scalar field to take imaginary values, though we require the metric remains to be real valued. This condition is satisfied when γ is imaginary i.e. $\gamma^2 < 0$. In this extended case, we can easily show $\mu_0 \leq \frac{\pi}{2}$. Thus the left and right boundary is now causally connected and it becomes a traversable wormhole as depicted in the lower right panel of Fig.35. In this case the scalar field takes imaginary values. This leads to the negative energy background which allows the traversable wormhole.²³ Notice that our metric is real. This is highly non-trivial. To see this for example let us consider the $T\bar{T}$ deformation with complex coupling $\mu \in \mathbb{C}$. Naively we will have complex finite cut-off $r_c \in \mathbb{C}$ and we have complex geometry.

10.2 Computing geodesic length

To explicitly confirm that the solution describes a traversable wormhole when γ takes imaginary value, let us compute the geodesic length between two points, each at one of the two different boundaries.

²³In [217], the Janus black hole solution for a real value of γ can be made traversable if we further deform it by a double trace deformation. In this paper we do not consider any double trace deformation when we talk about Janus solutions.

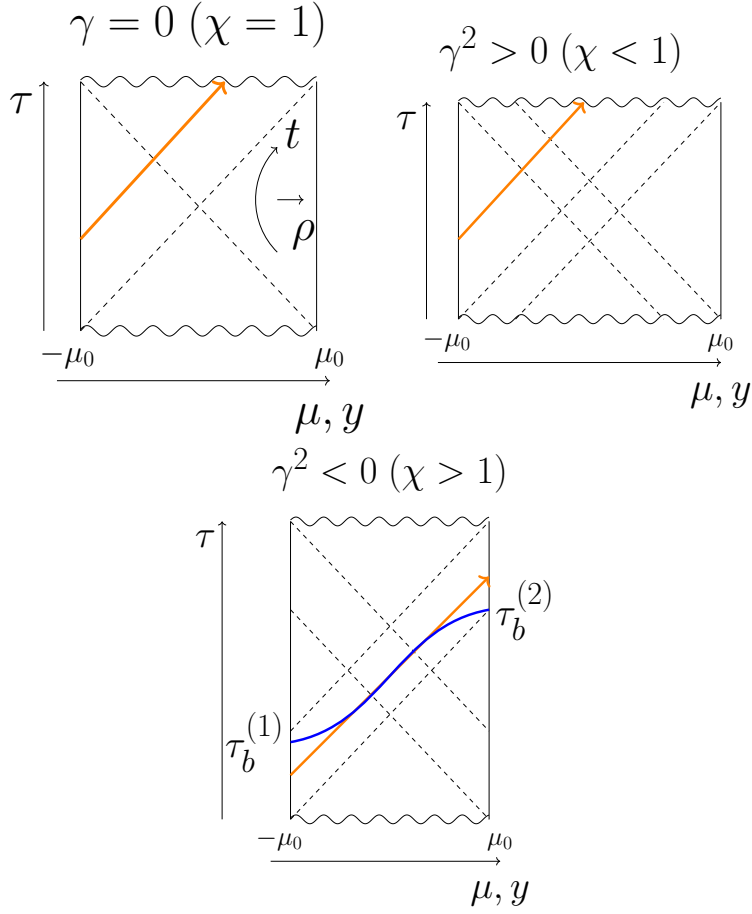


Figure 35: The Penrose diagram of the BTZ black hole (above, left, $\gamma^2 = 0$), time dependent Janus (above right $\gamma^2 > 0$) and the traversable wormhole (below $\gamma^2 < 0$). The orange line denote the null geodesics. For $\gamma^2 < 0$ case, the null geodesics from left reaches to the right boundary. The blue line denotes the geodesics connecting between two points locates in left and right boundaries.

We consider the geodesics connecting two point $(\tau_b^{(2)}, y_\infty)$ and $(\tau_b^{(1)}, -y_\infty)$. See Fig.35 for intuition. The asymptotic values τ_∞ and y_∞ are related to the boundary values $\frac{e^{\rho_\infty}}{2} \approx \frac{2}{\epsilon}$, $\tau_\infty \approx t_b$ by

$$\begin{aligned} f(y_\infty) &\approx \frac{1}{4} e^{2\rho_\infty} \cosh^2 r_H t_b \\ \tan \tau_\infty &\approx \sinh r_H t_b \end{aligned} \tag{10.2}$$

where t_b is the time at which the subsystem is defined. Then, we consider the geodesics which connects two asymptotic boundaries. The geodesics is along $\theta = \text{const.}$. Thus, we consider the curve $\tau = \tau(y)$. The geodesics is determined by

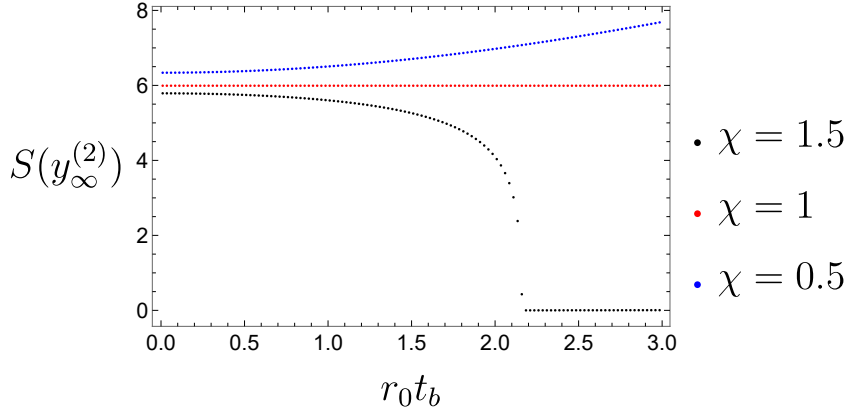


Figure 36: Plots of the geodesics length with $\chi = 1.5$ (black), $\chi = 1$ (red), $\chi = 0.5$ (blue). We set $\epsilon = 0.01$. For $\chi = 1.5 > 1$, we observed that the geodesic length getting zero when $r_H t_b^{\text{crit}} \approx 2.16$.

varying the functional

$$S_{\text{geodesics}}[y] = \int_{-y_\infty}^{y_\infty} dy \sqrt{1 - f(y) \left(\frac{d\tau}{dy} \right)^2}. \quad (10.3)$$

The equation of motion is simply obtained by $\delta S_{\text{geodesics}}[y] = 0$ and

$$\frac{d}{dy} \left(\frac{f(y) \tau'(y)}{\sqrt{1 - f(y) (\tau'(y))^2}} \right) = 0, \Leftrightarrow \frac{f(y) \tau'}{\sqrt{1 - f(y) \tau'^2}} = \sqrt{C} \quad (10.4)$$

where C is a positive integration constant. By solving the equation, we obtain,

$$\begin{aligned} \tau(y) &= B + A(y; C), \\ A(y; C) &= \int^y dy \sqrt{\frac{C}{f(y)(f(y) + C)}} \\ &= \frac{2\sqrt{C}}{\sqrt{(1+\chi)(1+\chi+2C)}} \int_0^{\tanh y} \frac{ds}{\sqrt{\left(1 - \frac{1-\chi}{1+\chi} s^2\right) \left(1 - \frac{1-\chi+2C}{1+\chi+2C} s^2\right)}}. \end{aligned} \quad (10.5)$$

Importantly, $A(-y, C) = -A(y, C)$ and we find that

$$\begin{aligned} \tau(y \rightarrow \infty) &= B + A(\infty; C) = \tau_b^{(2)} \\ \tau(y \rightarrow -\infty) &= B - A(\infty; C) = \tau_b^{(1)} \\ \Leftrightarrow B &= \frac{\tau_b^{(2)} + \tau_b^{(1)}}{2}, \quad A(\infty; C) = \frac{\tau_b^{(2)} - \tau_b^{(1)}}{2}. \end{aligned} \quad (10.6)$$

The conditions with function $A(\infty; C)$ determines the value of C in terms of the

boundary time. The simple case will be $\tau_b^{(1)} = \tau_b^{(2)} = \tau_b$ and in this case $B = \tau_b, C = 0$ and thus $\tau(y) = \tau_0$ even for non zero γ .

Next we discuss the on-shell action. It is analytically obtained as

$$\begin{aligned}\underline{S}_{\text{geodesics}} &= \int_{-y_\infty^{(2)}}^{y_\infty^{(2)}} dy \sqrt{\frac{f(y)}{f(y) + C}} \\ &= 2S(y_\infty^{(2)}; C)\end{aligned}\tag{10.7}$$

where we introduce

$$\begin{aligned}S(y; C) &= \int^y dy \sqrt{\frac{f(y)}{f(y) + C}} \\ &= \sqrt{\frac{1 + \chi}{1 + \chi + 2C}} \int_0^{\tanh y_\infty} \frac{ds}{1 - s^2} \sqrt{\frac{1 - \frac{1-\chi}{1+\chi} s^2}{1 - \frac{1-\chi+2C}{1+\chi+2C} s^2}}.\end{aligned}\tag{10.8}$$

One simple case is $t_b^{(1)} = t_b^{(2)} = t_b$ and we have $C = 0$ and $\tau(y) = \tan^{-1} \sinh r_H t_b$. In this case,

$$\underline{S}_{\text{geodesics}} = 2y_\infty^{(2)} = 2 \log \left(\frac{1}{\chi} \frac{2}{\epsilon} \cosh r_H t_b \right).\tag{10.9}$$

This results is similar to the linear growth in [204]. The interesting case is when we set $t_b^{(1)} = -t_b^{(2)} = t_b$. In this case, we find that at some critical time the geodesics length approaching to zero (See Fig. 36) for $\chi > 1$. This is because that we have a traversable wormhole in the bulk and we can connect the two points causally. When $\chi = 1$, $S(y_\infty; C) = \rho_\infty = \text{const.}$ because of the isometry of the BTZ blackhole or thermofield double state is invariant under the time translation with opposite sign. On the other hand $\chi < 1$ case (e.g. real valued Janus deformation), we find the geodesic length monotonically increasing because the ER bridge getting long.

For $\chi > 1$, the critical time can be analytically derived as follows. Since the critical time is the time that the geodesics getting null, we demand $\frac{f(y)}{f(y)+C} = 0$ for any $y \in [-y_\infty^{(2)}, y_\infty^{(2)}]$. Since $f(y) > 0$ for $\chi > 0$, this is a case when $C = +\infty$. In this case, the expression of $\tau_b = A(\infty, C)$ simplifies to

$$\tau_b^{\text{crit}} = A(\infty; C \rightarrow \infty) = \sqrt{\frac{2}{1 + \chi}} K \left(\frac{1 - \chi}{1 + \chi} \right) = \mu_0.\tag{10.10}$$

Thus the critical time is given by

$$r_H t_b^{\text{crit.}} = \text{arcsinh}(\tan \mu_0).\tag{10.11}$$

We see that $t_b^{\text{crit.}}$ is positive finite only when $\chi > 1$.

10.3 Thermodynamic entropy and pseudo entropy

We previously pointed out that a black hole possesses entropy, which corresponds to the thermodynamic entropy of the boundary theory when it is in thermal equilibrium. Moreover, this entropy can alternatively be interpreted as the entanglement entropy between two replicated systems in the thermofield double state $|\text{TFD}(\beta)\rangle$. From this viewpoint, the entropy is given by the minimal cross section of the Einstein–Rosen bridge.

As ER=EPR, it has been widely conjectured that wormholes are closely tied to certain patterns of quantum entanglement. A natural question then arises: what sort of entanglement structure underlies a traversable wormhole? To explore this, we investigate a deformation of the thermofield double state induced by an imaginary Janus deformation.

Analogous to the standard Janus configuration, this deformation introduces a conformal interface into the thermofield double state. However, a crucial difference is that the interface Lagrangian $\gamma\mathcal{L}_{\text{interface}}$ is now non-Hermitian. Furthermore, the deformation causes the two CFTs to acquire non-Hermitian Hamiltonians,

$$H_{\text{CFT}_{(1/2)}} = H_{\text{undeformed CFT}_{(1/2)}} \pm \gamma V \quad (10.12)$$

for a certain Hermitian operator V . Defining $H = H_{\text{CFT}_{(1)}}$, we then have $H^\dagger = H_{\text{CFT}_{(2)}}$. The Euclidean boundary path integral dual to the traversable wormhole described in the previous section is expressed as

$$\text{Tr} \left[e^{-\frac{\beta}{2}H} e^{-\frac{\beta}{2}H^\dagger} \right]. \quad (10.13)$$

See Figure 37.

These features substantially modify the structure of the thermofield double states, leading to

$$\langle \text{TFD}(\beta) | \text{TFD}(\beta) \rangle \rightarrow \langle \overline{\text{TFD}}(\beta; \gamma) | \text{TFD}(\beta; \gamma) \rangle. \quad (10.14)$$

Here $|\text{TFD}(\beta, \gamma)\rangle$ and $|\overline{\text{TFD}}(\beta, \gamma)\rangle$ are defined as purifications of

$$\rho_{(2)} \propto e^{-\frac{\beta}{4}H^\dagger} e^{-\frac{\beta}{2}H} e^{-\frac{\beta}{4}H^\dagger} \quad (10.15)$$

and

$$\rho_{(1)} \propto e^{-\frac{\beta}{4}H} e^{-\frac{\beta}{2}H^\dagger} e^{-\frac{\beta}{4}H}, \quad (10.16)$$

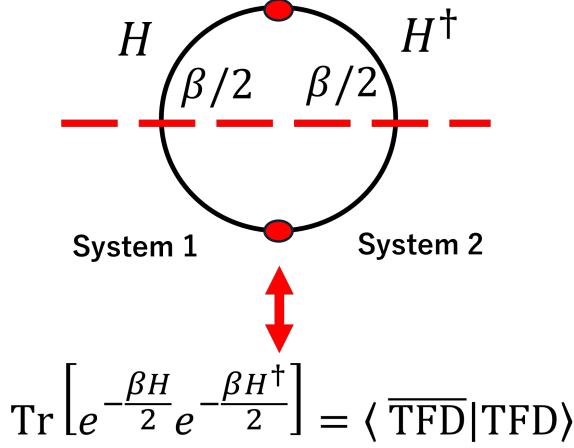


Figure 37: Schematic representation of the Euclidean path integral for $\text{Tr}\left[e^{-\frac{\beta H}{2}} e^{-\frac{\beta H^\dagger}{2}}\right]$. The red dots indicate the defects connecting the two distinct Hamiltonians H and $H^\dagger$. The red line prepares the state $|\text{TFD}\rangle$ and its conjugate $\langle \overline{\text{TFD}}|$.

respectively [218]. More explicitly,

$$\begin{aligned} |\text{TFD}(\beta, \gamma)\rangle &= e^{-\frac{\beta H}{4}} \otimes e^{-\frac{\beta H^\dagger}{4}} \sum_n |E_n^{(r)}\rangle_1 |E_n^{(r)}\rangle_2, \\ \langle \overline{\text{TFD}}(\beta, \lambda)| &= \sum_n \langle E_n^{(l)}|_1 \langle E_n^{(l)}|_2 e^{-\frac{\beta H}{4}} \otimes e^{-\frac{\beta H^\dagger}{4}}, \end{aligned} \quad (10.17)$$

where $|E_n^{(r)}\rangle$ and $\langle E_n^{(l)}|$ denote the right and left eigenvectors of the non-Hermitian Hamiltonian H .

A crucial observation is that

$$(|\text{TFD}(\beta, \gamma)\rangle)^\dagger \neq \langle \overline{\text{TFD}}(\beta, \gamma)|, \quad (10.18)$$

so the total density matrix $\rho_{12} = |\text{TFD}(\beta, \gamma)\rangle \langle \overline{\text{TFD}}(\beta, \gamma)|$ is non-Hermitian. In general, the entanglement entropy defined for such non-Hermitian density matrices or transition matrices is known as the pseudo entropy. A more formal definition and interpretation will be presented in Section 12.

The pseudo entropy between the two CFTs can be computed via the minimal surface dividing the time slice $t = \tau = 0$, leading to

$$S[\rho_{(1)}] = S[\rho_{(2)}] = \frac{\pi c r_H}{6} \sqrt{1 + \sqrt{1 - \gamma^2}}. \quad (10.19)$$

Even though the reduced density matrices (10.15) and (10.16) are non-Hermitian, the resulting entropy is nevertheless real, since $\text{Tr}[(\rho_{(1/2)})^n]$ is real. This follows

from the replica method,

$$\mathrm{Tr}[\rho_1^n] = \mathrm{Tr}[\rho_2^n] = \mathrm{Tr} \left[e^{-\frac{\beta H}{2}} e^{-\frac{\beta H^\dagger}{2}} \cdots e^{-\frac{\beta H}{2}} e^{-\frac{\beta H^\dagger}{2}} \right] \quad (10.20)$$

which, owing to the cyclicity of the trace, yields a real quantity. The von Neumann pseudo entropy is also real because

$$S_{\mathrm{v.N.}}[\rho_1] = S_{\mathrm{v.N.}}[\rho_2] = \lim_{n \rightarrow 1} \frac{1}{1-n} \log \frac{\mathrm{Tr}[\rho_1^n]}{\mathrm{Tr}[\rho_1]^n}. \quad (10.21)$$

Nevertheless, the pseudo entropy exhibits a qualitatively different behavior from conventional entanglement entropy. The expression (10.19) is a monotonically increasing function for $\gamma^2 < 0$ and can become arbitrarily large. In contrast, ordinary entanglement entropy is always bounded above by $\log \dim \mathcal{H}_1$.

11 Gluing two AdS

In this section we study a geometric construction in which two AdS spacetimes are glued along a codimension-one hypersurface [4]. The resulting configuration induces a dynamical theory on the gluing surface, providing a possible realization of “open holography” without assuming an asymptotic boundary.

The motivations for considering such a construction are the following:

1. By taking a flat limit $L_{\mathrm{AdS}} \rightarrow 0$ on one side, one recovers the AdS–bath systems widely used in studies of black-hole evaporation and information dynamics.
2. Real cosmological spacetimes such as de Sitter do not possess timelike boundaries. Holography for half of dS has been recently proposed; by gluing two halves, one may aim to construct holography for full dS.

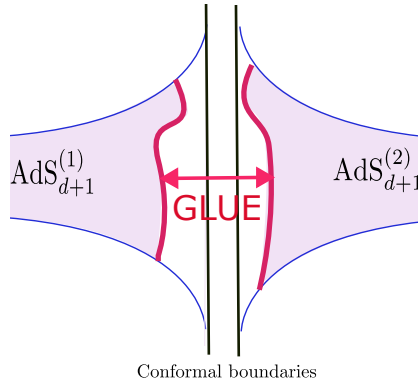


Figure 38: Gluing two AdS geometries along a hypersurface.

11.1 Gluing AdS Spacetimes

We begin with some geometric preliminaries. Consider a codimension-one hypersurface Σ with outward normal n^A , tangent vectors e_a^A , and induced metric h_{ab} . The extrinsic curvature is defined by

$$K_{ab} = e_a^A e_b^B \nabla_A n_B, \quad K = h^{ab} K_{ab}. \quad (11.1)$$

As an example, consider vacuum AdS_{d+1} in Poincaré coordinates with a constant- η surface. The metric is

$$ds^2 = \frac{d\eta^2 + h_{ab} dx^a dx^b}{\eta^2}. \quad (11.2)$$

A direct computation gives

$$\begin{aligned} K_{ab} &= -\frac{1}{2} \frac{\partial}{\partial \ln \eta} h_{ab}(\eta) \Big|_{\eta_0} = \frac{1}{\eta_0^2} h_{ab}, \\ K &= d, \\ K_{ab} - K h_{ab} + (d-1) h_{ab} &= 0. \end{aligned} \quad (11.3)$$

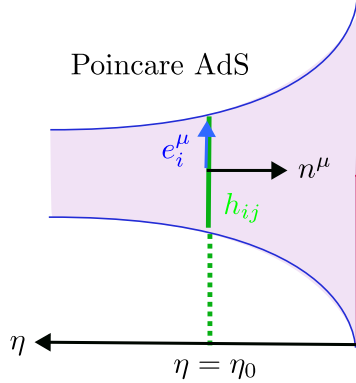


Figure 39: Constant- η slice in Poincaré AdS.

We now consider two AdS_{d+1} spaces (denoted $\text{AdS}_{d+1}^{(1)}$ and $\text{AdS}_{d+1}^{(2)}$), each cut by hypersurfaces Σ_1 and Σ_2 . Gluing them requires satisfying the Israel junction conditions:

$$\begin{aligned} h_{ab}^{(1)} &= h_{ab}^{(2)}, \\ K_{ab}^{(1)} - K^{(1)} h_{ab} + K_{ab}^{(2)} - K^{(2)} h_{ab} + 2\mathcal{T} h_{ab} &= 0. \end{aligned} \quad (11.4)$$

The action consists of two bulk contributions and two boundary terms,

$$\begin{aligned} I_{\text{bulk}} &= \sum_{a=1,2} \frac{1}{16\pi G_N} \int d^{d+1}y \sqrt{-G} \left(R[G] + \frac{d(d-1)}{L_{\text{AdS}_a}^2} \right), \\ I_{\text{bdry}} &= \sum_{a=1,2} \frac{1}{8\pi G_N} \int_{\Sigma_a} d^d x \sqrt{-h^{(a)}} (K^{(a)} - \mathcal{T}). \end{aligned} \quad (11.5)$$

When both bulks are AdS vacua, the metrics are

$$ds_{(1)}^2 = \frac{d\eta^2 + dx_i^2}{\eta^2}, \quad ds_{(2)}^2 = \frac{d\tilde{\eta}^2 + d\tilde{x}_i^2}{\tilde{\eta}^2}. \quad (11.6)$$

The Hamiltonian constraint on each side takes the form

$$R = K^2 - K^{ab}K_{ab} - \frac{d(d-1)}{L_{\text{AdS}}^2}. \quad (11.7)$$

Using the junction conditions yields constant mean curvature:

$$\begin{aligned} K^{(1)} &= \frac{dT}{d-1} + \frac{d(d-1)}{4\mathcal{T}} \left(\frac{1}{L_{\text{AdS}_1}^2} - \frac{1}{L_{\text{AdS}_2}^2} \right), \\ K^{(2)} &= \frac{dT}{d-1} + \frac{d(d-1)}{4\mathcal{T}} \left(\frac{1}{L_{\text{AdS}_2}^2} - \frac{1}{L_{\text{AdS}_1}^2} \right). \end{aligned} \quad (11.8)$$

11.2 Stress Tensor and $T\bar{T}$ Structure

In AdS_3 , the Brown-York tensor on each side is

$$\tau_{ab}^{(a)} = K_{ab}^{(a)} - K^{(a)}h_{ab} + \mathcal{T}h_{ab}, \quad a = 1, 2. \quad (11.9)$$

The Israel condition requires the total stress tensor to vanish:

$$\tau_{ab}^{(1)} + \tau_{ab}^{(2)} = 0. \quad (11.10)$$

If we think this is kind of Virasoro constraint in two dimensional gravity, we can think the holographic theory on junction surface is two coupled CFTs with dynamical two dimensional gravity²⁴. Since the Brown York tensor can measure the energy of the bulk like blackhole mass, the constraint is actually also the constraint of the three dimensional bulk gravity. Possibly this is related we have closed universe and no local gauge invariant expectations which carry mass. From the definitions,

$$\tau^{(a)} := h^{ab}\tau_{ab}^{(a)} = 0, \quad D^a\tau_{ab}^{(a)} = 0. \quad (11.11)$$

²⁴Here we mean dynamical gravity that we need to do path integral over the metric.

The Hamiltonian constraint becomes

$$R[h] + \mu^{(a)} + \tau_{ab}^{(a)} \tau_{(a)}^{ab} - (\tau^{(a)})^2 = 0, \quad \mu^{(a)} = \frac{d(d-1)}{L_{\text{AdS}_a}^2} - \frac{d}{d-1} (\mathcal{T}^{(a)})^2. \quad (11.12)$$

which reproduces the flow equation of a $T\bar{T}$ deformation 4.52. The constant part $\mu^{(a)}$ arises because of our stress tensor here is Brown York tensor.

A trivial $\mathbb{Z}_2$ -symmetric solution obeys

$$K_{ab}^{(a)} = \mathcal{T} h_{ab}, \quad (11.13)$$

whose general form is

$$A(x^2 + \eta^2 - t^2) + B\eta + Cx + Dt + E = 0. \quad (11.14)$$

Depending on $\mathcal{T}$, these are completely classified into three types:

1. AdS-type ($\mathcal{T}L_{\text{AdS}} < 1$),
2. flat ($\mathcal{T}L_{\text{AdS}} = 1$),
3. dS-type ($\mathcal{T}L_{\text{AdS}} > 1$).

11.3 Chiral Gluing Solutions

To study nontrivial solutions, introduce light-cone coordinates:

$$G_{AB}^{(1)} dx^A dx^B = \frac{d\eta^2 - dU dV}{\eta^2}, \quad G_{AB}^{(2)} dx^A dx^B = \frac{d\tilde{\eta}^2 - d\tilde{U} d\tilde{V}}{\tilde{\eta}^2}. \quad (11.15)$$

Consider the surfaces

$$\eta = e^{-\phi(U)}, \quad \tilde{\eta} = e^{-\tilde{\phi}(\tilde{U})}. \quad (11.16)$$

The induced metrics become

$$ds^2|_{\Sigma^{(1)}} = (\phi')^2 dU^2 - e^{2\phi} dU dV, \quad ds^2|_{\Sigma^{(2)}} = (\tilde{\phi}')^2 d\tilde{U}^2 - e^{2\tilde{\phi}} d\tilde{U} d\tilde{V}, \quad (11.17)$$

and the stress tensors are

$$\tau_{UU}^{(1)} = -\phi'' + \phi'^2, \quad \tau_{\tilde{U}\tilde{U}}^{(2)} = -\tilde{\phi}'' + \tilde{\phi}'^2. \quad (11.18)$$

The brane curvature satisfies $R[h] = 0$.

The two coordinate systems may be related by

$$\tilde{U} = P(U), \quad \tilde{V} = V + Q(U). \quad (11.19)$$

Matching the induced metrics gives

$$e^{2\phi(U)} = e^{2\tilde{\phi}(P(U))} P'(U), \quad (11.20)$$

and determines $Q(U)$:

$$Q'(U) = e^{-2\phi} \left(-\frac{P''}{P'} \phi' + \frac{1}{4} \frac{P''^2}{P'^2} \right). \quad (11.21)$$

The Israel condition becomes

$$-\phi'' + \phi'^2 = \frac{1}{2} T_+(U) = -\frac{1}{4} \text{Sch}(\tilde{U}, U), \quad (11.22)$$

where

$$T_+(U) = -\frac{1}{2} \text{Sch}(P, U) = \frac{3}{4} \left(\frac{P''}{P'} \right)^2 - \frac{1}{2} \frac{P'''}{P'}. \quad (11.23)$$

Special examples of these equations include:

- $T_+(U) = 0$: $P(U) \in SL(2, \mathbb{R})$.
- constant $\tau_{UU}^{(1)} = \alpha^2$: $P(U) = e^{2\sqrt{2}\alpha U}$.
- small deformation: $P(U) = U + \frac{a}{1+U^4}$.

These are non-trivial solution does not posses $\mathbb{Z}_2$ symmetry.

11.4 Non-Chiral Solutions (Perturbative)

A general surface can be written as

$$\eta = e^{-\phi(U,V)}, \quad \tilde{\eta} = e^{-\tilde{\phi}(\tilde{U}, \tilde{V})}, \quad \tilde{U} = A(U, V), \quad \tilde{V} = B(U, V). \quad (11.24)$$

Expand ϕ as

$$\phi = \epsilon f + \epsilon^2 g + \epsilon^3 h + O(\epsilon^4). \quad (11.25)$$

The curvature and extrinsic curvature expand as

$$R^{(1)} = 8(\partial_U \partial_V f) \epsilon - 8 \left[2f(\partial_U \partial_V f) - (\partial_U \partial_V f)^2 - \partial_U \partial_V g + (\partial_U^2 f)(\partial_V^2 f) \right] \epsilon^2 + \dots, \quad (11.26)$$

$$K^{(1)} = 2 + 4(\partial_U \partial_V f) \epsilon + (-8f \partial_U \partial_V f + 4\partial_U \partial_V g) \epsilon^2 + \dots. \quad (11.27)$$

Requiring $K = 2$ yields

$$f(U, V) = P(U) + Q(V), \quad g(U, V) = \alpha(U) + \beta(V). \quad (11.28)$$

The Brown–York tensor becomes

$$\begin{aligned} \tau_{UU}^{(1)} &= -P'' \epsilon + [(P')^2 - \alpha''] \epsilon^2 + [2P' \alpha' - 2P' Q' P'' - \partial_U^2 h] \epsilon^3 + O(\epsilon^4), \\ \tau_{VV}^{(1)} &= -Q'' \epsilon + [(Q')^2 - \beta''] \epsilon^2 + [2Q' \beta' - 2Q' P' Q'' - \partial_V^2 h] \epsilon^3 + O(\epsilon^4). \end{aligned} \quad (11.29)$$

To leading orders, the Israel conditions imply

$$\tau_{UU}^{(1)} + \left(\frac{d\tilde{U}}{dU} \right)^2 \tau_{\tilde{U}\tilde{U}}^{(2)} = 0, \quad \tau_{VV}^{(1)} + \left(\frac{d\tilde{V}}{dv} \right)^2 \tau_{\tilde{V}\tilde{V}}^{(2)} = 0. \quad (11.30)$$

This is equivalent to the Virasoro constraint again.

11.5 Application: Half-dS Holography

This gluing construction motivates a possible framework for holography in de Sitter space. A key component is the recently proposed holography for half of dS [8]:

$$(\text{brane theory at } \theta = \theta_0) \iff (\text{cut dS spacetime}). \quad (11.31)$$

The holographic entanglement entropy on generic time slices exhibits super-volume-law behavior,

$$S_A \propto |A|^p, \quad p > 1, \quad (11.32)$$

violating strong subadditivity. This sharply contrasts with ordinary QFT in dS.

One explanation is that the brane theory is highly nonlocal. For instance, a nonlocal model of the form

$$I = \int d^d x \phi e^{(-\partial^2)^q} \phi \quad (11.33)$$

predicts

$$S_A \propto |A|^{d-2+2q}. \quad (11.34)$$

As an interesting future problems, we can consider gluing two half dS spacetime and contain full dS holography.

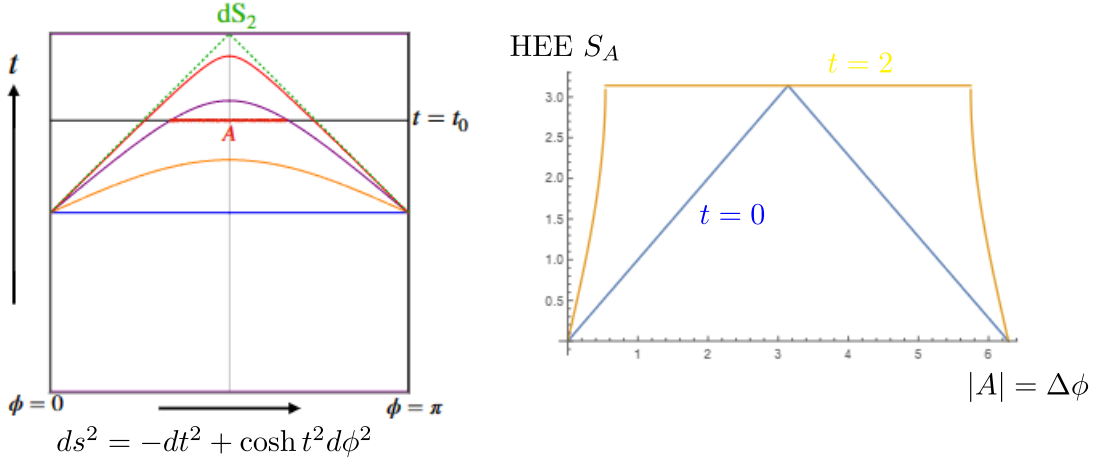


Figure 40: Super-volume-law entanglement in half-dS holography. The figures are cite from [8]

12 Measurement and holographic BCFT and ICFT

12.1 Final state projection and its relation to Page curve

Here we introduce another way of solving the information loss problems with black-hole final state projections based on [5]. In this chapter, we discuss one proposed resolution due to Horowitz and Maldacena and analyze the Page curve in that model.

The central question in the information loss problem is whether semiclassical gravity can exhibit unitarity. A key issue in the semiclassical picture is that physics at the black hole singularity is beyond the regime of validity of general relativity. Horowitz and Maldacena showed that by introducing the following modification to otherwise standard semiclassical gravity, the theory can be made unitary [219, 220]: At the black hole singularity, impose a fixed final state via post-selection. To see how unitarity is preserved under this modification, consider quantum teleportation in quantum mechanics with a fixed final state (post-selection) [221, 222].

12.1.1 Post-selection and quantum teleportation

Quantum mechanics is a framework for the time evolution of a given initial state $|\Psi_i\rangle$ and for measurements performed on that evolving state. What we can access are the outcomes of a sequence of measurements $\{P_{\alpha_k}(t_k)\}_k$ at times $t_1, t_2, \dots, t_n$ and the probabilities for those outcomes. Define the composite measurement operator $\Pi(\alpha) = P_{\alpha_n}(t_n) \cdots P_{\alpha_1}(t_1)$. The decoherence functional $D(\alpha, \alpha')$ for two outcome

histories $\alpha = (\alpha_k)$ and $\alpha' = (\alpha'_k)$ is

$$D(\alpha, \alpha') := \text{Tr}[\Pi(\alpha) |\Psi_i\rangle \langle \Psi_i| \Pi(\alpha')]. \quad (12.1)$$

The probability that the outcome is α is then

$$p(\alpha) = D(\alpha, \alpha). \quad (12.2)$$

By the definition of measurements, $\sum_{\alpha} p(\alpha) = 1$.

We may also consider quantum mechanics in which a fixed final state is imposed at time $t = T$. Such a setup applies, for example, when among all processes starting at $t = 0$ we are only interested in those that yield a specified measurement outcome for some observable at $t = T$. Imposing a fixed final state $|\Psi_f\rangle$ is referred to as post-selection on $|\Psi_f\rangle$. In a theory with both a fixed initial state $|\Psi_i\rangle$ and a fixed final state $\langle \Psi_f|$, the state at times $0 < t < T$ is characterized by two vectors

$$|\Psi_1(t)\rangle := e^{-i \int_0^t dt H} |\Psi_i\rangle, \quad \langle \Psi_2(t)| := \langle \Psi_f| e^{i \int_T^t dt H}, \quad (12.3)$$

see [223, 224]. One can formulate measurements on states specified by the pair $|\Psi_1\rangle, |\Psi_2\rangle$. A natural generalization of the decoherence functional is

$$D(\alpha, \alpha') = \frac{\text{Tr}[\Pi(\alpha) \rho_2 \Pi(\alpha') \rho_1]}{\text{Tr}[\rho_2 \rho_1]}, \quad (12.4)$$

$$\rho_2 := |\Psi_2\rangle \langle \Psi_2|, \quad \rho_1 := |\Psi_1\rangle \langle \Psi_1|. \quad (12.5)$$

The probability for outcome α is then

$$p(\alpha) = D(\alpha, \alpha). \quad (12.6)$$

Another definition is the Aharonov–Bergmann–Lebowitz (ABL) rule,

$$p(\alpha) = \frac{\text{Tr}[\Pi(\alpha) \rho_2 \Pi(\alpha) \rho_1]}{\sum_{\alpha'} \text{Tr}[\Pi(\alpha') \rho_2 \Pi(\alpha') \rho_1]}. \quad (12.7)$$

These two agree, e.g., when ρ_2 commutes with all $\Pi(\alpha')$.

Furthermore, defining the transition matrix from $|\Psi_1\rangle$ to $|\Psi_2\rangle$ by

$$\tau^{1|2} := \frac{|\Psi_1\rangle \langle \Psi_2|}{\langle \Psi_2 | \Psi_1 \rangle}, \quad (12.8)$$

one may generalize the probability as

$$p(\alpha) = \frac{\text{Tr}[\Pi(\alpha) \tau^{1|2}]}{\text{Tr}[\tau^{1|2}]}.$$

In general, $p(\alpha)$ defined this way is complex.

Now consider quantum teleportation with a fixed final state. Suppose Alice post-selects onto the Bell state $|\text{Bell}\rangle$. This corresponds to the Bell measurement outcome $(0, 0)$ being predetermined in standard teleportation. In this case, Bob's final quantum state equals the state $|\psi\rangle$ that we wished to transmit. Unlike standard teleportation, no classical communication from Alice to Bob is needed.

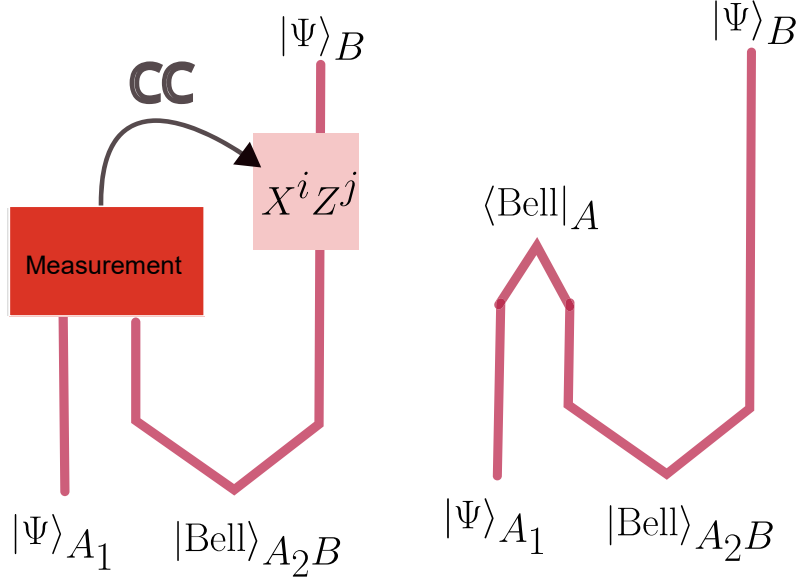


Figure 41: Standard quantum teleportation (left) vs. post-selected teleportation (right).

12.1.2 Pseudo-entropy

In quantum theories with post-selection on a specific state, the measure of entanglement for pure states is given by the pseudo-entropy [225, 226]. Consider two states $|\Psi_1\rangle, |\Psi_2\rangle$ in a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and the transition matrix

$$\tau^{1|2} = \frac{|\Psi_1\rangle \langle \Psi_2|}{\langle \Psi_2 | \Psi_1 \rangle}. \quad (12.9)$$

The reduced transition matrix on subsystem A and the pseudo-entropy are defined by

$$\tau_A^{1|2} := \text{Tr}_B[\tau^{1|2}] , \quad (12.10)$$

$$S_A^{1|2} := -\text{Tr}_A\left[\tau_A^{1|2} \log \tau_A^{1|2}\right] . \quad (12.11)$$

Only when $|\Psi_1\rangle = |\Psi_2\rangle$ do these reduce to the usual reduced density matrix and the entanglement entropy. In general, $\tau_A^{1|2}$ is not Hermitian and $S_A^{1|2}$ is complex. Next, we show that the real part $\text{Re}[S_A^{1|2}]$ quantifies entanglement in quantum mechanics with post-selection on $|\Psi_2\rangle$. For simplicity, consider a qubit system

$$|\Psi_1\rangle = c_1 |00\rangle_{AB} + s_1 |11\rangle_{AB} , \quad (12.12)$$

$$|\Psi_2\rangle = c_2 |00\rangle_{AB} + s_2 |11\rangle_{AB} , \quad (12.13)$$

and we try to distill Bell pairs. The average number of distillable Bell pairs is

$$n = \sum_{k\ell} D_{k,\ell} \frac{\log {}_n C_k + \log {}_n C_\ell}{2} . \quad (12.14)$$

Define the “pseudo-probabilities” Π_k by

$$\Pi_k = \frac{\langle \Psi_f | \Pi_k | \Psi_i \rangle}{\langle \Psi_f | \Psi_i \rangle} , \quad (12.15)$$

so that $D_{k,\ell} = \Pi_k p_\ell^*$. Then

$$\lim_{n \rightarrow \infty} \frac{n}{m} = \text{Re} \left[- \sum_k \Pi_k \log \Pi_k \right] = \text{Re} [S_A^{1|2}] . \quad (12.16)$$

Pseudo-entropy can also serve as an order parameter distinguishing whether two states belong to different phases.

12.1.3 The Horowitz–Maldacena proposal

Set up. Consider matter fields Φ , a classical gravitational solution that undergoes collapse to form a black hole, small quantum fluctuations of the gravitational field around that solution, and quantum field theory of the matter on that classical background. Decompose the total Hilbert space as $\mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$, where:

- $\mathcal{H}_M$: the Hilbert space obtained by embedding the classical configuration of the infalling matter that forms the BH into a quantum Hilbert space;
- $\mathcal{H}_{\text{in}}$: the Hilbert space of quantum fluctuations around the classical solution

on the portion of a Cauchy slice inside the horizon (obtained by splitting a Cauchy surface of the classical solution into interior and exterior parts);

- $\mathcal{H}_{\text{out}}$: the Hilbert space of quantum fluctuations on the exterior portion of that slice.

Take the initial state

$$|\Psi\rangle_{\text{tot}} = |\psi\rangle_M \otimes |U\rangle_{\text{in,out}}, \quad |U\rangle = \sum_{i=1}^N \frac{1}{\sqrt{N}} |i\rangle_{\text{in}} |i\rangle_{\text{out}}, \quad N = e^{S_{BH}}. \quad (12.17)$$

That is, we consider a situation in which, after the BH forms and enough time passes for Hawking radiation to occur, the interior and exterior modes, $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$, are highly entangled. In what follows, “interior degrees of freedom” will refer to $\mathcal{H}_M \otimes \mathcal{H}_{\text{in}}$. If the information prior to BH formation is to be recoverable after evaporation, the S -matrix $S : \mathcal{H}_M \rightarrow \mathcal{H}_{\text{out}}$ must be unitary. We further assume that this S -matrix can be treated as a highly chaotic, complicated random unitary.

Next, at the singularity we post-select onto a state called the black hole state $|\text{BH}\rangle$, defined on the interior Hilbert space by

$$\langle \text{BH} | := \frac{1}{N} \sum_{m,i} S_{mi} \langle E_m |_M \langle i |_{\text{in}}. \quad (12.18)$$

■ Restoration of unitarity and relation to Hawking’s calculation.

After performing the measurement (post-selection), the state becomes

$$|\tilde{\Psi}\rangle_{\text{out}} = N^{3/2} {}_{M,\text{in}}\langle \text{BH} | |\psi\rangle_M |U\rangle_{\text{in,out}} = \sum_{m,i} S_{mi} (\langle E_m |_M |\psi\rangle_M) |i\rangle_{\text{out}}. \quad (12.19)$$

Since $S : \mathcal{H}_M \rightarrow \mathcal{H}_{\text{in}}$ is unitary, $|\tilde{\Psi}\rangle_{\text{out}}$ represents a unitary evolution from $|\Psi\rangle$. We may also include an interaction U between $\mathcal{H}_M$ and $\mathcal{H}_{\text{in}}$. In that case, unitarity is recovered only if $\langle \text{BH} | U$ is a maximally entangled state [220], precisely as in post-selected teleportation, which works only when the post-selected final state is maximally entangled.

Moreover, recalling the assumption that the S -matrix can be approximated by a random unitary, we find for the coarse-grained post-measurement state

$$\langle i | \tilde{\rho} | j \rangle := \langle i | \tilde{\Psi} \rangle \langle \tilde{\Psi} | j \rangle \sim \sum_{m,n} \overline{S_{mi} S_{nj}^*} (\langle m |_M |\psi\rangle_M) (\langle n |_M |\psi\rangle_M)^* \quad (12.20)$$

$$= \frac{1}{N} \delta_{ij}. \quad (12.21)$$

Thus the coarse-grained state is microcanonical, reproducing the result of Hawking’s

calculation.

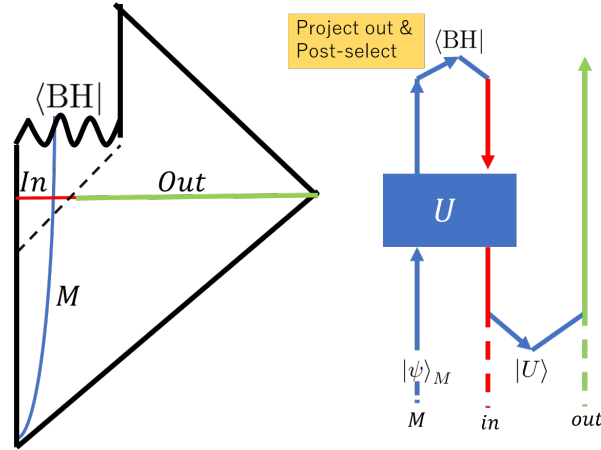


Figure 42: Setup of the Horowitz–Maldacena proposal.

12.1.4 A CFT toy model of the Horowitz–Maldacena proposal

As a toy model of the proposal, consider the time evolution of pseudo-entropy in a two-dimensional CFT with post-selection. We study three models. The first applies

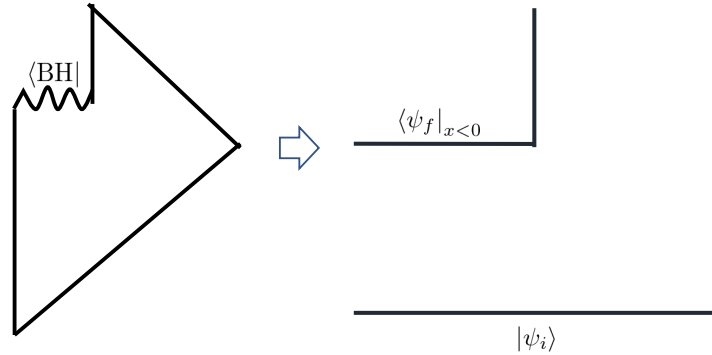


Figure 43: Post-selected 2d CFT and its relation to the Horowitz–Maldacena proposal.

post-selection uniformly over the whole system; the second applies post-selection nonuniformly in a manner closer to the final-state proposal setup.

■ Uniform post-selection model.

At time $t = T$, post-select onto a fixed conformal boundary state $e^{-\delta H} |B\rangle$. For simplicity, take the same state as the initial condition. The transition matrix at

time t is

$$\tau^{1|2} = \frac{|\Psi_1\rangle\langle\Psi_2|}{\langle\Psi_2|\Psi_1\rangle}, \quad (12.22)$$

$$|\Psi_1\rangle = e^{-iHt} e^{-\delta H} |B\rangle, \quad (12.23)$$

$$|\Psi_2\rangle = e^{iH(T-t)} e^{-\delta H} |B\rangle. \quad (12.24)$$

The inner product $\langle\Psi_2|\Psi_1\rangle$ is given by a path integral on a strip of width $L = 2\delta + iT$. Introduce lightcone coordinates on the strip $w_- = x - t$, $w_+ = x + t$. After Wick rotation and defining complex coordinates $w = x + i\tau$, $\bar{w} = x - i\tau$, the strip is mapped to the upper half-plane by the conformal transformation $z = e^{\frac{\pi w}{L}}$. The reduced transition matrix corresponds to a path integral with a cut along A , and the pseudo-entropy is obtained, as in the usual entanglement entropy, by inserting twist operators at the endpoints of A and computing their correlators. To evaluate the two-point function of twist operators, assume the CFT is holographic. Recalling the gravity dual of a strip, the dual geometry is

$$ds^2 = -\frac{T^2}{\pi^2} \cosh^2 \rho dt^2 + d\rho^2 + \frac{T^2}{\pi^2} \sinh^2 \rho dx^2, \quad (12.25)$$

with an EOW brane

$$\cosh \rho \sin\left(\frac{\pi t}{T}\right) = \cosh \eta_0. \quad (12.26)$$

The EOW brane tension is $\mathcal{T} = -\coth \eta_0 < -1$, and the induced metric is

$$ds^2|_Q = \sinh^2 \eta_0 \left(-d\tau^2 + \frac{\pi^2}{T^2} \cosh^2 \tau dx^2 \right), \quad (12.27)$$

which is that of dS_2 . The pseudo-entropy is computed from geodesics in this geometry. There are two candidate geodesics: one connecting the endpoints of the subsystem (connected) and one that attaches to the EOW brane (disconnected). Since the HRT formula arises from a saddle-point evaluation in gravity, we choose the saddle with the smaller real part. Hence,

$$\text{Re}[S_A^{1|2}] = \min\left\{\text{Re}[S_{A,\text{con}}^{1|2}], \text{Re}[S_{A,\text{dis}}^{1|2}]\right\}, \quad (12.28)$$

In the present model, the time dependence is

$$S_{\text{con}}^{1|2} = \frac{c}{3} \log \left[\frac{2T}{\pi\epsilon} \sin \left(\frac{\pi(x_2 - x_1)}{2T} \right) \right], \quad (12.29)$$

$$S_{\text{dis}}^{1|2} = \frac{c}{3} \log \left[\frac{2T}{\pi\epsilon} \sin \left(\frac{\pi t}{T} \right) \right] + i \frac{\pi c}{6} + 2S_{\text{bdry}}. \quad (12.30)$$

The boundary entropy, determined by the EOW brane tension, is

$$S_{\text{bdry}} = \frac{c}{6} \log \frac{|\mathcal{T}| - 1}{|\mathcal{T}| + 1} - i \frac{\pi c}{12}, \quad (12.31)$$

so $S_{\text{dis}}^{1|2}$ becomes real:

$$S_{\text{dis}}^{1|2} = \frac{c}{3} \log \left[\frac{2T}{\pi \epsilon} \sin \left(\frac{\pi t}{T} \right) \right] + \frac{c}{6} \log \frac{|\mathcal{T}| - 1}{|\mathcal{T}| + 1}. \quad (12.32)$$

The connected branch $S_{\text{con}}^{1|2}$ coincides with the vacuum pseudo-entropy and is constant; depending on the boundary entropy and the subsystem size, it is selected between times t_* and $T - t_*$ as the physical pseudo-entropy.

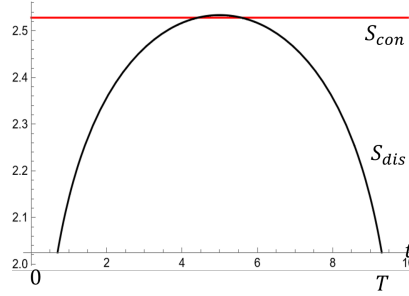


Figure 44: Behavior of pseudo-entropy in the uniform post-selection model.

■ Nonuniform post-selection model.

Consider a nonuniform final state $|\Psi_f\rangle = e^{-\delta H} |B\rangle_{x<0} \otimes |0\rangle_{x>0}$. The time evolution is represented by the path integral in Fig. 45. This path integral again maps

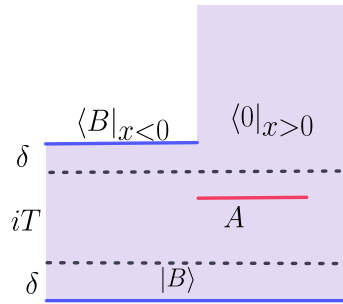


Figure 45: Path integral for the nonuniform post-selection model.

to the upper half-plane via the conformal transformation

$$w = \sqrt{z} - \frac{L}{2} \log \left\{ \frac{1 + \frac{\sqrt{z}}{L}}{\frac{\sqrt{z}}{L} - 1} \right\} - \frac{L}{2} \pi i, \quad (12.33)$$

$$\bar{w} = \sqrt{\bar{z}} - \frac{L}{2} \log \left\{ \frac{1 + \frac{\sqrt{\bar{z}}}{L}}{\frac{\sqrt{\bar{z}}}{L} - 1} \right\} + \frac{L}{2} \pi i, \quad (12.34)$$

with $L = \frac{2}{\pi}(2\delta + iT)$. Using this map, one can compute the pseudo-entropy. Although the functional form is not simple, it can be plotted numerically; see Fig. 12.1.4. For a subsystem $A = [0, x_A > 0]$, the real part of the pseudo-entropy follows a Page-curve behavior. When $x_A = \infty$, the disconnected geodesic is selected at all times. For finite x_A , both connected and disconnected geodesics are finite; which one is selected depends on parameters and time, but both exhibit qualitatively similar behavior.

For $t \sim 0$, pseudo-entropy grows due to entanglement production from the boundary state, with a logarithmic growth $\log(2t/\epsilon)$. Near $t \sim T$, entanglement decreases; this can be understood as the effect of knowing the final state, which increases the information about the reduced transition matrix. For $x_A < \infty$, the pseudo-entropy remains finite at $t = T$ because the entanglement between A and $B = [x_A, \infty]$ does not vanish.

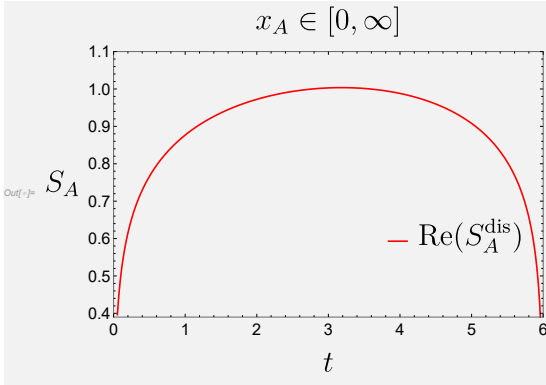


Figure 46: Time evolution of pseudo-entropy for $A = [0, \infty]$. Parameters: $T = 6$, $\epsilon = \delta = 0.01$, $S_{\text{bdry}} = 0$.

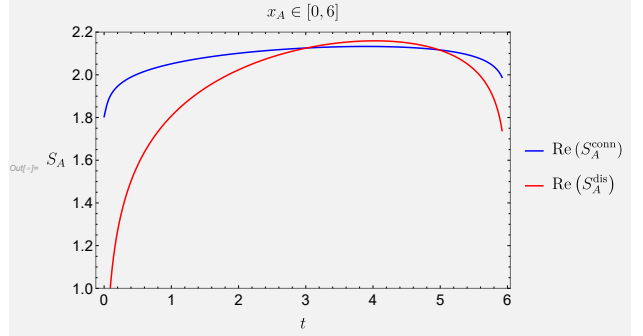


Figure 47: Time evolution of pseudo-entropy for $A = [0, 6]$. Parameters: $T = 6$, $\epsilon = \delta = 0.01$, $S_{\text{bdry}} = 0$.

■ Partial post-selection model.

In the two models above, post-selection at $t = T$ was imposed over the entire space. In contrast, the Horowitz–Maldacena proposal considers post-selection only at the black hole singularity. As a toy model closer to their setup, take the initial

and final states to be

$$|\Psi_i\rangle = \mathcal{N}_i e^{-\delta H} |B\rangle, \quad |\Psi_f\rangle = \mathcal{N}_f (|B\rangle \langle B|_{x<0} \otimes \mathbb{I}_{x>0}) e^{-\delta H} |B\rangle. \quad (12.35)$$

This corresponds to the path integral shown in Fig. 48. The manifold for this path

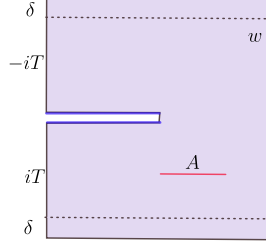


Figure 48: Path integral for the partial post-selection model.

integral can again be mapped to the upper half-plane via

$$w(z) = L \log(1 - z) + LM \log\left(\frac{z + M}{M}\right), \quad (12.36)$$

where $L = \frac{\delta + iT}{\pi}$ and $LM = \frac{\delta - iT}{\pi}$. In this case, as in the nonuniform post-selection model, the pseudo-entropy follows a Page-curve behavior.

12.2 Conformal Defect and measurement like behavior

12.2.1 Possible relation between indirect measurement and spacelike branes

One characteristic feature of quantum measurements is that they can reduce entanglement entropy in Lorentzian path integrals. In the previous section, we saw that the final-state projection reduces entropy. In actual quantum measurement protocols, however, we often need to discuss the time evolution that follows the measurement process. This naturally happens, for example, in the case of indirect measurements.

■ Indirect measurement

In general, it is often difficult to access the system of interest directly. In such situations, it is useful to introduce an auxiliary bath system and correlate it with the system in order to extract information. This indirect measurement can be implemented using the following procedure:

1. Prepare an initial quantum state that is a direct product of the system and bath Hilbert spaces:

$$\rho_{\text{tot}} = \rho_S \otimes \rho_B. \quad (12.37)$$

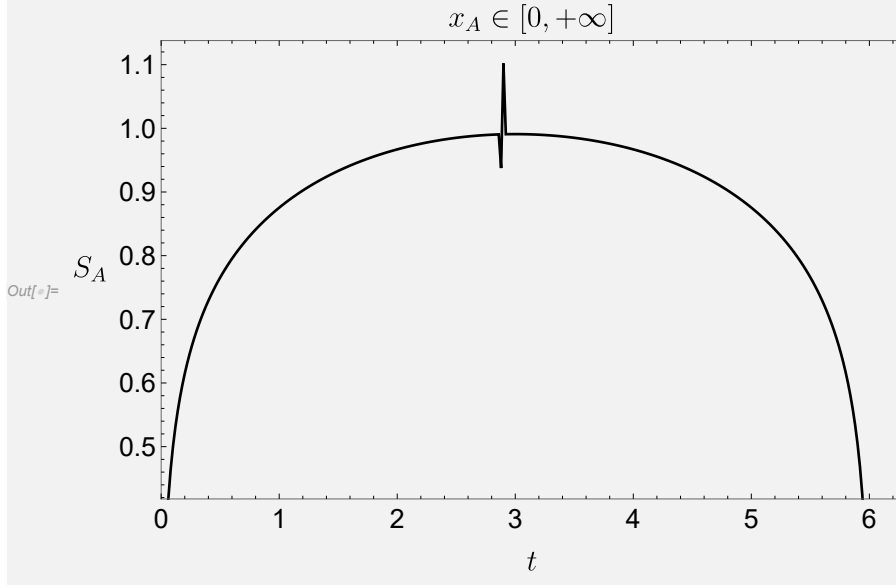


Figure 49: Pseudo-entropy in the partial post-selection model ($T = 6$, $\epsilon = \delta = 0.01$).

2. Apply a unitary operator U to generate correlations between the system and the bath:

$$\rho_{\text{tot}} \rightarrow U \rho_S \otimes \rho_B U^\dagger. \quad (12.38)$$

3. Perform a projective measurement $\{\Pi_k\}_k$ on the bath. The probability of obtaining outcome k is

$$\Pi_k = \text{Tr}_{SB} [\Pi_k U \rho_S \otimes \rho_B U^\dagger \Pi_k]. \quad (12.39)$$

The post-measurement state of the total system conditioned on outcome k is then

$$\rho_{\text{tot}}^{(k)} = \frac{1}{\Pi_k} \Pi_k U \rho_S \otimes \rho_B U^\dagger \Pi_k. \quad (12.40)$$

4. Introduce an orthonormal basis $\{|\Psi_m\rangle\}$ for the bath, and write the spectral decomposition of the initial bath state as $\rho_B = q_l |\Phi_l\rangle \langle \Phi_l|$. Then the Kraus operators and POVM elements are defined as

$$\begin{aligned} M_{ki} &= \sqrt{q_l} \langle \Psi_m | \Pi_k U | \Phi_l \rangle, \quad i = (l, m), \\ E_k &= \sum_i M_{ki}^\dagger M_{ki}. \end{aligned} \quad (12.41)$$

The resulting system state after obtaining outcome k becomes

$$\begin{aligned}\rho_S^{(k)} &= \frac{1}{\Pi_k} \sum_i M_{ki} \rho_S M_{ki}^\dagger, \\ \Pi_k &= \text{Tr}[E_k \rho_S]\end{aligned}\tag{12.42}$$

5. If the measurement outcome is not recorded, the resulting system state is

$$\rho'_S = \sum_k \Pi_k \rho_S^{(k)} = \sum_{k,i} M_{ki} \rho_S M_{ki}^\dagger.\tag{12.43}$$

■ Spacelike branes from tracing out the bath

From the viewpoint of the total system, the dynamics can be described using the Schwinger–Keldysh path integral. Suppose we are only interested in the system S and trace out the bath. If the total unitary U acts instantaneously, we expect that the effective action of the system acquires a spacelike brane contribution:

$$L_{S,\text{eff}} = L_S + \delta(t_0) \gamma L_{\text{sp. brane}}.\tag{12.44}$$

This follows in the same spirit as the Caldeira–Leggett model or, more generally, the Feynman–Vernon influence functional formulation [227, 228].

Here, γ can be interpreted as the strength of the measurement. This interpretation arises because the spacelike brane term emerges upon tracing out the bath: if the unitary U factorizes into a tensor product, then $\gamma = 0$. In measurement terms, if the unitary is decoupled, the bath carries no information about the system, and effectively no measurement has been performed.

Here we introduce an instantaneous spacelike brane, i.e. a thin brane localized at $t = 0$ in the Lorentzian path integral. One may also consider a thick brane. It is known that certain soliton configurations in quantum field theory and string theory correspond to thick spacelike branes [229]. In such cases, the system fields exhibit dissipative radiation associated with tachyon condensation. It is plausible to interpret this radiation as representing the energy flow between the system and the bath in the indirect measurement process.

Of course, not all spacelike branes provide an effective description of indirect measurement. For example, in the models discussed in Sec. 12.1, the pseudo entropy satisfies $S_A = S_{A^c}$, which holds because the geodesic is determined solely by the entanglement surface. This implies that no decoherence occurs, and the path integral should instead be interpreted as producing a two-state amplitude.

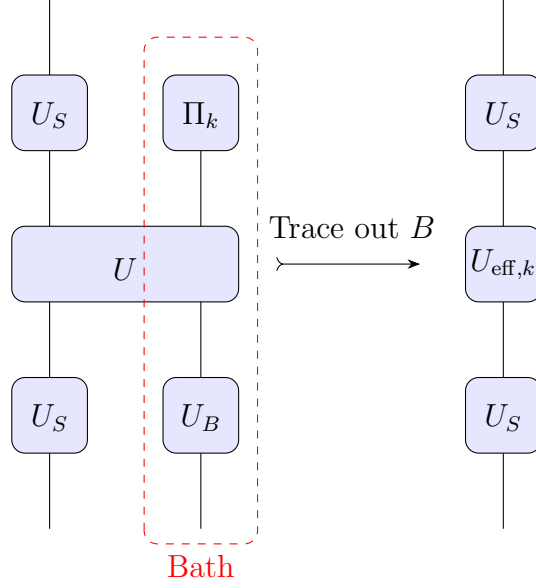


Figure 50: A schematic illustration of the indirect measurement process. After tracing out the bath system, the system dynamics is modified by an effective spacelike brane term.

12.2.2 Janus solution for spacelike branes: Analytic continuation from Euclidean path integrals

We now discuss the entanglement evolution in the presence of spacelike branes. To this end, we consider another variant of the Janus solution. As we have seen in previous sections, the gravity dual of Janus solutions is given by soft-wall geometries [7]. In this sense, the present setup is a generalization of that in Section 12.1. We have two strategies for obtain the spacelike branes in Lorentzian path integrals and their gravity duals

1. We firstly consider branes in Euclidean path integrals. This is usual ICFT and the gravity dual is usual Janus solution. We compute some physical quantities like pseudo entropy with Euclidean time. Then, we do the Wick rotation.
2. From the first sight we consider a Lorentzian path integral directly. Formally, we can construct the time-like Janus solution via double Wick rotation.

■ Euclidean path integrals

First, we consider the Euclidean solution to gain some intuition. We place the brane at $\tau = 0$; see figure 51 for the boundary path integrals dual to the present bulk-field-based model. Because the Euclidean path integrals extend over an infinite range, the path integral prepares a transition matrix between two vacuum states of CFTs that differ by a marginal deformation. More concretely, we consider two CFTs

with Hamiltonians

$$H_{\pm} = H_0 \pm \phi_{\pm} \int dx \mathcal{O}(x), \quad (12.45)$$

and denote their vacua by $|0_{\pm}\rangle$. Here $\phi_{\pm}$ are the boundary values of the dilaton field, which are holographically dual to the deformations on the brane. We then consider a constant Euclidean time slice $\tau = \tau_0$, on which the transition matrix is defined. In this setup, the transition matrix $\mathcal{T}$ for $\tau_0 > 0$ can be written as

$$\begin{aligned} \mathcal{T}(\tau_0 > 0) &= \mathcal{N} \cdot \lim_{T \rightarrow \infty} e^{-\tau_0 H_+} e^{-T H_-} |0_- \rangle \langle 0_+| e^{-H_+(T-\tau_0)} \\ &= \mathcal{N} \cdot e^{-\tau_0 H_+} |0_- \rangle \langle 0_+|, \end{aligned} \quad (12.46)$$

where $\mathcal{N}$ is a normalization constant. Similarly, for $\tau_0 < 0$, we obtain

$$\mathcal{T}(\tau_0 < 0) = \mathcal{N} \cdot |0_- \rangle \langle 0_+| e^{\tau_0 H_-}. \quad (12.47)$$

Compared with more standard holographic constructions of interface CFTs, our setup is slightly different in that the defect is placed on a time slice, and we consider the pseudo entropy of a subregion that is parallel to this defect.

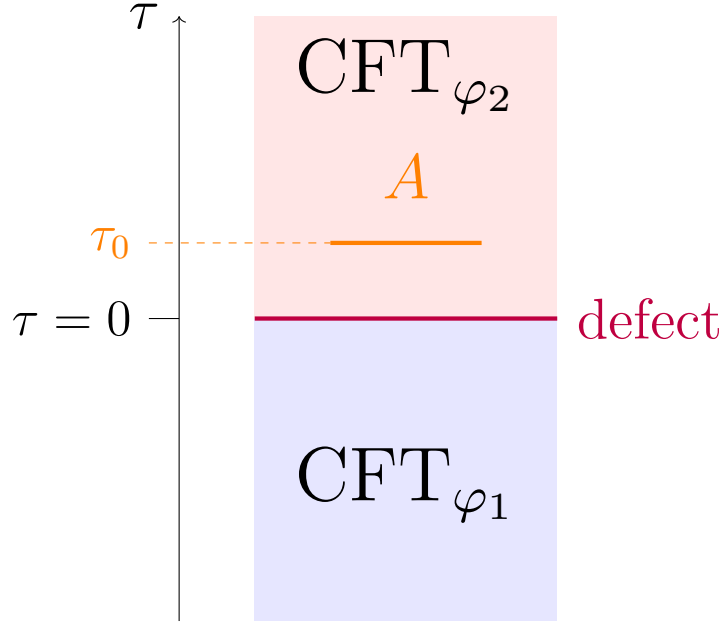


Figure 51: The model with bulk fields. The bulk dilaton interpolates between two CFTs, CFT_{φ_1} and CFT_{φ_2} . On the CFT side, this corresponds to a defect along $\tau = 0$.

We now discuss the gravity dual in more detail. Explicitly, the Euclidean Janus

solution with the defect on the time slice $\tau = 0$ can be written as

$$\begin{aligned} ds^2 &= dy^2 + f(y) \frac{dx^2 + d\xi^2}{\xi^2}, \quad f(y) = \frac{1}{2}(1 + \sqrt{1 - 2\gamma^2} \cosh 2y), \\ \varphi(y) &= \frac{1}{\sqrt{2}} \log \left(\frac{1 + \sqrt{1 - 2\gamma^2} + \sqrt{2}\gamma \tanh y}{1 + \sqrt{1 - 2\gamma^2} - \sqrt{2}\gamma \tanh y} \right), \end{aligned} \quad (12.48)$$

where we have adopted the hyperbolic slice coordinates (y, ξ, x) . The parameter γ controls the strength of the deformation by the spacelike defect, as in standard defect setups. In particular, for $\gamma = 0$ the defect is absent on the CFT side, and the dual geometry reduces to vacuum AdS_3 . We restrict to the range $|\gamma| < \gamma_c = \frac{1}{\sqrt{2}}$. The asymptotic values of the dilaton field are related to γ as

$$\varphi(y \rightarrow \pm\infty) = \frac{1}{\sqrt{2}} \log \left(\frac{1 + \sqrt{1 - 2\gamma^2} \pm \sqrt{2}\gamma}{1 + \sqrt{1 - 2\gamma^2} \mp \sqrt{2}\gamma} \right). \quad (12.49)$$

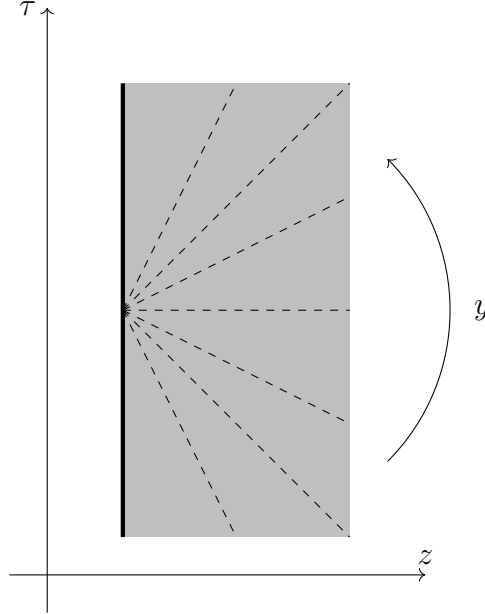


Figure 52: Janus solution (12.48). The dashed lines denote constant- y surfaces, $y = \text{const.}$.

■ Pseudo entropy from the Euclidean Janus solution

We now move on to the computation of the pseudo entropy in this setup. As illustrated in figure 51, we choose the subsystem A to be an interval parallel to the defect. This differs from earlier studies such as [230, 231], where the interval is taken perpendicular to the defect. We compute both the Euclidean and Lorentzian time evolution of the pseudo entropy by essentially the same strategy as in the previous

sections: we take A to be the interval $[-l, l]$ on a constant Euclidean time slice $\tau = \text{const.}$, perform the computation, and then analytically continue to Lorentzian time via Wick rotation. Thus, our pseudo entropy is simply the entropy of the transition matrix constructed from two distinct vacuum states, together with its Lorentzian time evolution.

A subtlety here is that the bulk geometry is no longer pure AdS, so the natural time coordinate in the bulk is different from the standard one. We relate the bulk time coordinate and the boundary time coordinate by using an analogue of the coordinate transformation used for vacuum AdS₃. We will explain this in more detail below.

Finally, we turn to the holographic computation of the pseudo entropy. Our task is to evaluate the geodesic length connecting the endpoints of the entangling surface ∂A in the Janus geometry (12.48).

The geodesic equation

To find the geodesic $x = x(y)$, $\xi = \xi(y)$ and its length, we consider the Lagrangian

$$\mathcal{L} = \sqrt{1 + f(y) \frac{\dot{x}^2 + \dot{\xi}^2}{\xi^2}}, \quad (12.50)$$

where the dot denotes a derivative with respect to y . Because of the nice AdS₂ isometry, we can solve the geodesic equation of motion for any $f(y)$. Concretely, the Lagrangian (12.50) has the following two symmetries:

1. AdS₂ isometry

$$x \rightarrow x + \lambda x, \quad \xi \rightarrow \xi + \lambda \xi. \quad (12.51)$$

The associated Noether charge is

$$Q_{\text{AdS}_2} = \frac{f(y)}{\mathcal{L}} \frac{\xi \dot{\xi} + x \dot{x}}{\xi^2}. \quad (12.52)$$

2. x -translation

$$x \rightarrow x + c. \quad (12.53)$$

The corresponding Noether charge is

$$Q_T = \frac{f(y)}{\mathcal{L}} \frac{\dot{x}}{\xi^2}. \quad (12.54)$$

By dividing (12.52) by (12.54), we obtain

$$\begin{aligned}\frac{Q_{\text{AdS}_2}}{Q_T} &= \frac{1}{2} \frac{d}{dx} (\xi^2(x) + x^2), \\ \xi^2(x) + x^2 &= 2 \frac{Q_{\text{AdS}_2}}{Q_T} x + C,\end{aligned}\tag{12.55}$$

where C is an integration constant. Thanks to the $\mathbb{Z}_2$ symmetry $x \rightarrow -x$, we have $Q_{\text{AdS}_2} = 0$. Thus, we obtain

$$\xi^2 + x^2 = C,\tag{12.56}$$

with some constant $C > 0$. Notice that this relation does not rely on the explicit form of $f(y)$; it only uses the AdS_2 structure.

Combining (12.50), (12.56) and (12.54), we obtain the geodesic equations

$$\begin{aligned}\left(\frac{\sqrt{C}\dot{x}}{C - x^2}\right)^2 &= \frac{\alpha}{f(y)(f(y) - \alpha)}, \\ \mathcal{L}^2 &= \frac{f(y)}{f(y) - \alpha}, \quad \alpha = CQ_T^2.\end{aligned}\tag{12.57}$$

From the first equation in (12.57), we obtain the solution

$$\begin{aligned}\frac{x(y)}{\sqrt{C}} &= \tanh\left(-\sqrt{\frac{-k}{\chi}} F(W(y)|k) + B\right), \\ W(y) &:= \text{Arcsin}\left(\frac{1}{2}\sqrt{\frac{(2\alpha - \chi - 1)(\chi \cosh 2y + 1) \text{csch}^2 y}{\alpha\chi}}\right), \\ k &= -\frac{4\alpha\chi}{(1 - \chi)(2\alpha - \chi - 1)},\end{aligned}\tag{12.58}$$

where $\chi = \sqrt{1 - 2\gamma^2}$, B is an integration constant and $-\frac{\pi}{2} < \text{Arcsin}(\cdot) < \frac{\pi}{2}$. Here $F(z|m)$ denotes the incomplete elliptic integral of the first kind.

Cutoff surface

Now we take some UV cut-off surface. We have some remark. In the case of the Janus geometry dual to interface CFTs, it is known that a single FG coordinate chart cannot cover the entire spacetime [232]. Practically, this means that there can be several inequivalent choices for the coordinate transformation from the H_2 slice coordinates (y, ξ, x) to a Poincaré-like coordinate system (z, t, x) . Various prescriptions for choosing the cutoff surface in such geometries have been discussed in [233].

In our analysis, we define the regularized AdS boundary by

$$\xi = \epsilon \sqrt{f(y)}. \quad (12.59)$$

This definition is motivated by two considerations. First, in the pure AdS limit with $\gamma = 0$ and $\chi = 1$, this surface exactly coincides with $z = \epsilon$ in Poincaré coordinates. Second, with an appropriate definition of the boundary time τ , the induced metric on this cutoff surface (which is identified with the CFT metric) becomes flat.

Indeed, from (12.59), the induced metric takes the form

$$ds^2|_{\text{ind}} = \frac{dx^2}{\epsilon^2} + \frac{1}{\epsilon^2} \left(1 + \frac{4f}{f'^2} \right) d\xi^2. \quad (12.60)$$

We now introduce the boundary time τ via

$$d\tau^2 = \left(1 + \frac{\epsilon^2}{\xi'^2} \right) d\xi^2 = \left(1 + \frac{\left(\frac{\xi}{\epsilon} \right)^2}{-\chi^2 + \left(2 \left(\frac{\xi}{\epsilon} \right)^2 - 1 \right)^2} \right) d\xi^2, \quad (12.61)$$

so that the boundary metric becomes

$$ds^2 = \frac{dx^2 + d\tau^2}{\epsilon^2}. \quad (12.62)$$

We can in principle obtain $\tau = \tau(\xi)$ by solving (12.61). For $\tau \gg O(\epsilon)$, one finds $\tau \sim \xi$, whereas for $\tau \sim O(\epsilon)$ there is a small deviation from ξ .

Determination of the constants C, B, α

We first determine C . As mentioned above, the relation (12.56) follows only from the AdS₂ structure and does not depend on the explicit form of $f(y)$. It is therefore natural to assume that the constant C is universal and coincides with the one in the vacuum AdS case, namely

$$C = l^2 + \tau_0^2. \quad (12.63)$$

This should be reliable at least when $\tau_0 \gg \epsilon$. Next, we determine B using the boundary condition at the turning point $x = 0, y = y_t$. At the turning point, $\dot{x}(y)$ diverges. From the first equation in (12.57), we find that $f(y_t) = \alpha$, i.e.

$$1 + \chi \cosh 2y_t = 2\alpha. \quad (12.64)$$

Using this, we also obtain

$$W(y_t) = \frac{\pi}{2}. \quad (12.65)$$

Substituting $y = y_t, x = 0$ into (12.58) gives

$$B = \sqrt{\frac{-k}{\chi}} K(k), \quad (12.66)$$

where $K(k) = F\left(\frac{\pi}{2} | k\right)$ is the complete elliptic integral of the first kind.

To fix α , we introduce y_∞ as the value of y where the geodesic intersects the cutoff surface (12.59). We thus have

$$\frac{\xi(y_\infty)}{\sqrt{f(y_\infty)}} = \epsilon. \quad (12.67)$$

Using (12.58) at the cutoff surface $y \rightarrow y_\infty$ with $x \rightarrow l$, we determine α from

$$\frac{l}{\sqrt{C}} = -\tanh\left(\sqrt{\frac{-k}{\chi}}(F(W(\infty) | k) - K(k))\right). \quad (12.68)$$

The geodesic length

Next, we evaluate the length of the geodesic.

Since the geodesic equation can be solved in terms of elliptic integrals, the geodesic length can also be written explicitly. The length of the HRT surface Γ_A is given by

$$\begin{aligned} \text{Area}[\Gamma_A] &= 2 \int_{y_t}^{y_\infty} dy \sqrt{\frac{f(y)}{f(y) - \alpha}} \\ &= \left[\sqrt{\frac{m}{\alpha\chi}} ((\chi + 1)F(H(y) | m) + (2\alpha - \chi - 1)\Pi(n; H(y) | m)) \right]_{y_t}^{y_\infty}, \end{aligned} \quad (12.69)$$

where $\Pi(n; z | m)$ is the incomplete elliptic integral of the third kind and

$$\begin{aligned} H(y) &:= \text{Arcsin}\left(\frac{1}{2}\sqrt{\frac{(\chi + 1)(-2\alpha + \chi \cosh(2y) + 1)\text{csch}^2(y)}{\alpha\chi}}\right), \\ m &:= \frac{4\alpha\chi}{(\chi + 1)(2\alpha + \chi - 1)}, \quad n := \frac{2\alpha}{\chi + 1}. \end{aligned} \quad (12.70)$$

At the turning point (12.64), we have $H(y_t) = 0$ and hence

$$F(H(y_t)|m) = 0, \quad \Pi(n; H(y_t)|m) = 0. \quad (12.71)$$

We now simplify the contribution at $y = y_\infty$ in (12.69). For this purpose, we use the following identity:

$$\begin{aligned} \Pi(n; z|m) &= F(z|m) - \Pi\left(\frac{m}{n}; z|m\right) \\ &+ \frac{1}{2\sqrt{\frac{(n-m)(n-1)}{n}}} \log \left(\frac{\sqrt{\frac{(n-m)(n-1)}{n}} \tan z + \sqrt{1 - m \sin^2 z}}{-\sqrt{\frac{(n-m)(n-1)}{n}} \tan z + \sqrt{1 - m \sin^2 z}} \right). \end{aligned} \quad (12.72)$$

Using this, we obtain the final expression for the geodesic length:

$$\begin{aligned} \text{Area}[\Gamma_A] &= \sqrt{\frac{m}{\alpha\chi}} \left(2\alpha F(H(y_\infty)|m) - (2\alpha - \chi - 1) \Pi\left(\frac{m}{n}; H(y_\infty)|m\right) \right) \\ &+ \log \left(\frac{\sqrt{\frac{(n-m)(n-1)}{n}} \tan H(y_\infty) + \sqrt{1 - m \sin^2 H(y_\infty)}}{-\sqrt{\frac{(n-m)(n-1)}{n}} \tan H(y_\infty) + \sqrt{1 - m \sin^2 H(y_\infty)}} \right). \end{aligned} \quad (12.73)$$

The pseudo entropy S_A is then determined by the standard HRT formula (3.91). As in the previous subsection, we also perform an analytic continuation at $\tau_0 = 0$ to obtain the Lorentzian time evolution, formally by replacing $\tau_0 = it_0 + \epsilon'$.

We show the Euclidean and Lorentzian time evolution of the pseudo entropy S_A in figure 53 and figure 54, respectively.²⁵ In Euclidean time, the pseudo entropy grows mildly as $S_A \sim \log \frac{t_0}{l}$, and in Lorentzian time it shows similar behavior up to times of order $t_0 \sim l/2$. At late times, S_A approaches the vacuum entanglement entropy

$$S_0 = \frac{c}{3} \log \frac{2l}{\epsilon}. \quad (12.74)$$

For Euclidean evolution, this is natural: the long Euclidean evolution projects the state onto the vacuum of the Hamiltonian. For Lorentzian evolution, we can interpret this as follows: the reduced transition matrix on A loses information about the presence of one of the defects, even though the full transition matrix still retains this information. The pseudo entropy becomes small near the defect, reflecting the fact that for $\gamma \neq 0$, the two vacuum states are distinct.

In Lorentzian time, the pseudo entropy exhibits oscillatory behavior around $t_0 \sim l$. This feature should be cross-checked from the boundary CFT side, and we leave

²⁵One must keep track of the branch of $\sqrt{\frac{(n-m)(n-1)}{n}}$.

this point as an interesting open problem.

To analyse the dependence on the cutoff and time more systematically, we introduce the parameter

$$\delta = \frac{2(2\alpha - \chi - 1)}{\chi} e^{-2y_\infty} = 2(2\alpha - \chi - 1) \left(\frac{\epsilon}{2\tau_0} \right)^2. \quad (12.75)$$

This definition is motivated by the expression for y_∞ . For time scales satisfying $\epsilon \ll |\tau_0|$, we can safely assume that δ is small. Since $H(y_\infty) \approx \text{Arcsin} \sqrt{\frac{1-\delta}{n}}$ for small δ , we find

$$\Pi(n; H(y_\infty) | m) = F(H_\infty | m) - \Pi\left(\frac{m}{n}; H_\infty | m\right) + \frac{\log\left(\frac{4}{\delta} \left(\frac{n}{n-1} + \frac{m}{n-m}\right)^{-1}\right)}{2\sqrt{\frac{(n-1)(n-m)}{n}}} + O(\delta), \quad (12.76)$$

where we have introduced $H_\infty = \text{Arcsin} \sqrt{\frac{1}{n}}$. Note that $F(H_\infty | m)$ and $\Pi\left(\frac{m}{n}; H_\infty | m\right)$ are of order $O(\delta^0)$.

Plugging this into (12.73), we finally obtain

$$\begin{aligned} \text{Area}[\Gamma_A] = & \sqrt{\frac{m}{\alpha\chi}} \left(2\alpha F(H_\infty | m) - (2\alpha - \chi - 1) \Pi\left(\frac{m}{n}; H_\infty | m\right) \right) \\ & + \log\left(\frac{4}{\delta} \left(\frac{n}{n-1} + \frac{m}{n-m}\right)^{-1}\right) + O(\delta). \end{aligned} \quad (12.77)$$

We see that the pseudo entropy exhibits a logarithmic dependence on the cutoff, just like the vacuum entanglement entropy in CFT₂. The formula is valid in the regime $|\tau_0| \gg \epsilon$, i.e. $\delta \ll 1$.

■ Late time behavior of the pseudo entropy

Late Euclidean time $\tau_0 \gg l$

In the limit $\tau_0 \rightarrow \infty$, the entangling region is located far away from the defect, and we naturally expect the situation to approach that of the vacuum. In the vacuum case $\gamma = 0$ (so that $\chi = 1$), we have $\alpha = 1 + \frac{\tau_0^2}{l^2}$ and thus $\alpha \rightarrow \infty$ as $\tau_0 \rightarrow \infty$. For general γ , we can confirm similar behavior by analyzing the large τ_0 and large α limits of (12.68). A short calculation yields

$$\begin{aligned} (\text{l.h.s}) &= \frac{l}{\tau_0} + O\left(\left(\frac{l}{\tau_0}\right)^3\right), \\ (\text{r.h.s}) &= \sqrt{\frac{1}{\alpha}} + O\left(\frac{1}{\alpha^{3/2}}\right). \end{aligned} \quad (12.78)$$

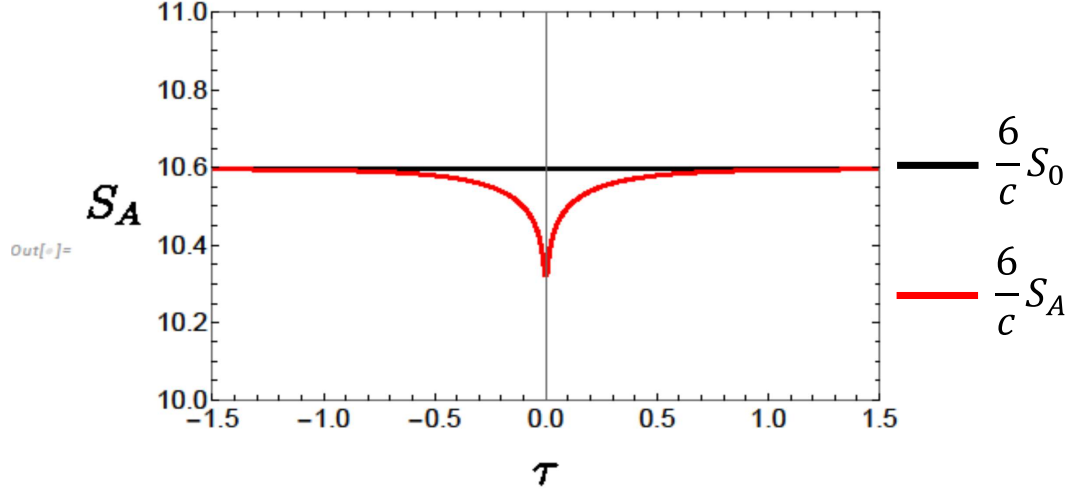


Figure 53: Pseudo entropy $S_A(\tau_0)$ under Euclidean time evolution. We set $\gamma = 0.35, \epsilon = 0.01, l = 1$. The black line represents the entanglement entropy in the undeformed CFT, $S_0 = \frac{c}{3} \log \frac{2l}{\epsilon}$.

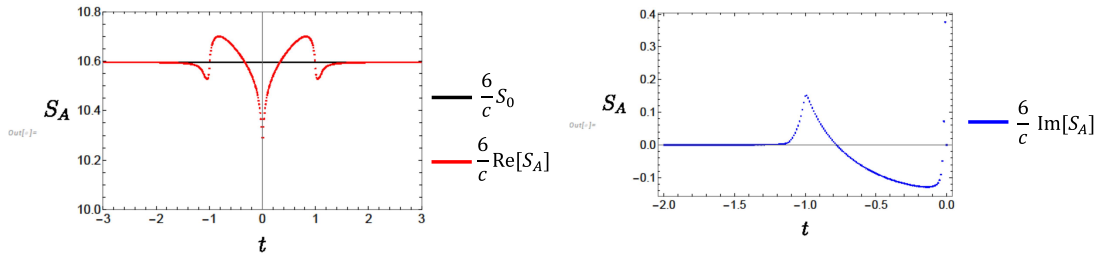


Figure 54: Pseudo entropy under Lorentzian time evolution obtained by Wick rotation. We plot $S_A(t_0)$ with $\gamma = 0.4, \epsilon = \epsilon' = 0.01, l = 1$ for $t_0 \in [0, 5]$. The left panel shows the real part, and the right panel the imaginary part of $S_A(t_0)$.

Thus, we again obtain $\alpha = \left(\frac{\tau_0}{l}\right)^2 + \dots$ for general γ . Evaluating the geodesic length (12.77) in the regime $\tau_0 \gg l \gg \epsilon$, we find

$$S_A = \frac{c}{3} \log \left(\frac{2l}{\epsilon} \right) + O \left(\left(\frac{l}{\tau_0} \right)^2 \right), \quad (12.79)$$

i.e. the pseudo entropy relaxes to the vacuum entanglement entropy at late Euclidean times, as expected.

Late Lorentzian time $t_0 \gg l$

We now consider real-time dynamics obtained by the Wick rotation $\tau_0 = it_0 + \epsilon'$, $t_0 > 0$. The late-time behavior can be analyzed in a similar spirit, although one must be more careful with branch choices. As in the Euclidean case, we expect $\alpha \sim \frac{\tau_0^2}{l^2} \sim -\frac{t_0^2}{l^2} \left(1 - \frac{2i\epsilon'}{t_0} \right)$. Introducing $y_\alpha = \frac{1+\chi}{2\alpha}$, we find

$$\begin{aligned} -\tanh \left(\sqrt{\frac{-k}{\chi}} \left(F \left(\text{Arcsin} \sqrt{1 - y_\alpha} \middle| k \right) - K(k) \right) \right) &= -\sqrt{\frac{2y_\alpha}{1+\chi}} + O(y_\alpha^{5/2}), \\ \frac{1}{\sqrt{1 + \frac{\tau_0^2}{l^2}}} &= \frac{1}{\sqrt{-\frac{t_0^2}{l^2}}} + O\left(\frac{l^3}{t_0^3}\right), \end{aligned} \quad (12.80)$$

in the limit $\alpha \rightarrow -\infty, t_0 \rightarrow \infty$. We thus confirm $\alpha = -\frac{t_0^2}{l^2} + \dots$.

Next we examine the late-time limit of the pseudo entropy. We first evaluate the expression inside the square root in (12.70) at $y = y_\infty$:

$$q := \frac{(\chi + 1)(-2\alpha + \chi \cosh(y_\infty) + 1) \text{csch}^2(y_\infty)}{2\alpha\chi} = -\frac{1+\chi}{2t_0^2} (l^2 - \epsilon^2) \left(1 + i \frac{4l^2}{l^2 - \epsilon^2} \frac{\epsilon'}{t_0} \right) + \dots \quad (12.81)$$

Since q is complex, we must choose the branch of the square root carefully.²⁶ We choose the positive branch; the other choice would simply give the pseudo entropy

²⁶We use $\sqrt{q} = \exp \left(\frac{1}{2} \log |q| + \frac{i}{2} \text{Arg} q + i n \pi \right) = \pm \sqrt{|q|} \exp \left(\frac{i}{2} \text{Arg} q \right)$, where $\text{Arg} \cdot \in (-\pi, \pi]$.

with the opposite sign. Using $\text{Arg}q = -\pi + \frac{2\epsilon'}{t_0}$ for $t_0 > 0$, we find

$$\begin{aligned}\sqrt{|q|} &= \frac{l}{t_0} \sqrt{\frac{1+\chi}{2}} \sqrt{\left(1 - \frac{\epsilon^2}{l^2}\right) \left(1 + \left(\frac{4}{1 - \frac{\epsilon^2}{l^2}}\right)\right)^2 \left(\frac{\epsilon'}{t_0}\right)^2 + \dots}, \\ \sqrt{q} &= -i \frac{l}{t_0} \sqrt{\frac{1+\chi}{2}} \sqrt{\left(1 - \frac{\epsilon^2}{l^2}\right) \left(1 + \left(\frac{4}{1 - \frac{\epsilon^2}{l^2}}\right)\right)^2 \left(\frac{\epsilon'}{t_0}\right)^2 \exp\left(i \frac{2\epsilon'}{t_0}\right) + \dots}, \\ H(y_\infty) &= \text{Arcsin} \sqrt{q} = -i \frac{\sqrt{l^2 - \epsilon^2}}{\sqrt{2}t_0} \sqrt{1 + \chi} \sqrt{1 + \left(\frac{4l^2}{l^2 - \epsilon^2}\right)^2 \left(\frac{\epsilon'}{t_0}\right)^2 \exp\left(i \frac{2\epsilon'}{t_0}\right) + \dots}.\end{aligned}\tag{12.82}$$

We similarly take the same branch for $\sqrt{\alpha}$:

$$\sqrt{\alpha} = \sqrt{|\alpha|} \exp\left(\frac{i}{2} \text{Arg} \alpha\right) = i \frac{t_0}{l} \sqrt{1 + \frac{4\epsilon'^2}{t_0^2}} \exp\left(-i \frac{2\epsilon'}{t_0}\right).\tag{12.83}$$

For small z , one has

$$2\alpha F(z|m) - (2\alpha - \chi - 1) \Pi\left(\frac{m}{n}; z|m\right) = (1 + 2\alpha + 2\alpha\chi)z + O(z^3).\tag{12.84}$$

Using this expansion, the first term of (12.73) behaves as

$$\sqrt{\frac{m}{\alpha\chi}} \left(2\alpha F(H(y_\infty)|m) - (2\alpha - \chi - 1) \Pi\left(\frac{m}{n}; H(y_\infty)|m\right)\right) = 2(1 + \chi) + \dots.\tag{12.85}$$

Similarly, we find

$$\log \left(\frac{\sqrt{\frac{(n-m)(n-1)}{n}} \tan H(y_\infty) + \sqrt{1 - m \sin^2 H(y_\infty)}}{-\sqrt{\frac{(n-m)(n-1)}{n}} \tan H(y_\infty) + \sqrt{1 - m \sin^2 H(y_\infty)}} \right) = \log \left(\frac{4l^2}{\epsilon^2} \right) + O\left(\frac{l^2}{t_0^2}, \frac{\epsilon^2}{t_0^2}\right).\tag{12.86}$$

Combining these results, we confirm that in the Lorentzian late-time limit, the pseudo entropy again approaches the vacuum entanglement entropy.

Early time behavior We now consider the regime $\epsilon \ll |\tau_0| \ll l$, i.e. early times compared with the subsystem size. We still take the subsystem A to be the interval $[-l, l]$ at Euclidean time $\tau_0 \simeq \xi(\infty)$. The relation between the turning point y_t and

the ratio $l/\xi(\infty)$ is given by

$$\frac{l}{\xi(\infty)} = \sinh \left[\int_{y_t}^{\infty} \frac{dy}{\sqrt{f(y) \left(\frac{f(y)}{f(y_t)} - 1 \right)}} \right]. \quad (12.87)$$

Let us take the limit $y_t \rightarrow 0$, or equivalently $\frac{l}{\xi(\infty)} \gg 1$. Since the dominant contribution to the integral comes from the region around $y \sim y_t$, we can approximate $f(y)$ by $f(y_t)$ in a rough estimate. For small y_t , we have $f(y_t) = \alpha = \frac{1+\chi}{2}$. Then

$$\int_{y_t}^{\infty} \frac{dy}{\sqrt{f(y) \left(\frac{f(y)}{f(y_t)} - 1 \right)}} \simeq -\frac{1}{\chi} \log \left[\sqrt{\frac{1-\chi}{1+\chi}} \frac{y_t}{4} \right] + O(y_t^2). \quad (12.88)$$

Using $\tau_0 \simeq \xi(\infty)$, this gives

$$\frac{l}{\tau_0} \simeq \frac{1}{2} \left(\sqrt{\frac{1-\chi}{1+\chi}} \frac{y_t}{4} \right)^{-\frac{1}{\sqrt{\chi}}}. \quad (12.89)$$

The UV cutoff can be estimated for $\tau_0 \gg \epsilon$ from $\epsilon \simeq \frac{\tau_0}{\sqrt{f(y_\infty)}}$, which implies

$$y_\infty = \log \left(\frac{2}{\sqrt{\chi}} \frac{\tau_0}{\epsilon} \right). \quad (12.90)$$

We can now estimate the area of Γ_A in the early-time limit $y_t \rightarrow 0$:

$$\begin{aligned} \text{Area}[\Gamma_A] &= 2 \int_{y_t}^{y_\infty} dy \sqrt{\frac{f(y)}{f(y) - \alpha}} \\ &\simeq 2y_\infty - \frac{\sqrt{2(1+\chi)}}{\chi} \log \left(\sqrt{\frac{1-\chi}{1+\chi}} \frac{y_t}{4} \right) + O(1) \\ &\simeq -2 \log \epsilon + \left(2 - \frac{\sqrt{2(1+\chi)}}{\chi} \right) \log \tau_0 + \frac{\sqrt{2(1+\chi)}}{\chi} \log l. \end{aligned} \quad (12.91)$$

The first term originates from the integral near y_∞ , where $f(y_\infty)$ is large. This shows that, in the early-time regime $\epsilon \ll |\tau_0| \ll l$, the pseudo entropy grows logarithmically with time for both Euclidean and Lorentzian evolutions.

Entanglement entropy on the defect; at $\tau_0 = t_0 = 0$ At $\tau_0 = t_0 = 0$, time-reversal symmetry simplifies the computation. In particular, the geodesic corresponding to the subsystem on the defect lies entirely on the $y = 0$ slice. On this

slice, the metric takes the same form as the constant-time slice of Poincaré AdS₃:

$$ds^2|_{y=0} = f(0) \frac{d\xi^2 + dx^2}{\xi^2}. \quad (12.92)$$

The cutoff surface is located at $\xi = \epsilon f(0)$. Consequently, the entanglement entropy on the defect is identical in form to the vacuum entanglement entropy, but with an effective central charge and cutoff,

$$c_{\text{eff}} = \frac{2l_{\text{AdS}}}{3G_N} \sqrt{f(0)}, \quad \epsilon_{\text{eff}} = \epsilon f(0). \quad (12.93)$$

Thus we obtain

$$S_A(\tau_0 = t_0 = 0) = \frac{c_{\text{eff}}}{3} \log \frac{2l}{\epsilon_{\text{eff}}} = \sqrt{\frac{1+\chi}{2}} \frac{c}{3} \log \left(\frac{2}{1+\chi} \frac{2l}{\epsilon} \right). \quad (12.94)$$

Note that the effective central charge c_{eff} is independent of the choice of the cutoff and decreases monotonically as the deformation strength γ is increased. This supports the interpretation that the defect plays a role similar to that of a measurement, effectively reducing entanglement.

12.2.3 Lorentzian path integrals: time like Janus

■ The setup

In this subsection, instead of obtaining the Lorentzian dynamics by analytic continuation from the Euclidean setup, we directly consider a Lorentzian path integral in the holographic CFT with a spacelike brane at $t = 0$. The path integral again produces a transition matrix between two distinct vacuum states of the two CFTs. More explicitly, on a constant time slice $t = t_0$, we define the transition matrix

$$\begin{aligned} \mathcal{T}(t_0) &= \lim_{T \rightarrow \infty} e^{-i(T-|t_0|)H_-} |0_- \rangle \langle 0_+| e^{+iT H_+} e^{i|t_0| H_-} \\ &= |0_- \rangle \langle 0_+| e^{+i|t_0| H_-}. \end{aligned} \quad (12.95)$$

The second line is identical to the analytic continuation of the Euclidean model discussed previously. We will confirm later that this indeed leads to the same pseudo entropy profiles. Note that if the initial and final states are not vacuum eigenstates of the Hamiltonians, the resulting transition matrix would no longer coincide with the one obtained by analytic continuation; see figure 56 for a schematic comparison. As we now explain, this Lorentzian path integral is holographically dual to the time-like Janus solution, which describes a thick spacelike brane in the bulk gravity theory.

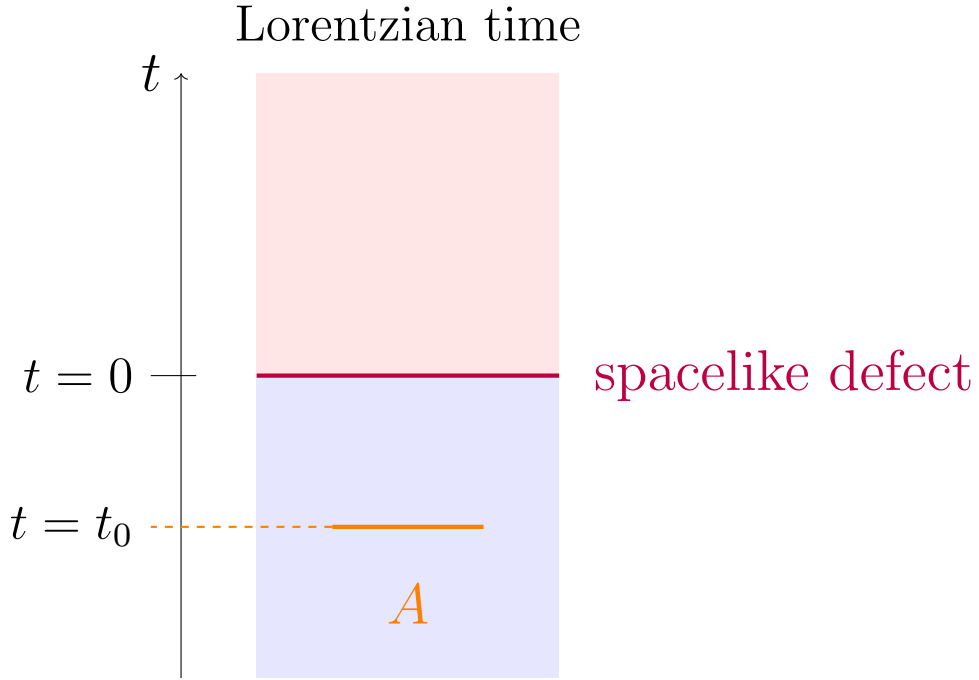


Figure 55: Schematic Lorentzian path integral for the spacelike brane model. We expect that the spacelike defect reduces entanglement in a measurement-like way. The left panel would correspond to a vacuum process, while additional quenches/defects could lead to competing entanglement creation and suppression, suggesting possible entanglement transitions.

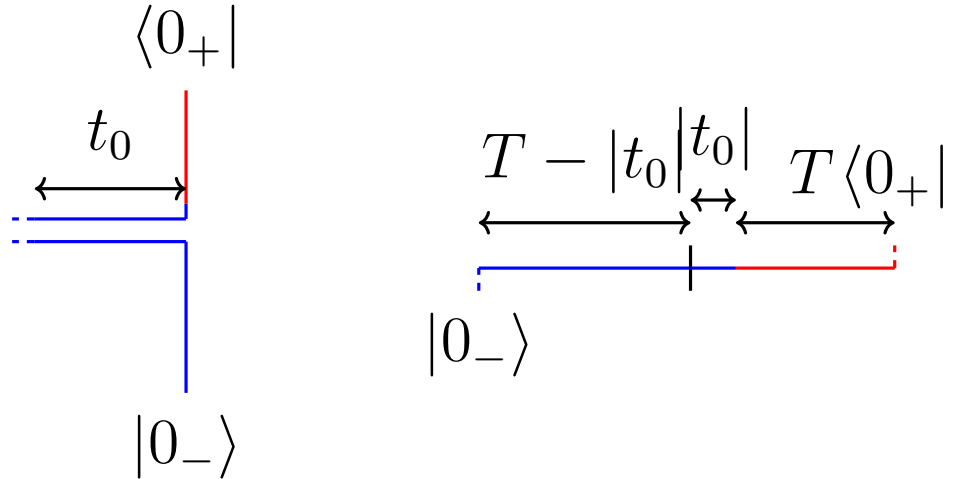


Figure 56: Contours of the path integrals in the complex time plane. Vertical segments correspond to Lorentzian evolution and horizontal ones to Euclidean evolution. Left: analytic continuation of the Euclidean model. Right: Lorentzian model.

To construct the gravity dual of this Lorentzian time evolution in a CFT with a spacelike interface, we begin with a convenient coordinate description of vacuum AdS_3 . In Poincaré coordinates, the metric is

$$ds^2 = \frac{-dt^2 + dz^2 + dx^2}{z^2}. \quad (12.96)$$

We now introduce new coordinates (y, ξ, x) via

$$\begin{aligned} z &= \frac{\xi}{\cos y}, & t &= \xi \tan y, \\ \sin y &= \frac{t}{z}, & z^2 - t^2 &= \xi^2. \end{aligned} \quad (12.97)$$

In these coordinates, the metric becomes

$$ds^2 = -dy^2 + \cos^2 y \frac{d\xi^2 + dx^2}{\xi^2}. \quad (12.98)$$

If we restrict (y, ξ) to

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \quad \xi \geq 0, \quad (12.99)$$

this covers the region

$$-z \leq t \leq z \quad (12.100)$$

in the Poincaré patch. We will refer to this as region II.

The other regions of AdS_3 can be covered by analytically continuing (y, ξ) to complex values. For the region $t \geq z$ (called region I), we define

$$\begin{aligned} y &= -i\rho + \frac{\pi}{2}, & \rho &\geq 0, & \xi &= i\eta, & \eta &\geq 0, \\ z &= \frac{\eta}{\sinh \rho}, & t &= \eta \coth \rho. \end{aligned} \quad (12.101)$$

Similarly, the region $t \leq -z$ (called region III) is described by

$$\begin{aligned} y &= i\rho - \frac{\pi}{2}, & \rho &\geq 0, & \xi &= i\eta, & \eta &\geq 0, \\ z &= \frac{\eta}{\sinh \rho}, & t &= -\eta \coth \rho. \end{aligned} \quad (12.102)$$

The gravity dual of the Lorentzian time evolution (12.95) is obtained by solving the Einstein equations with a dilaton field. This solution can be constructed by a

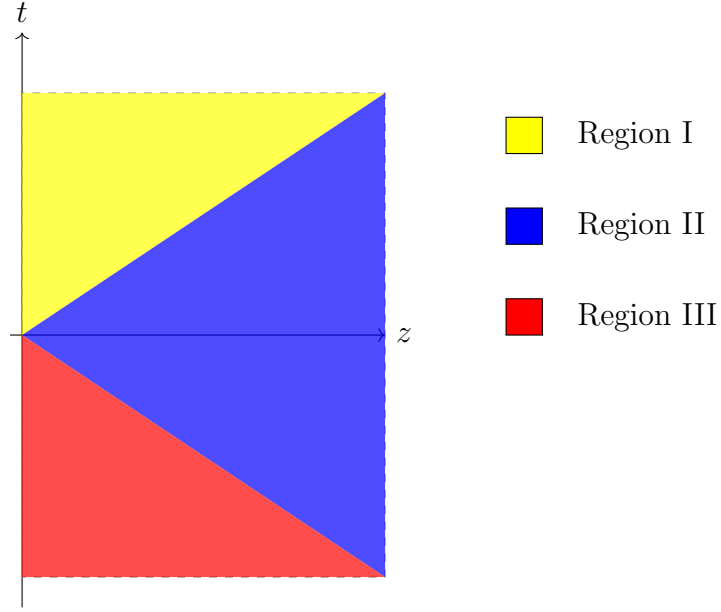


Figure 57: Three regions in vacuum AdS_3 with dS_2/H_2 slice coordinates. The blue region is II, the yellow is I, and the red is III.

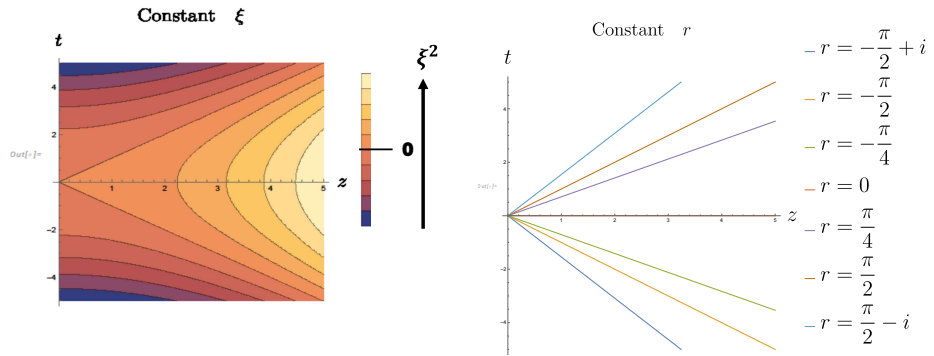


Figure 58: Contour plots of constant ξ^2 surfaces (left) and constant y surfaces (right). The figure is cited from [7]

double Wick rotation of the original Janus solution [115]. The final form is

$$\begin{aligned}
ds^2 &= -dy^2 + f(y) \frac{d\xi^2 + dx^2}{\xi^2}, \\
\text{Region II : } f(y) &= \frac{1}{2} \left(1 + \sqrt{1 - 2\gamma^2} \cos 2y \right), \\
\varphi(y) &= \frac{1}{\sqrt{2}} \log \left(\frac{1 + \sqrt{1 - 2\gamma^2} + i\sqrt{2}\gamma \tan y}{1 + \sqrt{1 - 2\gamma^2} - i\sqrt{2}\gamma \tan y} \right).
\end{aligned} \tag{12.103}$$

If we focus on region II where y is real, and take γ real with $|\gamma| < \frac{1}{\sqrt{2}}$, the dilaton is purely imaginary:

$$\varphi(y) = i \frac{2\gamma \tan y}{1 + \sqrt{1 - 2\gamma^2}}. \tag{12.104}$$

This solution coincides with the Wick-rotated Euclidean Janus geometry (12.48) under the particular continuation $y \rightarrow iy$. By analytically continuing further, we obtain the solution in the other two regions:

$$\begin{aligned}
ds^2 &= d\rho^2 - f(\rho) \frac{-d\eta^2 + dx^2}{\eta^2}, \\
\text{Region I : } f(\rho) &= \frac{1}{2} \left(1 - \sqrt{1 - 2\gamma^2} \cosh 2\rho \right), \\
\varphi(\rho) &= \frac{1}{\sqrt{2}} \log \left(\frac{1 + \sqrt{1 - 2\gamma^2} + \sqrt{2}\gamma \coth \rho}{1 + \sqrt{1 - 2\gamma^2} - \sqrt{2}\gamma \coth \rho} \right),
\end{aligned} \tag{12.105}$$

$$\begin{aligned}
ds^2 &= d\rho^2 - f(\rho) \frac{-d\eta^2 + dx^2}{\eta^2}, \\
\text{Region III : } f(\rho) &= \frac{1}{2} \left(1 - \sqrt{1 - 2\gamma^2} \cosh 2\rho \right), \\
\Phi(\rho) &= \frac{1}{\sqrt{2}} \log \left(\frac{1 + \sqrt{1 - 2\gamma^2} - \sqrt{2}\gamma \coth \rho}{1 + \sqrt{1 - 2\gamma^2} + \sqrt{2}\gamma \coth \rho} \right).
\end{aligned} \tag{12.106}$$

We make two remarks about this geometry. First, there exist lines $\rho = \rho_c$ such that

$$f(\rho_c) = 0, \tag{12.107}$$

where the dS_2 slices shrink to a point. On this line, we also have

$$1 + \sqrt{1 - 2\gamma^2} - \sqrt{2}\gamma \coth \rho_c = 0, \tag{12.108}$$

so the dilaton diverges, and the solution becomes singular at $\rho = \rho_c$. Nevertheless, since we are effectively describing a non-Hermitian evolution, we assume that such

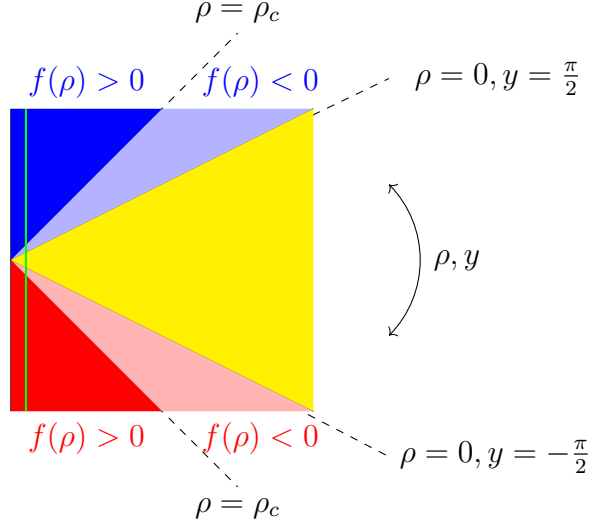


Figure 59: Sketch of the Janus spacetime dual to a CFT with a spacelike interface and nonzero γ . The red and light red regions represent III, the yellow region is II, and the blue and light blue regions are I. The green line indicates the schematic position of the cutoff surface.

exotic behavior in the gravity dual is acceptable and proceed. Moreover, for $0 \leq \rho < \rho_c$, we have $f(\rho) > 0$ and $\Phi(\rho)$ is complex, reminiscent of our brane-localized scalar model.

Second, the asymptotic structure is as follows. In region II, the warp factor of the H_2 slices does not diverge as $y \rightarrow \pm \frac{\pi}{2}$ due to the Wick rotation. In contrast, in regions I and III, the factor $-f(\rho)$ in (12.104) and (12.106) grows exponentially as $\rho \rightarrow \infty$, so each of these regions is asymptotically AdS. We therefore place the holographic CFT on these asymptotic boundaries.

■ Time evolution of pseudo entropy

We now compute the holographic pseudo entropy in this Lorentzian solution. We take the subsystem A to be the interval $x \in [-l, l]$ on the boundary time slice $t = t_0$; for definiteness, we focus on $t_0 < 0$ (the $t_0 > 0$ case is analogous). In this situation, the HRT surface starts in region III. We denote this codimension-two surface by $\eta = \eta(\rho)$, $x = x(\rho)$, and write the corresponding Lagrangian as

$$\mathcal{L}_{\text{III}} = \sqrt{1 - f(\rho) \frac{-\dot{\eta}^2 + \dot{x}^2}{\eta^2}}. \quad (12.109)$$

As before, this Lagrangian has two symmetries:

1. AdS_2 scaling:

$$\eta \rightarrow (1 + \lambda)\eta, \quad x \rightarrow (1 + \lambda)x. \quad (12.110)$$

The associated Noether charge is

$$Q_{\text{AdS}_2} = -\frac{f(\rho)}{\mathcal{L}_{\text{III}}} \frac{-\dot{\eta}\eta + \dot{x}x}{\eta^2}. \quad (12.111)$$

Due to the $\mathbb{Z}_2$ symmetry $x \rightarrow -x$, there is a turning point at $x = 0$ where $\frac{d\eta}{dx} = 0$. Hence $Q_{\text{AdS}_2} = 0$, leading to

$$\eta^2 = x^2 - C, \quad (12.112)$$

with some constant C . As before, the derivation does not rely on the explicit form of $f(\rho)$, so it is natural to assume that A is independent of γ and agrees with the vacuum AdS_3 result,

$$C = l^2 - t_0^2. \quad (12.113)$$

Equivalently, we impose the boundary condition $\eta(l) = t_0$.

2. x -translation:

$$\eta \rightarrow \eta, \quad x \rightarrow x + c, \quad (12.114)$$

with Noether charge

$$Q_T = -\frac{f(\rho)}{\mathcal{L}_{\text{III}}} \frac{\dot{x}}{\eta^2}. \quad (12.115)$$

Combining (12.109), (12.112) and (12.115), we obtain

$$\sqrt{C} \left(\frac{\dot{x}}{x^2 - C} \right)^2 = \frac{\alpha}{f(\rho)(f(\rho) - \alpha)}, \quad (12.116)$$

$$\mathcal{L}_{\text{III}}^2 = \frac{f(\rho)}{f(\rho) - \alpha}, \quad (12.117)$$

where we introduced $\alpha = CQ_T^2$. The first equation integrates to

$$x(\rho) = \sqrt{C} \tanh \left(-\sqrt{\frac{-k}{\chi}} F(W(\rho)|k) + B \right), \quad (12.118)$$

with

$$\begin{aligned} W(\rho) &= \pi - \tilde{W}(\rho), \\ \tilde{W}(\rho) &= \text{Arcsin} \left(\sqrt{\frac{(1-\chi)(2\alpha + \chi \cosh 2\rho - 1) \text{csch}^2 \rho}{4\alpha\chi}} \right), \\ k &= \frac{4\alpha\chi}{(1-\chi)(-2\alpha + \chi + 1)}. \end{aligned} \quad (12.119)$$

The constants α and B are determined from boundary conditions on $x(\rho)$:

- Because of the $\mathbb{Z}_2$ symmetry, $\rho(x)$ is an even function of x and has a turning point at $x = 0$ where $\frac{d\rho}{dx} = 0$. From (12.116), this occurs when either $f(\rho_t) = 0$ or $f(\rho_t) = \alpha$. We assume the correct choice is

$$2\alpha + \chi \cosh 2\rho_t - 1 = 0, \quad (12.120)$$

in analogy with the vacuum AdS_3 case, where $f(\rho) = 0$ corresponds to $x = \pm\sqrt{C}$. There are two possible branches for $\sin^{-1}(0)$ in (12.119). Choosing $\sin^{-1}(0) = 0$ gives $B = 0$, while choosing $\sin^{-1}(0) = \pi$ leads to

$$B = 2\sqrt{\frac{-k}{\chi}} K(k). \quad (12.121)$$

In the vacuum AdS_3 limit $\chi \rightarrow 1$ (so $k \rightarrow +\infty$), this second branch reproduces the correct behavior: using the asymptotic expansion of $K(k)$, we obtain

$$B \rightarrow \frac{1}{\sqrt{\chi}} \log(-k) \rightarrow \pm\infty. \quad (12.122)$$

- At the boundary $\rho \rightarrow \infty$, we require $x \rightarrow l$. This fixes α :

$$\begin{aligned} W(\rho \rightarrow \infty) &= \pi - \text{Arcsin} \sqrt{\frac{1-\chi}{2\alpha}}, \\ l &= \sqrt{C} \tanh \left(\sqrt{\frac{-k}{\chi}} F \left(\tilde{W}(\rho_\infty) \middle| k \right) \right). \end{aligned} \quad (12.123)$$

In figure 60, we show the resulting $\alpha(t_0)$ for representative parameters. We see that α generally becomes complex around $t_0 \gtrsim -l$. The small imaginary part $-i\epsilon'$ ensures that the state is prepared as the vacuum and yields smooth numerical plots.

Thus we obtain the geodesic

$$x(\rho; t_0) = \sqrt{C} \tanh \left(\sqrt{\frac{-k}{\chi}} F \left(\tilde{W}(\rho) \middle| k \right) \right). \quad (12.124)$$

A few comments are in order. First, the turning point ρ_t depends on t_0 for generic $\gamma < \gamma_c$. For small γ , we expect that $\rho_t(t_0)$ remains real and positive until it approaches ρ_c , which happens roughly around $t_0 \sim -l$. In this regime, ρ_t and the entire geodesic lie in the part of region III where $f(\rho) > 0$; see figure 59. For later

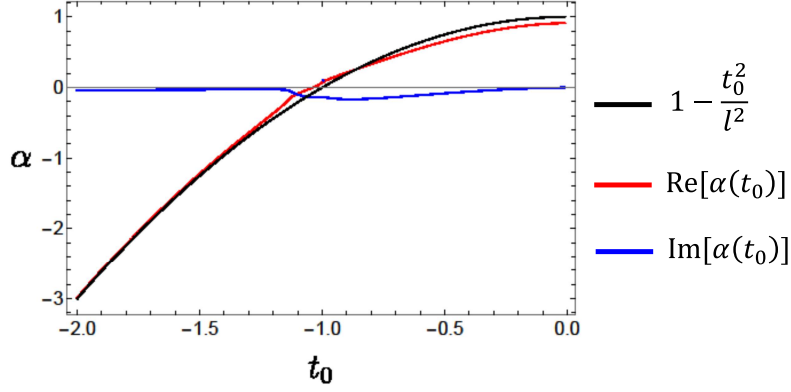


Figure 60: Dependence of $\alpha(t_0 - i\epsilon')$ on t_0 for $l = 1, \gamma = 0.4, \epsilon = \epsilon' = 0.01$. The solid curve $1 - \frac{t_0^2}{l^2}$ shows the corresponding vacuum AdS_3 result ($\gamma = 0, \chi = 1$). The figure is cited from [7].

times, the turning point enters the region where $f(\rho) < 0$, and may eventually reach region II where ρ_t becomes purely imaginary. This picture is accurate for sufficiently small γ . However, for larger γ (stronger coupling between system and bath), ρ_t can become a generic complex number after some time $t_0 \sim O(l)$, reflecting the fact that α itself becomes complex. In this case, the geodesic moves into another analytic continuation of region III, rather than simply transitioning into region II.

Second, for sufficiently early times the geodesic can have time-like segments because parts of it lie in regions with $f(\rho) < 0$, leading to complex-valued holographic pseudo entropy. In figure 62 we plot representative profiles of $x(\rho)$.

The geodesic length is given by

$$\begin{aligned} \text{Area}[\Gamma_A] &= -2 \int d\rho \sqrt{\frac{f(\rho)}{f(\rho) - f(\rho_t)}} \\ &= \left[\sqrt{\frac{-k}{\alpha\chi}} ((1 - \chi)F(W(\rho)|k) + (2\alpha + \chi - 1)\Pi(j; W(\rho)|k)) \right]_{\rho_\infty}^{\rho_t}, \end{aligned} \quad (12.125)$$

where

$$j = \frac{2\alpha}{1 - \chi}. \quad (12.126)$$

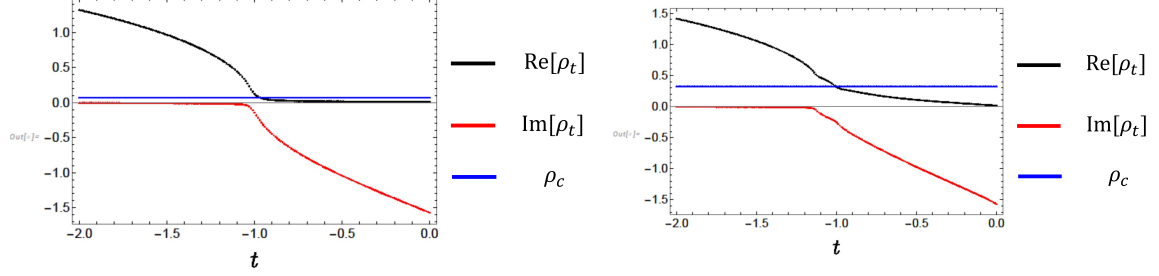


Figure 61: Turning point ρ_t as a function of t_0 . Left: $\gamma = 0.1$; right: $\gamma = 0.4$. We use $l = 1, \epsilon = \epsilon' = 0.01$. For small γ , we see a transition from real to purely imaginary ρ_t around $t_0 \sim -l$, whereas for larger γ , ρ_t attains generic complex values. The figure is cited from [7]

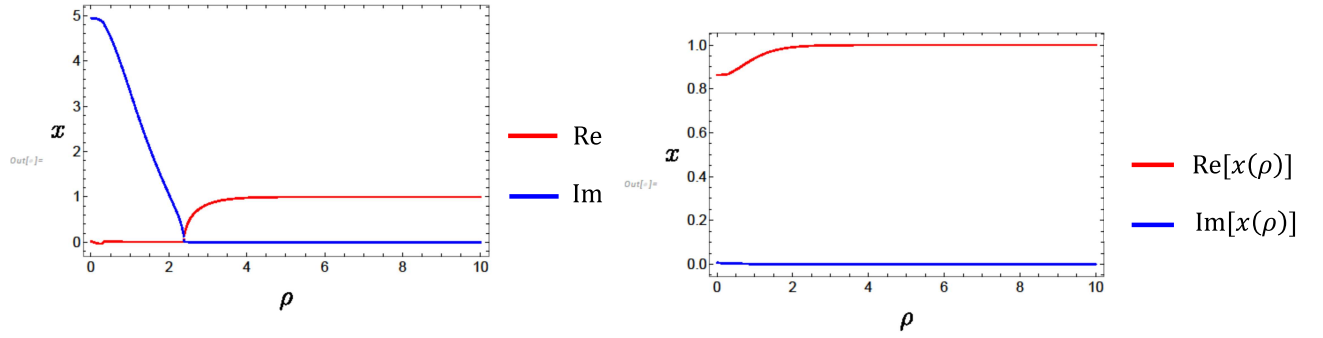


Figure 62: Profiles of $x(\rho)$ for $t_0 = 5$ (left) and $t_0 = 0.5$ (right). Parameters are $\gamma = 0.4, l = 1, \epsilon = \epsilon' = 0.01$. In both cases, $x(\rho)$ approaches zero at $\rho = \rho_t$. The figure is cited from [7]

We introduce a cutoff surface at $\rho = \rho_\infty$ via

$$\frac{\epsilon}{-t_0} = \frac{1}{\sqrt{1 - f(\rho_\infty)}}. \quad (12.127)$$

This choice is convenient because the quantity under the square root is always positive and reduces to the vacuum AdS result when $\gamma = 0$. It provides a good cutoff as long as $|t_0| \gg \epsilon$.

Using $W(\rho_t) = \pi$ and the identities

$$\begin{aligned} F(\pi - \theta | k) &= 2K(k) - F(\theta | k), \\ \Pi(j; \pi - \theta | k) &= 2\Pi(j | k) - \Pi(j; \theta | k), \end{aligned} \quad (12.128)$$

we arrive at

$$\text{Area}[\Gamma_A] = -\sqrt{\frac{-k}{\alpha\chi}} \left((1 - \chi)F\left(\tilde{W}(\rho_\infty) \middle| k\right) + (2\alpha + \chi - 1)\Pi\left(j; \tilde{W}(\rho_\infty) \middle| k\right) \right). \quad (12.129)$$

As in the Euclidean case, we can use the identity (12.72) to obtain

$$\begin{aligned} \text{Area}[\Gamma_A] &= \sqrt{\frac{-k}{\alpha\chi}} \left(2\alpha F\left(\tilde{W}(\rho_\infty) \middle| k\right) - (2\alpha + \chi - 1)\Pi\left(\frac{k}{j}; \tilde{W}(\rho_\infty) \middle| k\right) \right) \\ &\quad + \log \left(\frac{\sqrt{\frac{(j-k)(j-1)}{j}} \tan \tilde{W}(\rho_\infty) + \sqrt{1 - k \sin^2 \tilde{W}(\rho_\infty)}}{-\sqrt{\frac{(j-k)(j-1)}{j}} \tan \tilde{W}(\rho_\infty) + \sqrt{1 - k \sin^2 \tilde{W}(\rho_\infty)}} \right). \end{aligned} \quad (12.130)$$

Using (12.123), this can also be written as

$$\begin{aligned} \text{Area}[\Gamma_A] &= \sqrt{\frac{-k}{\alpha\chi}} \left(2\alpha \sqrt{\frac{\chi}{-k}} \text{arctanh}\left(\frac{l}{\sqrt{C}}\right) - (2\alpha + \chi - 1)\Pi\left(\frac{k}{j}; \tilde{W}(\rho_\infty) \middle| k\right) \right) \\ &\quad + \log \left(\frac{\sqrt{\frac{(j-k)(j-1)}{j}} \tan \tilde{W}(\rho_\infty) + \sqrt{1 - k \sin^2 \tilde{W}(\rho_\infty)}}{-\sqrt{\frac{(j-k)(j-1)}{j}} \tan \tilde{W}(\rho_\infty) + \sqrt{1 - k \sin^2 \tilde{W}(\rho_\infty)}} \right). \end{aligned} \quad (12.131)$$

As illustrated in figure 63, the pseudo entropy obtained in this Lorentzian construction exhibits the same behavior as that obtained by analytic continuation of the Euclidean model. This is consistent with the fact that, in our setup, the transition matrix is the same in both descriptions.

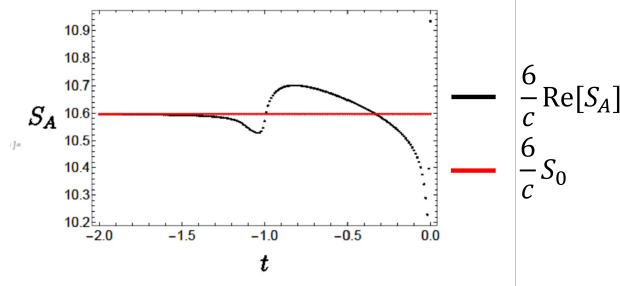


Figure 63: Time evolution of the pseudo entropy. We set $\gamma = 0.4, \epsilon = 0.01, l = 1$. The reference value S_0 denotes the vacuum entanglement entropy without defects, $S_0 = \frac{c}{3} \log \frac{2l}{\epsilon}$. The figure is cited from [7]

12.3 Measurement-induced phase transition–like behavior in holographic BCFT and its distinction from actual measurement

In the above two examples we see that the interface or boundaries spreading in spatial directions decrease the entanglement entropy, and this behavior is similar to that of measurements. In condensed matter theories, there are nonequilibrium phase transitions in chaotic Hamiltonians under continuous measurement, called measurement-induced phase transitions. The measurement-induced phase transition is an entanglement phase transition. Suppose we have some many-body Hamiltonians and some measurement with a parameter which expresses the “strength” of measurements. As a concrete example, in the original reference [234], they consider unitary random circuit models with local Haar-random unitaries and measurements. The measurement happens with rate p . They found that

1. for small $0 \leq p < p_c$, the entanglement entropy for some subsystems grows linearly $S_A \propto t$, similar to the quantum quench, and saturates after some time with volume-law entanglement;
2. for the critical value $p = p_c$, the entanglement entropy grows logarithmically $S_A \propto \log t$;
3. for large enough measurement rate $p > p_c$, the entanglement growth becomes constant $S_A = \text{const.}$

Similar behavior can be seen in a free fermion chain with stochastic Schrödinger equations [235].

In these discussions, the essential point is the competition between entanglement growth by unitary time evolution and entanglement decrease by measurement.

At this moment we may think that a similar entanglement phase transition can be realized by holographic BCFTs or ICFTs. Since we know that the unitary time

evolution from a smeared boundary state gives linear growth of entanglement, and on the other hand we know that the final-state projection, i.e. the bra boundary states in the Lorentzian time evolution, gives entanglement decrease. In [7], we considered Lorentzian path integrals with different boundary states $|\mathbf{B}(\phi)\rangle$ and $|\mathbf{B}(-\phi)\rangle$. The parameter ϕ is an analogue of the measurement strength and is obtained by inserting a boundary operator on the boundary. We then considered the following path integrals in Figure 64.

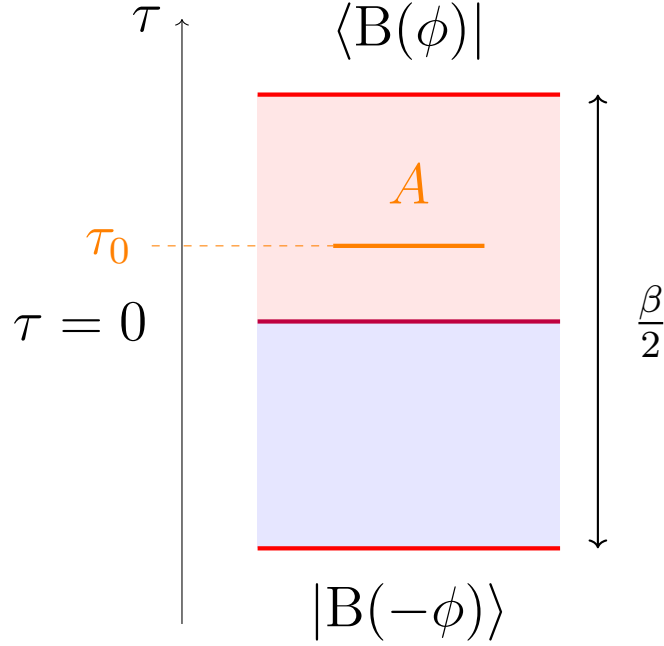


Figure 64: The model with different boundary states discussed in the above sections.

We consider the analytic continuation to the Lorentzian path integral, which is one kind of generalization of the quench setup in 7.2. The holographic duals of these path integrals are given by bulk AdS spacetime with an EOW brane and a brane-localized matter theory. We compute the pseudo entanglement entropy defined in the Lorentzian path integral and find that the following entanglement phase transitions occur:

1. for $\phi < \phi_{\text{crit}}$, we see linear growth of entanglement $S_A \propto t$;
2. for $\phi = \phi_{\text{crit}}$, we see logarithmic growth $S_A \propto \log t$;
3. for $\phi > \phi_{\text{crit}}$, instead of linear growth, we see constant entanglement $S_A = \text{const.}$

This is very similar to measurement-induced phase transitions. However, it is not the same, because

- Measurement-induced phase transitions probe quantities that are nonlinear in the density matrix, which follows a stochastic Schrödinger equation. Concretely, we need to consider $\mathbb{E}[S_{\text{v.N.}}[\rho]]$ rather than $S_{\text{v.N.}}[\mathbb{E}[\rho]]$ [235]. The latter appears in the entropy of Lindblad equations. However, in our setup, at first sight, there is no randomness, so it should be a different type of phase transition.
- Moreover, our phase transition concerns the pseudo entropy and not the entanglement entropy. To be honest, it is not clear how the pseudo entanglement discussion is related to that with continuous measurement.

However, at least one may possibly relate this to pseudo-Hermitian or non-unitary CFTs. In [10, 11], we consider CFTs with non-Hermitian Hamiltonians. In this case, we need to distinguish the ket and bra boundary states, denoted by $\langle \bar{\text{B}} |$ and $|\text{B}\rangle$. Similar to the TFD states in imaginary Janus deformation, it holds that $(|\text{B}\rangle)^\dagger \neq \langle \bar{\text{B}} |$. In pseudo-Hermitian systems, interestingly, there exists a linear operator V such that

$$V H V^{-1} = H^\dagger, \quad (12.132)$$

and we can map the right eigenvector to the left eigenvector by

$$\langle E_n^{(l)} | = (V K | E_n^{(r)} \rangle)^\dagger, \quad (12.133)$$

where K makes c -numbers complex conjugate. Then, with this operation, $(V K |\text{B}\rangle)^\dagger = \langle \bar{\text{B}} |$.

The above discussion of the pseudo-entropy phase transition is for a Hermitian Hamiltonian. If we consider some operator V that effectively flips $\phi \rightarrow -\phi$, we can say

$$(|\text{B}(\phi)\rangle)^\dagger \neq \langle \text{B}(-\phi) |, \quad (V |\text{B}(\phi)\rangle)^\dagger = \langle \text{B}(-\phi) |, \quad (12.134)$$

and obtain a similar boundary structure for the pseudo-Hermitian case. A more concrete discussion will be an interesting future work.

Part IV

Summary

13 Future problems and Conclusion

In this thesis, we have explored the deep connection between two seemingly distinct problems in theoretical physics: the information loss paradox in quantum gravity and the mechanisms of thermalization in quantum many-body systems. The key framework that unifies these problems is the AdS/CFT correspondence, which relates gravitational dynamics to non-equilibrium dynamics.

In Part II, we established quantitative relation between the eigenstate thermalization hypothesis (ETH) and non-perturbative gravitational contributions such as wormholes without randomness. By employing large deviation techniques, we demonstrated that the long-time behavior of thermal correlation functions is universally governed by ETH, and that toy models of holographic many-body systems necessarily satisfy the strong form of ETH in any spacetime dimension. Note that our results based on the validity of inequalities (6.32) and (8.23) whose complete proof are still in open questions. However, we are missing is more strong evidence of existence of wormholes in higher dimensions. And also our treatment of degeneracy of spectrum in CFTs are not sophisticated. Especially, in two dimensional CFTs possess many degeneracies coming from infinite number of conserved charges called quantum KdV series [236]. To understand, the ETH with these charges and its implication to bulk spacetime is one of the most important topic for this directions [237]. This provides a resolution of Maldacena's information loss problem: the late-time deviation from semiclassical decay is not mysterious but encoded in the exponentially small off-diagonal ETH matrix elements, which capture the same non-perturbative gravitational physics responsible for wormhole saddles.

In Part III, we extended holographic duality to open quantum systems, motivated by Hawking's original formulation of the information loss problem in asymptotically flat spacetime. By coupling AdS gravity to a bath and introducing controlled non-Hermitian deformations such as the imaginary Janus interface, we identified the building blocks of a holographic correspondence for dissipative and measurement-induced dynamics. This provides a concrete setting in which the two essential components of open quantum systems acquire geometric interpretations in the bulk. The future problem is constricting the Lindblad holography by combining all of our results. For light primary case, it is easy but to see interesting evaporation behavior we need to consider back reaction or wait very long time. This is challenging problems.

Overall, the results of this thesis demonstrate that information preservation in quantum gravity is intimately tied to the microscopic structure of quantum chaos and thermalization. These insights not only contribute to the foundations of quantum gravity but also open new possibilities for studying non-equilibrium dynamics in strongly coupled quantum matter.

Acknowledgements

I would like to express my deepest gratitude to my advisor, Professor Tadashi Takayanagi, for his continuous guidance, encouragement, and insightful comments throughout my doctoral research. His support has been invaluable to both my academic development and the completion of this thesis.

I also appreciate Yu Nakayama for checking the draft and pointing out mistakes in the paper [1].

I am also grateful to Professors Seiji Terashima, Johanna Erdmenger, Shinichi Sasa, Masamichi Miyaji, Masataka Watanabe, Masahito Yamazaki, Tokiro Numasawa, Gabriele Di Ubaldo and Rathindra Nath Das for their helpful discussions, suggestions, and warm encouragement. Their expertise has greatly enriched my research.

I would like to thank all the members of the YITP for creating a stimulating and supportive research environment. TK is supported by Grant-in-Aid for JSPS Fellows No. 23KJ1315.

Finally, I would like to express my heartfelt gratitude to my family and my fiancée Hiroko Sakuma for relaxed daily life, patience, and encouragement. Their support has sustained me throughout this journey.

A AdS spacetime, geodesic and its length

A.1 Geodesic in AdS spacetime via embedding method

■ AdS space

First we explain AdS space and asymptotically AdS space [69, 70, 238, 239]. The $(d+1)$ -dimensional anti-de Sitter space AdS_{d+1} is a maximally symmetric spacetime with negative curvature and has the isometry group $O(d, 2)$. Its Riemann curvature tensor is

$$R[G]_{ABCD} = \frac{1}{L_{\text{AdS}}^2} (G_{AD}G_{BC} - G_{AC}G_{BD}), \quad (\text{A.1})$$

and it satisfies the Einstein equation

$$R[G]_{AB} - \frac{1}{2}R[G]G_{AB} + \Lambda G_{AB} = 0, \quad \Lambda = -\frac{d(d-1)}{L_{\text{AdS}}^2}. \quad (\text{A.2})$$

AdS_{d+1} can be obtained as a hypersurface embedded in the flat space $\mathbb{R}^{d,2}$ with signature $(d, 2)$. The metric of $\mathbb{R}^{d,2}$ is

$$ds_{d+2}^2 = \eta_{\text{MN}} dX^{\text{M}} dX^{\text{N}} = -dX_{-1}^2 - dX_0^2 + \sum_{a=1}^d dX_a^2. \quad (\text{A.3})$$

AdS_{d+1} is described by the hypersurface

$$X^{\text{M}} X^{\text{N}} \eta_{\text{MN}} = -(X_{-1})^2 - (X_0)^2 + \sum_{a=1}^d (X_a)^2 = -L_{\text{AdS}}^2. \quad (\text{A.4})$$

■ Global coordinates

The embedding

$$X_{-1} = L_{\text{AdS}} \cosh r_g \cos \tau_g, \quad (\text{A.5})$$

$$X_0 = L_{\text{AdS}} \cosh r_g \sin \tau_g, \quad (\text{A.6})$$

$$X_a = L_{\text{AdS}} \sinh r_g \theta_a, \quad \sum_{a=1}^{d-1} \theta_a \theta_a = 1, \quad (\text{A.7})$$

gives the metric

$$ds^2 = L_{\text{AdS}}^2 (-\cosh^2 r_g d\tau_g^2 + dr_g^2 + \sinh^2 r_g d\Omega_{d-1}^2). \quad (\text{A.8})$$

At this point $\tau_g \in (0, 2\pi)$, so the geometry contains closed timelike curves. The global AdS metric is obtained by taking the universal cover $\tau_g \in \mathbb{R}$. The conformal boundary corresponds to $r_g \rightarrow \infty$.

■ Poincaré coordinates

The embedding

$$X_{-1} = \frac{1}{2z}(z^2 + L_{\text{AdS}}^2 + x^a x^b \eta_{ab}), \quad (\text{A.9})$$

$$X_a = \frac{L_{\text{AdS}} x^a}{z}, \quad (a = 0, \dots, d-1), \quad (\text{A.10})$$

$$X_d = \frac{1}{2z}(z^2 - L_{\text{AdS}}^2 + x^a x^b \eta_{ab}), \quad (\text{A.11})$$

gives the Poincaré metric

$$ds^2 = \frac{dz^2 + \eta_{ab} dx^a dx^b}{z^2}. \quad (\text{A.12})$$

The conformal boundary is at $z = 0$. The Poincaré patch covers only part of global AdS.

■ BTZ black hole

For the AdS₃ case, any vacuum solution of the Einstein equation is locally equivalent and obtained by a coordinate transformation and an appropriate compactification. Here we consider the black hole case. The BTZ black hole metric is

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dt^2 + \frac{L_{\text{AdS}}^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left(d\phi - \frac{r_+ r_-}{L_{\text{AdS}} r^2} dt \right)^2. \quad (\text{A.13})$$

The mass and angular momentum are given by

$$M = \frac{r_+^2 + r_-^2}{L_{\text{AdS}}^2}, \quad J = \frac{2r_+ r_-}{L_{\text{AdS}}}. \quad (\text{A.14})$$

The embedding into $\mathbb{R}^{2,2}$ with coordinates $X_{\mathbf{M}}$ satisfies

$$\begin{aligned} -(X_{-1})^2 - (X_0)^2 + (X_1)^2 + (X_2)^2 &= -L_{\text{AdS}}^2, \\ ds^2 &= -(dX_{-1})^2 - (dX_0)^2 + (dX_1)^2 + (dX_2)^2. \end{aligned} \quad (\text{A.15})$$

and

$$\begin{aligned} X_{-1} &= L_{\text{AdS}} \sqrt{A(r)} \cosh \hat{\phi}, \\ X_0 &= L_{\text{AdS}} \sqrt{B(r)} \sinh \hat{t}, \\ X_1 &= L_{\text{AdS}} \sqrt{A(r)} \sinh \hat{\phi}, \\ X_2 &= L_{\text{AdS}} \sqrt{B(r)} \cosh \hat{t}, \end{aligned} \quad (\text{A.16})$$

where

$$\begin{aligned} A(r) &= \frac{r^2 - r_-^2}{r_+^2 - r_-^2}, & B(r) &= A(r) - 1 = \frac{r^2 - r_+^2}{r_+^2 - r_-^2}, \\ \hat{t} &= \frac{r_+}{L_{\text{AdS}}} t - \frac{r_-}{L_{\text{AdS}}} \phi, & \hat{\phi} &= \frac{r_+}{L_{\text{AdS}}} \phi - \frac{r_-}{L_{\text{AdS}}} t. \end{aligned} \quad (\text{A.17})$$

For simplicity, set $r_- = 0$. The metric becomes

$$\begin{aligned} ds^2 &= -(r^2 - r_H^2) dt^2 + \frac{L_{\text{AdS}}^2}{r^2 - r_H^2} dr^2 + r^2 d\phi^2 \\ &= L_{\text{AdS}}^2 d\rho^2 + r_H^2 (-\sinh^2 \rho dt^2 + \cosh^2 \rho d\phi^2), \end{aligned} \quad (\text{A.18})$$

where $r = r_H \cosh \rho$.

The mass, angular momentum, and Hawking temperature are

$$M = \frac{r_H^2}{L_{\text{AdS}}^2}, \quad J = 0, \quad T_H = \beta^{-1} = \frac{r_H}{2\pi L_{\text{AdS}}}. \quad (\text{A.19})$$

The embedding becomes

$$\begin{aligned} X_{-1} &= L_{\text{AdS}} \frac{r}{r_H} \cosh \left(\frac{r_H \phi}{L_{\text{AdS}}} \right) = L_{\text{AdS}} \cosh \rho \cosh \left(\frac{r_H \phi}{L_{\text{AdS}}} \right), \\ X_0 &= L_{\text{AdS}} \sqrt{\frac{r^2 - r_H^2}{r_H^2}} \sinh \left(\frac{r_H t}{L_{\text{AdS}}} \right) = L_{\text{AdS}} \sinh \rho \sinh \left(\frac{r_H t}{L_{\text{AdS}}} \right), \\ X_1 &= L_{\text{AdS}} \frac{r}{r_H} \sinh \left(\frac{r_H \phi}{L_{\text{AdS}}} \right) = L_{\text{AdS}} \cosh \rho \sinh \left(\frac{r_H \phi}{L_{\text{AdS}}} \right), \\ X_2 &= L_{\text{AdS}} \sqrt{\frac{r^2 - r_H^2}{r_H^2}} \cosh \left(\frac{r_H t}{L_{\text{AdS}}} \right) = L_{\text{AdS}} \sinh \rho \cosh \left(\frac{r_H t}{L_{\text{AdS}}} \right). \end{aligned} \quad (\text{A.20})$$

Boundary metric:

For large ρ_∞ ,

$$ds_{\text{ind}}^2 \approx r_H^2 \frac{e^{2\rho_\infty}}{4} (-dt^2 + d\phi^2). \quad (\text{A.21})$$

To understand the global structure, introduce coordinates (U, V) :

$$\begin{aligned} ds^2 &= \frac{-4L_{\text{AdS}}^2 dU dV + r_H^2 (1 - UV)^2 d\phi^2}{(1 + UV)^2} \\ &= \frac{L_{\text{AdS}}^2}{\cos^2 \mu} \left(-d\tau^2 + d\mu^2 + \left(\frac{r_H}{L_{\text{AdS}}} \right)^2 \cos^2 \tau d\theta^2 \right) \\ &= L_{\text{AdS}}^2 \left(dy^2 + \cosh^2 y \left(-d\tau^2 + \left(\frac{r_H}{L_{\text{AdS}}} \right)^2 \cos^2 \tau d\theta^2 \right) \right). \end{aligned} \quad (\text{A.22})$$

The transformations are

$$V = \tan\left(\frac{\tau + \mu}{2}\right), \quad U = \tan\left(\frac{\tau - \mu}{2}\right), \quad (\text{A.23})$$

$$\sin \mu = \tanh y, \quad \cos \mu = \frac{1}{\cosh y}. \quad (\text{A.24})$$

The embedding coordinates become

$$\begin{aligned} X_{-1} &= L_{\text{AdS}} \frac{1 - UV}{1 + UV} \cosh\left(\frac{r_H \phi}{L_{\text{AdS}}}\right) = L_{\text{AdS}} \frac{\cos \tau}{\cos \mu} \cosh\left(\frac{r_H \phi}{L_{\text{AdS}}}\right) = L_{\text{AdS}} \cosh y \cos \tau \cosh\left(\frac{r_H \phi}{L_{\text{AdS}}}\right), \\ X_0 &= L_{\text{AdS}} \frac{V + U}{1 + UV} = L_{\text{AdS}} \frac{\sin \tau}{\cos \mu} = L_{\text{AdS}} \cosh y \sin \tau, \\ X_1 &= L_{\text{AdS}} \frac{1 - UV}{1 + UV} \sinh\left(\frac{r_H \phi}{L_{\text{AdS}}}\right) = L_{\text{AdS}} \frac{\cos \tau}{\cos \mu} \sinh\left(\frac{r_H \phi}{L_{\text{AdS}}}\right) = L_{\text{AdS}} \cosh y \cos \tau \sinh\left(\frac{r_H \phi}{L_{\text{AdS}}}\right), \\ X_2 &= L_{\text{AdS}} \frac{V - U}{1 + UV} = L_{\text{AdS}} \tan \mu = L_{\text{AdS}} \sinh y. \end{aligned} \quad (\text{A.25})$$

A.2 Geodesic length of AdS spacetimes

Recalling that AdS space can be obtained as a hypersurface embedded in a flat spacetime of one higher dimension, the problem of computing geodesics in AdS is equivalent to solving the following constrained variational problem:

$$S = \int ds L(X^{\mathbf{M}}(s)), \quad L = \frac{1}{2} \eta_{\mathbf{MN}} \dot{X}^{\mathbf{M}} \dot{X}^{\mathbf{N}} - \lambda(s) F(X^{\mathbf{M}}), \quad (\text{A.26})$$

$$F(X^{\mathbf{M}}) = \eta_{\mathbf{MN}} X^{\mathbf{M}} X^{\mathbf{N}} + L_{\text{AdS}}^2. \quad (\text{A.27})$$

Here $\dot{X}^{\mathbf{M}} = dX^{\mathbf{M}}/ds$, and the normalization condition

$$\dot{X}^{\mathbf{M}} \dot{X}_{\mathbf{M}} = \delta \quad (\text{A.28})$$

is imposed. The cases $\delta = 1, 0, -1$ correspond respectively to spacelike, null, and timelike geodesics. Below we consider the spacelike case.

The equations of motion for $X^{\mathbf{M}}$ and λ are

$$\ddot{X}^{\mathbf{M}} + 2\lambda X^{\mathbf{M}} = 0, \quad (\text{A.29})$$

$$X^{\mathbf{M}} X_{\mathbf{M}} + L_{\text{AdS}}^2 = 0. \quad (\text{A.30})$$

Differentiating the constraint twice gives

$$\ddot{X}^{\mathbf{M}} X_{\mathbf{M}} = -\dot{X}^{\mathbf{M}} \dot{X}_{\mathbf{M}} = -1. \quad (\text{A.31})$$

Using this together with the equation of motion, we obtain

$$\lambda(s) = -\frac{1}{2L_{\text{AdS}}^2}. \quad (\text{A.32})$$

Substituting this back into the equation of motion yields

$$\ddot{X}^{\mathbf{M}} - \frac{1}{L_{\text{AdS}}^2} X^{\mathbf{M}} = 0, \quad (\text{A.33})$$

$$X^{\mathbf{M}} = p^{\mathbf{M}} \cosh\left(\frac{s}{L_{\text{AdS}}}\right) + q^{\mathbf{M}} \sinh\left(\frac{s}{L_{\text{AdS}}}\right), \quad (\text{A.34})$$

where $p^{\mathbf{M}}$ and $q^{\mathbf{M}}$ are constant vectors in $(d+2)$ dimensions. The constraint implies that they are not independent:

$$p^{\mathbf{M}} p_{\mathbf{M}} = -q^{\mathbf{M}} q_{\mathbf{M}} = -L_{\text{AdS}}^2, \quad p^{\mathbf{M}} q_{\mathbf{M}} = 0. \quad (\text{A.35})$$

We shift the affine parameter s so that it vanishes at one endpoint P :

$$p^{\mathbf{M}} = X_P^{\mathbf{M}}. \quad (\text{A.36})$$

Let the other endpoint be Q , where the affine parameter takes the value s_Q . Taking the inner product of

$$X_Q^{\mathbf{M}} = p^{\mathbf{M}} \cosh\left(\frac{s_Q}{L_{\text{AdS}}}\right) + q^{\mathbf{M}} \sinh\left(\frac{s_Q}{L_{\text{AdS}}}\right) \quad (\text{A.37})$$

with $X_P^{\mathbf{M}}$ yields

$$\cosh\left(\frac{s_Q}{L_{\text{AdS}}}\right) = -\frac{X_P^{\mathbf{M}} X_Q^{\mathbf{N}} \eta_{\mathbf{MN}}}{L_{\text{AdS}}^2}. \quad (\text{A.38})$$

Furthermore, we obtain

$$q^{\mathbf{M}} = \frac{1}{\sinh\left(\frac{s_Q}{L_{\text{AdS}}}\right)} X_Q^{\mathbf{M}} - \cosh\left(\frac{s_Q}{L_{\text{AdS}}}\right) X_P^{\mathbf{M}}. \quad (\text{A.39})$$

The geodesic length is simply the on-shell action and is given by

$$\int_P^Q \sqrt{d\vec{X} \cdot d\vec{X}} = s_Q = L_{\text{AdS}} \cosh^{-1}\left(-\frac{X_P^{\mathbf{M}} X_Q^{\mathbf{N}} \eta_{\mathbf{MN}}}{L_{\text{AdS}}^2}\right). \quad (\text{A.40})$$

For example, for Poincare coordinate the geodesics distance connecting two points P and Q is given by

$$\text{Dist}(P : Q) = L_{\text{AdS}} \cosh^{-1} \left(\frac{z_P^2 + z_Q^2 + \eta_{ab}(x_P^a - x_Q^a)(x_P^b - x_Q^b)}{2z_P z_Q} \right). \quad (\text{A.41})$$

For blackhole coordinate (A.22), the geodesics length between the two points $P = (y_P, \tau_P, \theta_P), Q = (y_Q, \tau_Q, \theta_Q)$ is given by

$$\begin{aligned} & \cosh \frac{\text{Dist}(P : Q)}{L_{\text{AdS}}} \\ &= -\sinh y_P \sinh y_Q + \cosh y_P \cosh y_Q \left(\sin \tau_P \sin \tau_Q + \cos \tau_P \cos \tau_Q \cosh \left(\frac{r_H}{L_{\text{AdS}}} (\theta_P - \theta_Q) \right) \right). \end{aligned} \quad (\text{A.42})$$

B Proof of Canonical typicality

1. The key technical tool is the following version of Lévy's lemma: Let F be a continuous real-valued function defined on the sphere S^d . Let ξ denote its Lipschitz constant, i.e., the minimal number C such that, for all $\psi, \psi' \in S^d$,

$$|F(\psi) - F(\psi')| \leq C|\psi - \psi'|. \quad (\text{B.1})$$

Let $\text{Prob}[P]$ denote the probability of a condition P with respect to the uniform measure on S^d . Then for any $\epsilon > 0$,

$$\text{Prob}[|F(\psi) - F(\psi')| \geq \epsilon] \leq 2 \exp \left[-\frac{d+1}{9\pi^2 \xi^2} \epsilon^2 \right]. \quad (\text{B.2})$$

Applying this lemma to

$$F(\psi) := \|\rho_\psi^X - \rho_{\text{mc}}^X\|_1, \quad (\text{B.3})$$

we see that the Lipschitz constant satisfies $\xi \leq 2$. Indeed,

$$|F(\psi) - F(\psi')|^2 = \|\rho_\psi^X - \rho_{\text{mc}}^X\|_1 - \|\rho_{\psi'}^X - \rho_{\text{mc}}^X\|_1^2 \quad (\text{B.4})$$

$$\leq \|\rho_\psi^X - \rho_{\psi'}^X\|_1^2 \quad (\text{B.5})$$

$$= \|\text{Tr}_{X^c}(|\psi\rangle\langle\psi| - |\psi'\rangle\langle\psi'|)\|_1^2 \quad (\text{B.6})$$

$$\leq \| |\psi\rangle\langle\psi| - |\psi'\rangle\langle\psi'| \|_1^2 \quad (\text{B.7})$$

$$= 4 \left(1 - |\langle\psi|\psi'\rangle|^2 \right) = 4|\psi - \psi'|^2. \quad (\text{B.8})$$

The steps use, respectively, the triangle inequality for the trace norm, monotonicity of the trace norm under CPTP maps (in particular under partial trace), and the relation between trace distance and fidelity. Hence $\xi \leq 2$.

2. We next prove the bound

$$\mathbb{E}[F(\psi)] = \mathbb{E}[\|\rho_\psi^X - \rho_{\text{mc}}^X\|_1] \leq \sqrt{\frac{D_X}{D_{\text{eff mc}}}}, \quad (\text{B.9})$$

where

$$D_{\text{eff mc}} := \frac{1}{\text{Tr}_{X^c}[(\rho_{\text{mc}}^{X^c})^2]} = e^{S^{(2)}[\rho_{\text{mc}}^{X^c}]}. \quad (\text{B.10})$$

Using the inequality between the trace norm and the Hilbert–Schmidt norm,

$$\mathbb{E}[\|\rho_\psi^X - \rho_{\text{mc}}^X\|_1] \leq \sqrt{D_X} \mathbb{E}[\|\rho_\psi^X - \rho_{\text{mc}}^X\|_2], \quad (\text{B.11})$$

and by convexity of the square root,

$$\mathbb{E}[\|\rho_\psi^X - \rho_{\text{mc}}^X\|_2] = \mathbb{E}\left[\sqrt{\text{Tr}[(\rho_\psi^X - \rho_{\text{mc}}^X)^2]}\right] \quad (\text{B.12})$$

$$\leq \sqrt{\text{Tr}[\mathbb{E}[(\rho_\psi^X - \rho_{\text{mc}}^X)^2]]}. \quad (\text{B.13})$$

We now compute

$$\mathbb{E}[(\rho_\psi^X - \rho_{\text{mc}}^X)^2] = \mathbb{E}[(\rho_\psi^X)^2] - (\rho_{\text{mc}}^X)^2. \quad (\text{B.14})$$

Introduce a copy $T = X \otimes X^c$ and $T' = X' \otimes X'^c$. Then

$$\text{Tr}_X[(\rho_\psi^X)^2] = \text{Tr}_{XX'}[\rho_\psi^X \otimes \rho_\psi^{X'} \text{SWAP}_{XX'}], \quad (\text{B.15})$$

where $\text{SWAP}_{XX'}$ acts as

$$\text{SWAP}_{XX'} |\phi\rangle_X |\chi\rangle_{X'} = |\chi\rangle_X |\phi\rangle_{X'}. \quad (\text{B.16})$$

Averaging over the Haar measure and using

$$\mathbb{E}[c_n c_m^* c_p c_q^*] = \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} + 1)} (\delta_{nm} \delta_{pq} + \delta_{nq} \delta_{mp}), \quad (\text{B.17})$$

we obtain

$$\mathbb{E}[|\psi\rangle\langle\psi|_T \otimes |\psi\rangle\langle\psi|_{T'}] = \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} + 1)}(\mathbb{I}_{TT'} + \text{SWAP}_{TT'}). \quad (\text{B.18})$$

Hence,

$$\mathbb{E}[\text{Tr}_X[(\rho_\psi^X)^2]] = \frac{1}{D_{E,\Delta E}(D_{E,\Delta E} + 1)} \text{Tr}_{TT'}[\text{SWAP}_{XX'} + \text{SWAP}_{X^cX^c'}] \quad (\text{B.19})$$

$$\leq \text{Tr}_X[(\rho_{\text{mc}}^X)^2] + \text{Tr}_{X^c}[(\rho_{\text{mc}}^{X^c})^2]. \quad (\text{B.20})$$

Substituting this gives the bound (B.9).

3. We now prove the auxiliary bound

$$D_{\text{effmc}} \leq \frac{D_X}{D_{E,\Delta E}}. \quad (\text{B.21})$$

Indeed,

$$\text{Tr}_{X^c}[(\rho_{\text{mc}}^{X^c})^2] \leq \|\rho_{\text{mc}}^{X^c}\|_\infty \text{Tr}[\rho_{\text{mc}}^{X^c}] \quad (\text{B.22})$$

$$= \|\rho_{\text{mc}}^{X^c}\|_\infty \quad (\text{B.23})$$

$$= \sup_{|\phi\rangle \in \mathcal{H}_{X^c}} \langle\phi| \rho_{\text{mc}}^{X^c} |\phi\rangle \quad (\text{B.24})$$

$$= \frac{1}{D_{E,\Delta E}} \text{Tr}_X \left[\sup_{|\phi\rangle \in \mathcal{H}_{X^c}} \langle\phi| \mathbb{I}_T |\phi\rangle \right] \quad (\text{B.25})$$

$$= \frac{D_X}{D_{E,\Delta E}}, \quad (\text{B.26})$$

establishing (B.21).

4. Applying Lévy's lemma with $n = 2D_{E,\Delta E} - 1$, $F(\psi) = \|\rho_\psi^X - \rho_{\text{mc}}^X\|_1$, and $\xi = 2$, we have

$$\text{Prob}[\|\rho_\psi^X - \rho_{\text{mc}}^X\|_1 - \mathbb{E}[\|\rho_\psi^X - \rho_{\text{mc}}^X\|_1] \geq \epsilon] \leq 2 \exp \left[-\frac{1}{18\pi^2} D_{E,\Delta E} \epsilon^2 \right]. \quad (\text{B.27})$$

Hence,

$$\text{Prob}[\|\rho_\psi^X - \rho_{\text{mc}}^X\|_1 > \epsilon + \mathbb{E}[\|\rho_\psi^X - \rho_{\text{mc}}^X\|_1]] \quad (\text{B.28})$$

$$\leq \text{Prob}[\|\rho_\psi^X - \rho_{\text{mc}}^X\|_1 - \mathbb{E}[\|\rho_\psi^X - \rho_{\text{mc}}^X\|_1] \geq \epsilon]. \quad (\text{B.29})$$

Combining this inequality with Lemma (B.9) and Lemma (B.21) yields canonical typicality.

C Case without non-Resonant Condition

Up to this point, our analysis has relied on the non-resonance condition (5.46). In particular, any conformal field theory inevitably violates this assumption, since the underlying conformal algebra generates harmonic energy spacing. For instance, in two-dimensional conformal field theories, each primary state $|h_p, \bar{h}_p\rangle$ possesses an associated tower of descendant states with energies $h_p + \bar{h}_p + N + \bar{N}$ for $N, \bar{N} \in \mathbb{N} \cup \{0\}$. Consequently, there are numerous pairs of states with fixed energy differences such as $E_n - E_m = 1$.

In fact, we can extend the theorem (5.4) so that it remains valid even without imposing the non-resonance requirement.

Theorem 17. *Let $\epsilon > 0$ be arbitrary, and take any initial state $|\psi_0\rangle \in \mathcal{H}_{E,\Delta E}$ together with any bounded operator $\mathcal{O} \in \mathcal{B}(\mathcal{H})$. Assuming nondegeneracy of individual energy eigenvalues²⁷, the following bound holds:*

$$\left| \overline{\langle \psi(t) | \mathcal{O}(t) | \psi(t) \rangle} - \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T \right|^{2^T} \leq d_G \|\mathcal{O}\|_\infty^2 \left(\epsilon^2 + \frac{(D_{E,\Delta E})^{\frac{3}{2}}}{D_{\text{eff}}} \sqrt{\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}]} \right), \quad (\text{C.1})$$

where d_G denotes the maximal degeneracy of energy gaps:

$$d_G := \max_{(E_n, E_m) \in \mathcal{E}'_{E,\Delta E}} \{ (E_k, E_l) \in \mathcal{E}'_{E,\Delta E} \mid E_n - E_m = E_k - E_l \}. \quad (\text{C.2})$$

A second inequality also follows:

$$\left| \overline{\langle \psi(t) | \mathcal{O}(t) | \psi(t) \rangle} - \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T \right|^{2^T} \leq \inf_{\beta > 0} (\mathcal{S}(\beta)) \|\mathcal{O}\|_\infty^2 \left(\epsilon^2 + \frac{(D_{E,\Delta E})^{\frac{3}{2}}}{D_{\text{eff}}} \sqrt{\Delta_{\text{off,ETH}}^\epsilon[\mathcal{O}]} \right), \quad (\text{C.3})$$

where

$$\mathcal{S}(\beta) = e^{2\beta(E+\Delta E)} \sqrt{(|Z(\beta + it)|^2)^{2^T}}, \quad (\text{C.4})$$

and $Z(\beta + it)$ denotes the analytic continuation of the thermal partition function.

The proof follows essentially the same reasoning as the inequality derived in [240]. A detailed derivation is provided in Appendix C.1. The key outcome is that large degeneracies of energy gaps obstruct equilibration when probed through this type of inequality.

²⁷Equivalently, we may regard $\mathcal{O}$ as confined to an appropriate super-selection sector if the Hamiltonian displays degeneracy.

As an illustration, let us determine d_G in conformal field theories. In a generic CFT without special structure, d_G effectively counts the number of eigenstate pairs with energy difference $E_n - E_m = 1$. Each primary of conformal dimension Δ_p generates a conformal family, and such a family contains roughly $\lfloor E + \Delta E - \Delta_p \rfloor$ degenerate descendant contributions. This yields

$$d_G = \sum_{E < \Delta_p < E + \Delta E} d(\Delta_p) \lfloor E + \Delta E - \Delta_p \rfloor, \quad (\text{C.5})$$

with $d(\Delta_p)$ the multiplicity of primaries of weight Δ_p . An upper bound follows:

$$d_G \leq \sum_{E < \Delta_p < E + \Delta E} d(\Delta_p) (E + \Delta E - \Delta_p) < \Delta E \sum_{E < \Delta_p < E + \Delta E} d(\Delta_p). \quad (\text{C.6})$$

Using the known estimate [241],

$$d_G < \Delta E e^{S_{\text{mc}}(E) + O(\log S_{\text{mc}}(E))}, \quad (\text{C.7})$$

which holds for large E and large central charge.

In holographic settings, one may take $\epsilon^2 = \sigma^{2\eta} = O(e^{-\eta\gamma S_{\text{mc}}(E)})$ with $\eta\gamma < \frac{1}{2}$. Choosing $\Delta E < O(e^{-\gamma' S_{\text{mc}}(E)})$ for any $\gamma' > \frac{1}{2}$ guarantees equilibration for all initial states within the shell. Since the width of the energy shell must remain large enough to contain exponentially many states, we require $\gamma' < 1$, and suitable $\frac{1}{2} < \gamma' < 1$ can indeed be arranged.

A parallel reasoning may be applied to the second inequality (C.3). In a general chaotic theory, $|Z(\beta + it)|^2$ relaxes to $Z(2\beta)$, and one estimates $\mathcal{S}(\beta) \sim e^{2\beta(E-F)} \sim e^{2\beta S_{\text{mc}}(E)}$, where F is the free energy. In this situation, a yet smaller value of ΔE becomes necessary.

Throughout the above arguments we did not incorporate the influence of degeneracies in the energy eigenvalues themselves. When $\mathcal{O}$ carries no charge under any symmetry, including quantum KdV charges, the preceding treatment suffices. Actual CFTs, however, generally exhibit degeneracy, and thus a more refined analysis is required when considering operators of broader type.

C.1 Proof of Theorem 17

We now sketch the proof of Theorem 17. The logic closely mirrors that of [240]. Here we refrain from imposing the non-resonance condition, while still assuming non-degeneracy of the energy eigenvalues. We begin by considering the variance of

the long-time average:

$$\begin{aligned} \overline{\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T \right|^2}^T &= \sum_{\substack{(|E_n\rangle, |E_m\rangle) \in \mathcal{E}'_{E, \Delta E} \\ (|E_k\rangle, |E_l\rangle) \in \mathcal{E}'_{E, \Delta E}}} \overline{e^{-i((E_n - E_m) - (E_k - E_l))t}}^T \\ &\quad \times c_n c_m^* c_k c_l^* \langle E_m | \mathcal{O} | E_n \rangle \langle E_k | \mathcal{O} | E_l \rangle. \end{aligned} \quad (\text{C.8})$$

Introduce a composite index $\alpha = (n, m)$ with $n \neq m$, together with

$$\begin{aligned} M_{\alpha\beta} &:= \overline{e^{-i((E_n - E_m) - (E_k - E_l))t}}^T, \\ v_\alpha &= c_n c_m^* \langle E_n | \mathcal{O} | E_m \rangle. \end{aligned} \quad (\text{C.9})$$

Then the variance rewrites as

$$\overline{\left| \langle \psi(t) | \mathcal{O} | \psi(t) \rangle - \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle}^T \right|^2}^T = \sum_{\alpha, \beta} v_\alpha^* M_{\alpha\beta} v_\beta \leq \|M\| \sum_{\alpha} v_\alpha^* v_\alpha, \quad (\text{C.10})$$

where $\|M\|$ denotes the matrix norm, and from the definition of d_G we have the upper bound

$$\|M\| \leq \max_{\beta} \sum_{\alpha} |M_{\alpha\beta}| \leq d_G. \quad (\text{C.11})$$

The quantity $\sum_{\alpha} v_\alpha^* v_\alpha$ can be controlled exactly as in Appendix 5.4.1. Another estimate is obtained by

$$\sum_{\alpha, \beta} v_\alpha^* M_{\alpha\beta} v_\beta \leq \sqrt{\sum_{\alpha, \beta} |M_{\alpha\beta}|^2} \sum_{\alpha} v_\alpha v_\alpha^*. \quad (\text{C.12})$$

The squared Frobenius norm of M satisfies

$$\begin{aligned}
\|M\|_F^2 &= \sum_{\alpha,\beta} |M_{\alpha\beta}|^2 \\
&\leq \sum_{(|E_n\rangle, |E_m\rangle, |E_l\rangle, |E_k\rangle) \in \mathcal{E}_{E,\Delta E}^4} \overline{e^{-i((E_n-E_m)-(E_k-E_l))t} t^T} e^{i((E_n-E_m)-(E_k-E_l))t} T \\
&= \lim_{T, T' \rightarrow \infty} \frac{1}{TT'} \int_0^T dt \int_0^{T'} dt' \sum_{(|E_n\rangle, |E_m\rangle, |E_l\rangle, |E_k\rangle) \in \mathcal{E}_{E,\Delta E}^4} e^{-i((E_n-E_m)-(E_k-E_l))(t-t')} \\
&\leq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt e^{4\beta(E+\Delta E)} \sum_{(|E_n\rangle, |E_m\rangle, |E_l\rangle, |E_k\rangle) \in \mathcal{H}} e^{-\beta(E_n+E_m+E_k+E_l)} e^{-i((E_n-E_m)-(E_k-E_l))t} \\
&\leq e^{4\beta(E+\Delta E)} \overline{(|Z(\beta + it)|^2)^{2T}}.
\end{aligned} \tag{C.13}$$

D Numerical evidence for the upper bound (6.32)

We now comment on the bound (6.32)

$$\sum_{\{m_k\}: m \leq m_{\max}} \frac{\Gamma(1+m)s}{(1-s)^{m+1}} \prod_{k=2}^M \left(\frac{(-a_k)^{m_k}}{m_k!} \right) \leq \exp \left\{ 2s - \sum_{k=2}^M t_k a_k s^k \right\} \tag{D.1}$$

Since at $s = 0$, the left hand side is zero and right hand side is one we obtain the inequality. The problem is when the inequality break down since the left hand side is asymptotic series. We can numerically confirm the inequalities for small M with moderate $a_k, m_{\max}$ and t_k . For simplification we set $a_k = g_{2k} \sigma^{2k} = y^k$ with $y = \frac{\bar{G}^T}{S}$ where $S = \text{Poly}(S_{\text{mc}})$. Then the left hand side is simplified

$$(\text{LHS}) = \sum_{m=0}^{m_{\max}} \frac{\Gamma(1+m)s}{(1-s)^{m+1}} y^m \left(\sum_{\{m_k\}: \sum_{k=2}^M k m_k = m} \prod_{k=2}^M \left(\frac{(-1)^{m_k}}{m_k!} \right) \right). \tag{D.2}$$

With a choice $t_k = 6/2^k$ the right hand side is

$$(\text{RHS}) = \exp \left\{ 2s - 6 \left(\frac{ys}{2} \right)^2 \frac{1 - \left(\frac{ys}{2} \right)^{M-1}}{1 - \frac{ys}{2}} \right\}. \tag{D.3}$$

We show some numerical tests in Figure 65, Figure 66 and Figure 67. In the first plot, we see the RHS exceeds LHS for $s < 1/2$ with $y = 0.1$. When we increase M and $m_{\max}$, we observe the region where the inequality is valid is geeing narrow as in Figure 66. Once we make y smaller then the inequality recovered. From the usual discussion of the asyomtic series, we expect the inequality is valid for

$$M = 10, y = 0.1, m_{\max} = 10$$

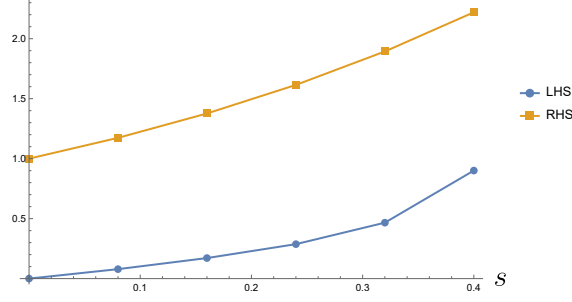


Figure 65: The plots of LHS and RHS with $M = 10, y = 0.1, m_{\max} = 10$.

$$M = 30, y = 0.1, m_{\max} = 30$$

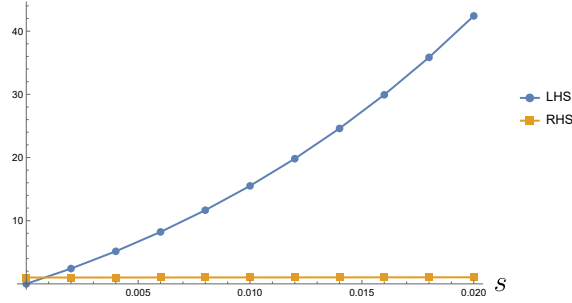


Figure 66: The plots of LHS and RHS with $M = 30, y = 0.1, m_{\max} = 30$.

$M < m_{\max} = O(1/y)$. In the physics discussion, we set $y = e^{-O(S_{\text{mc}})}$, and we expect we can set M and $m_{\max}$ to be $e^{O(S_{\text{mc}})}$.

E Numerical evidence of the inequality (8.23)

We now address the conjectured inequality (8.23). The statement is following: consider the two positive functions for vectors $x = (x_2, x_3, \dots, x_L)$

$$G_n^{(2q)}(x) := \frac{1}{2^{q+1}q!} \sum_{\{n_k\}: \sum_k kn_k=q} \frac{(2q)!}{\prod_{k=1}^q ((2m_k)!)^{n_k} (n_k)!} \prod_{k=2}^q (x_k)^{n_k} \quad (\text{E.1})$$

$$E_n^{(2q)}(x) := \sum_{\{m_k\}: m \leq m_{\max}} \frac{\Gamma(m+q)}{\Gamma(q)} \prod_{k=2}^M \frac{1}{m_k!} (-x_k)^{m_k}$$

where indices n mean the function is normalized by $1/2^{q+1}q!$. For sufficiently small x and for some positive integer $m_{\max}$, we have

$$G_n^{(2q)}(x) \leq E_n^{(2q)}(x). \quad (\text{E.2})$$

$$M = 30, y = 0.01, m_{\max} = 30$$

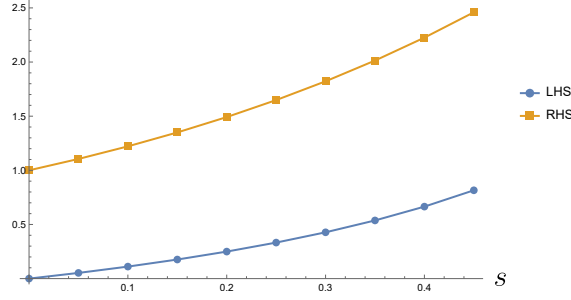


Figure 67: The plots of LHS and RHS with $M = 30, y = 0.01, m_{\max} = 30$.

For numerical calculation it is better to rewrite the summation to resolve the constraint. For function $G_n^{(2q)}(x)$, by using the relation $n_1 = q - \sum_{k=2}^q kn_k$ and for specific case $\Gamma_{2k} = \frac{1}{S^k}$, we have

$$G_n^{(2q)}(y) = \frac{(2q)!}{2^{q+1}q!} \sum_{n'_1=0}^q \frac{\text{Coeff}_G(n'_1)}{((2!)^{q-n'_1}(q-n'_1)!)} y^{n'_1} \quad (\text{E.3})$$

$$\text{Coeff}_G(n'_1) = \sum_{n_2, \dots, n_q: \sum_{k=2}^q kn_k = n'_1 = q-n_1} \frac{1}{\prod_{k=2}^q ((2n_k)!)^{n_k} (n_k)!}$$

where $y = \frac{\overline{G_{\text{can},\beta}^{(2)}}^T}{S}$ which is very small. Also for $E_n^{(2q)}$, with identification $g_{2k} = -\epsilon_k \Gamma_{2k} = -\frac{\epsilon_k}{S^k} = \frac{1}{S^k}$ and $\sigma^2 = \overline{G_{\text{can},\beta}^{(2)}}^T$, we have

$$E_n^{(2q)}(y) = \sum_{m=0}^{m_{\max}} \frac{\Gamma(q+m)}{\Gamma(q)} \text{Coeff}_E(m) y^m \quad (\text{E.4})$$

$$\text{Coeff}_E(m) = \sum_{m_2, \dots, m_M: \sum_{k=2}^M km_k = m} \frac{1}{\prod_{k=2}^q (m_k)!}.$$

We see $y = 0$

$$G_n^{(2q)}(y=0) = \frac{(2q-1)!!}{2^{q+1}q!} \leq 1 = E_n^{(2q)}(y=0). \quad (\text{E.5})$$

We set plots in Figure 68, 69, 70 and 71. We see the inequality is valid for large enough $m_{\max}$, M and small enough y . For fixed q , when we small $m_{\max}$ and M , we see the earlier violation of inequalities. By making $m_{\max}$ and M , we recover the inequalies. We put the plot of crossing points of two function in Figure 73. We find that the crossing points are given by $y_{\text{sol}}(q) \sim 1/q$. Note that we do not show the inequality for large $q = e^{O(S_{\text{mc}})}$ with physical value of S_{mc} and this is important

open question.

$$M = 100, m_{\max} = 5, q = 10$$

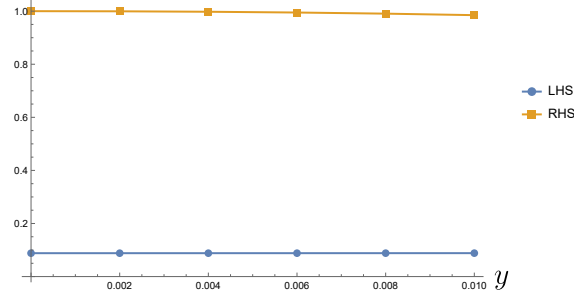


Figure 68: Plots of $G_n^{(2q)}$ and $E_n^{(2q)}$ with $M = 100, m_{\max} = 5, q = 10$.

$$M = 5, m_{\max} = 5, q = 20$$

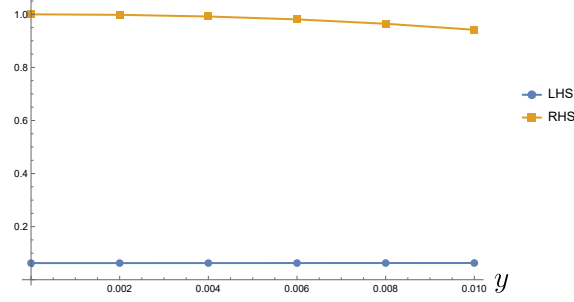


Figure 69: Plots of $G_n^{(2q)}$ and $E_n^{(2q)}$ with $M = 2, m_{\max} = 2, q = 10$.

$$M = 2, m_{\max} = 2, q = 20$$

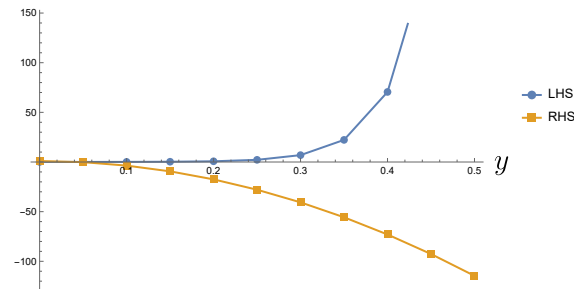


Figure 70: Plots of $G_n^{(2q)}$ and $E_n^{(2q)}$ with $M = 2, m_{\max} = 2, q = 20$.

$$M = 5, m_{\max} = 5, q = 20$$

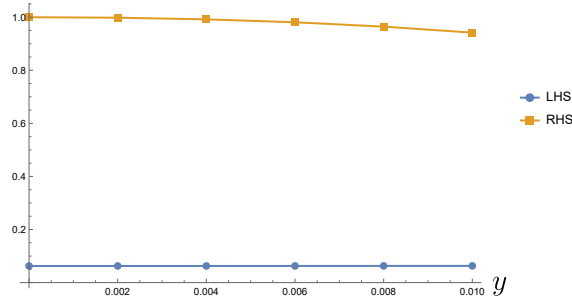


Figure 71: Plots of $G_n^{(2q)}$ and $E_n^{(2q)}$ with $M = 5, m_{\max} = 5, q = 20$.

$$M = 11, m_{\max} = 11, q = 22$$

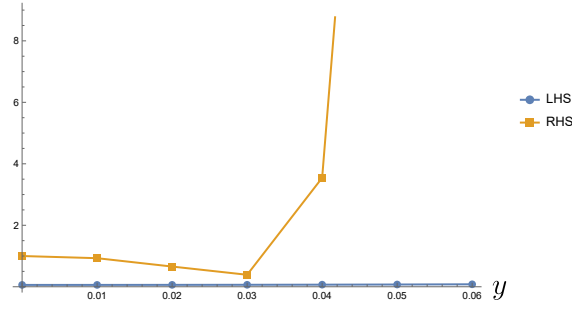


Figure 72: Plots of $G_n^{(2q)}$ and $E_n^{(2q)}$ with $M = 11, m_{\max} = 11, q = 22$.

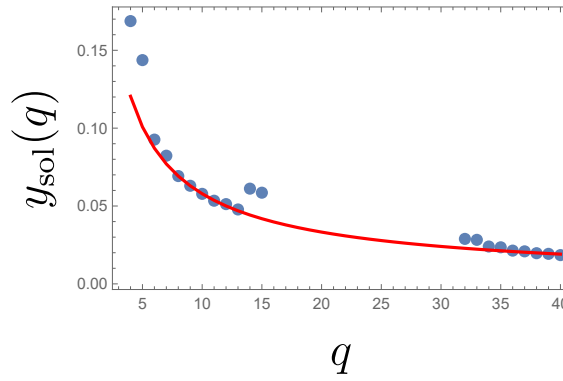


Figure 73: Plots of crossing points of $G_n^{(2q)}$ and $E_n^{(2q)}$ up to $q = 40$. We set $m_{\max} = M = \lfloor q/2 \rfloor$. The red curve denotes $y_{\text{sol}}(10) \frac{10}{q}$. In some q we cannot find solutions. For example there is no solution at $q = 22$. See Figure 72.

References

- [1] T. Kawamoto, *A strategy for proving the strong eigenstate thermalization hypothesis: chaotic systems and holography*, *JHEP* **01** (2025) 095 [[2411.09746](#)].
- [2] T. Kawamoto, R. Maeda, N. Nakamura and T. Takayanagi, *Holographic entanglement, pseudo-entropy and wormholes*, *International Journal of Modern Physics A* **0** (2025) 2548005 [<https://doi.org/10.1142/S0217751X25480057>].
- [3] T. Kawamoto, R. Maeda, N. Nakamura and T. Takayanagi, *Traversable AdS wormhole via non-local double trace or Janus deformation*, *JHEP* **04** (2025) 086 [[2502.03531](#)].
- [4] T. Kawamoto, S.-M. Ruan and T. Takayanagi, *Gluing AdS/CFT*, *JHEP* **07** (2023) 080 [[2303.01247](#)].
- [5] I. Akal, T. Kawamoto, S.-M. Ruan, T. Takayanagi and Z. Wei, *Page curve under final state projection*, *Phys. Rev. D* **105** (2022) 126026 [[2112.08433](#)].
- [6] T. Kawamoto and Y.-k. Suzuki, *Entanglement entropy via double-cone regularization*, *Phys. Rev. D* **110** (2024) 046011 [[2401.00219](#)].
- [7] H. Kanda, T. Kawamoto, Y.-k. Suzuki, T. Takayanagi, K. Tasuki and Z. Wei, *Entanglement phase transition in holographic pseudo entropy*, *JHEP* **03** (2024) 060 [[2311.13201](#)].
- [8] T. Kawamoto, S.-M. Ruan, Y.-k. Suzuki and T. Takayanagi, *A half de Sitter holography*, *JHEP* **10** (2023) 137 [[2306.07575](#)].
- [9] P.-X. Hao, T. Kawamoto, S.-M. Ruan and T. Takayanagi, *Non-extremal island in de Sitter gravity*, *JHEP* **03** (2025) 004 [[2407.21617](#)].
- [10] Y. Fukusumi and T. Kawamoto, *Generalizing fusion rules by shuffle: Symmetry-based classifications of nonlocal systems constructed from similarity transformations*, [2512.02139](#).
- [11] Y. Fukusumi and T. Kawamoto, *Generalizing quantum dimensions: Symmetry-based classification of local pseudo-Hermitian systems and the corresponding domain walls*, [2511.11059](#).
- [12] T. Kawamoto, T. Mori, Y.-k. Suzuki, T. Takayanagi and T. Ugajin, *Holographic local operator quenches in BCFTs*, *JHEP* **05** (2022) 060 [[2203.03851](#)].

- [13] I. Akal, T. Kawamoto, S.-M. Ruan, T. Takayanagi and Z. Wei, *Zoo of holographic moving mirrors*, *JHEP* **08** (2022) 296 [[2205.02663](#)].
- [14] J. Loschmidt *Wien.Ber.* **73** (1876) 128.
- [15] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7** (1973) 2333.
- [16] S. W. Hawking, *Breakdown of predictability in gravitational collapse*, *Phys. Rev. D* **14** (1976) 2460.
- [17] E. H. Lieb and J. Yngvason, *The physics and mathematics of the second law of thermodynamics*, **310** (1999) 1 [[cond-mat/9708200](#)].
- [18] L. Susskind, *The World as a hologram*, *J. Math. Phys.* **36** (1995) 6377 [[hep-th/9409089](#)].
- [19] G. 't Hooft, *Dimensional reduction in quantum gravity*, *Conf. Proc. C* **930308** (1993) 284 [[gr-qc/9310026](#)].
- [20] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [[hep-th/9711200](#)].
- [21] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [[hep-th/9905111](#)].
- [22] V. Balasubramanian, A. Bernamonti, J. de Boer, N. Copland, B. Craps, E. Keski-Vakkuri et al., *Holographic Thermalization*, *Phys. Rev. D* **84** (2011) 026010 [[1103.2683](#)].
- [23] S. W. Hawking, *Particle creation by black holes*, *Communications in Mathematical Physics* **43** (1975) 199.
- [24] J. Maldacena, *Eternal black holes in anti-de sitter*, *Journal of High Energy Physics* **2003** (2003) 021.
- [25] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)].
- [26] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)].
- [27] Y. Sekino and L. Susskind, *Fast Scramblers*, *JHEP* **10** (2008) 065 [[0808.2096](#)].

- [28] S. H. Shenker and D. Stanford, *Black holes and the butterfly effect*, *JHEP* **03** (2014) 067 [[1306.0622](#)].
- [29] J. Maldacena, S. H. Shenker and D. Stanford, *A bound on chaos*, *JHEP* **08** (2016) 106 [[1503.01409](#)].
- [30] J. v. Neumann, *Beweis des ergodensatzes und desh-theorems in der neuen mechanik*, *Zeitschrift für Physik* **57** (1929) 30.
- [31] J. M. Deutsch, *Quantum statistical mechanics in a closed system*, *Phys. Rev. A* **43** (1991) 2046.
- [32] M. Srednicki, *Chaos and Quantum Thermalization*, *Phys. Rev. E* **50** (1994) [[cond-mat/9403051](#)].
- [33] A. Kitaev, “A simple model of quantum holography.” <https://online.kitp.ucsb.edu/online/entangled15/kitaev/>, 2015.
- [34] J. Maldacena and D. Stanford, *Remarks on the Sachdev-Ye-Kitaev model*, *Phys. Rev. D* **94** (2016) 106002 [[1604.07818](#)].
- [35] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker et al., *Black Holes and Random Matrices*, *JHEP* **05** (2017) 118 [[1611.04650](#)].
- [36] P. Saad, S. H. Shenker and D. Stanford, *A semiclassical ramp in SYK and in gravity*, [1806.06840](#).
- [37] P. Saad, S. H. Shenker and D. Stanford, *JT gravity as a matrix integral*, [1903.11115](#).
- [38] P. Saad, *Late Time Correlation Functions, Baby Universes, and ETH in JT Gravity*, [1910.10311](#).
- [39] H. Kim, T. N. Ikeda and D. A. Huse, *Testing whether all eigenstates obey the eigenstate thermalization hypothesis*, *Phys. Rev. E* **90** (2014) 052105.
- [40] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, *The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole*, *JHEP* **12** (2019) 063 [[1905.08762](#)].
- [41] A. Almheiri, R. Mahajan, J. Maldacena and Y. Zhao, *The page curve of hawking radiation from semiclassical geometry*, *Journal of High Energy Physics* **2020** (2020) .

- [42] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*. Oxford University Press, 01, 2007, [10.1093/acprof:oso/9780199213900.001.0001](https://doi.org/10.1093/acprof:oso/9780199213900.001.0001).
- [43] R. M. Wald, *Black hole entropy is the noether charge*, *Physical Review D* **48** (1993) R3427.
- [44] G. Compère and A. Fiorucci, *Advanced Lectures on General Relativity*, [1801.07064](https://doi.org/10.1007/978-3-319-07064-4).
- [45] T. Jacobson, G. Kang and R. C. Myers, *On black hole entropy*, *Phys. Rev. D* **49** (1994) 6587 [[gr-qc/9312023](https://arxiv.org/abs/gr-qc/9312023)].
- [46] S. W. Hawking and G. T. Horowitz, *The gravitational hamiltonian, action, entropy and surface terms*, *Classical and Quantum Gravity* **13** (1996) 1487.
- [47] G. W. Gibbons and S. W. Hawking, *Action integrals and partition functions in quantum gravity*, *Phys. Rev. D* **15** (1977) 2752.
- [48] S. W. Hawking and D. N. Page, *Thermodynamics of Black Holes in anti-De Sitter Space*, *Commun. Math. Phys.* **87** (1983) 577.
- [49] J. W. York, *Black-hole thermodynamics and the euclidean einstein action*, *Phys. Rev. D* **33** (1986) 2092.
- [50] J. D. Brown and J. W. York, *Microcanonical functional integral for the gravitational field*, *Physical Review D* **47** (1993) 1420.
- [51] S. A. Fulling and S. Ruijsenaars, *Temperature, periodicity and horizons*, *Physics Reports* **152** (1987) 135.
- [52] T. Jacobson, *Introduction to quantum fields in curved space-time and the Hawking effect*, in *School on Quantum Gravity*, pp. 39–89, 8, 2003, DOI [[gr-qc/0308048](https://arxiv.org/abs/gr-qc/0308048)].
- [53] H. Umezawa, *Advanced Field Theory: Micro, Macro, and Thermal Physics*. American Institute of Physics, 1993.
- [54] H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and Condensed States Dynamics*. North-Holland, Amsterdam, Netherlands, Jan., 1982.
- [55] W. Israel, *Thermo-field dynamics of black holes*, *Physics Letters A* **57** (1976) 107.

- [56] G. 't Hooft, *On the quantum structure of a black hole*, *Nuclear Physics B* **256** (1985) 727.
- [57] H. Araki, *Type of von neumann algebra associated with free field*, *Progress of Theoretical Physics* **32** (1964) 956 [<https://academic.oup.com/ptp/article-pdf/32/6/956/5311286/32-6-956.pdf>].
- [58] H. Liu, *Lectures on entanglement, von Neumann algebras, and emergence of spacetime*, in *Theoretical Advanced Study Institute in Elementary Particle Physics 2023: Aspects of Symmetry*, 10, 2025, **2510.07017**.
- [59] S. N. Solodukhin, *Entanglement entropy of black holes*, *Living Rev. Rel.* **14** (2011) 8 [**1104.3712**].
- [60] K. Ohmori and Y. Tachikawa, *Physics at the entangling surface*, *J. Stat. Mech.* **1504** (2015) P04010 [**1406.4167**].
- [61] L. Susskind and J. Uglum, *Black hole entropy in canonical quantum gravity and superstring theory*, *Phys. Rev. D* **50** (1994) 2700 [[hep-th/9401070](#)].
- [62] E. Witten, *Gravity and the crossed product*, *JHEP* **10** (2022) 008 [**2112.12828**].
- [63] H. Tasaki, *Mechanics, Statistical, (in Japanese)*, Baifukan (2008) .
- [64] S. W. Hawking, *Breakdown of predictability in gravitational collapse*, *Phys. Rev. D* **14** (1976) 2460.
- [65] J. Kaplan, “Lectures on ads/cft from the bottom up.” Unpublished course notes, Johns Hopkins University, 2016.
- [66] I. R. Klebanov, *World volume approach to absorption by nondilatonic branes*, *Nucl. Phys. B* **496** (1997) 231 [[hep-th/9702076](#)].
- [67] V. Balasubramanian and P. Kraus, *A Stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [[hep-th/9902121](#)].
- [68] K. Skenderis, *Asymptotically Anti-de Sitter space-times and their stress energy tensor*, *Int. J. Mod. Phys. A* **16** (2001) 740 [[hep-th/0010138](#)].
- [69] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849 [[hep-th/0209067](#)].
- [70] K. Skenderis and S. N. Solodukhin, *Quantum effective action from the ads/cft correspondence*, *Physics Letters B* **472** (2000) 316–322.

- [71] I. Papadimitriou and K. Skenderis, *AdS / CFT correspondence and geometry*, *IRMA Lect. Math. Theor. Phys.* **8** (2005) 73 [[hep-th/0404176](#)].
- [72] I. Papadimitriou and K. Skenderis, *Correlation functions in holographic RG flows*, *JHEP* **10** (2004) 075 [[hep-th/0407071](#)].
- [73] I. Papadimitriou, *Lectures on Holographic Renormalization*, *Springer Proc. Phys.* **176** (2016) 131.
- [74] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun. Math. Phys.* **104** (1986) 207.
- [75] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602 [[hep-th/0603001](#)].
- [76] V. E. Hubeny, M. Rangamani and T. Takayanagi, *A Covariant holographic entanglement entropy proposal*, *JHEP* **07** (2007) 062 [[0705.0016](#)].
- [77] P. Calabrese and J. L. Cardy, *Entanglement entropy and quantum field theory*, *J. Stat. Mech.* **0406** (2004) P06002 [[hep-th/0405152](#)].
- [78] P. Calabrese and J. Cardy, *Entanglement entropy and conformal field theory*, *J. Phys. A* **42** (2009) 504005 [[0905.4013](#)].
- [79] A. Lewkowycz and J. Maldacena, *Generalized gravitational entropy*, *JHEP* **08** (2013) 090 [[1304.4926](#)].
- [80] X. Dong, *Holographic Entanglement Entropy for General Higher Derivative Gravity*, *JHEP* **01** (2014) 044 [[1310.5713](#)].
- [81] X. Dong, A. Lewkowycz and M. Rangamani, *Deriving covariant holographic entanglement*, *JHEP* **11** (2016) 028 [[1607.07506](#)].
- [82] J. Maldacena, D. Stanford and Z. Yang, *Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space*, *PTEP* **2016** (2016) 12C104 [[1606.01857](#)].
- [83] I. Aref'eva, M. Khramtsov, M. Tikhonovskaya and I. Volovich, *Replica-nondiagonal solutions in the SYK model*, *JHEP* **07** (2019) 113 [[1811.04831](#)].
- [84] A. Eberlein, V. Kasper, S. Sachdev and J. Steinberg, *Quantum quench of the Sachdev-Ye-Kitaev Model*, *Phys. Rev. B* **96** (2017) 205123 [[1706.07803](#)].

- [85] G. Tarnopolsky, *Large q expansion in the Sachdev-Ye-Kitaev model*, *Phys. Rev. D* **99** (2019) 026010 [[1801.06871](#)].
- [86] C. Choi, M. Mezei and G. Sárosi, *Exact four point function for large q SYK from Regge theory*, *JHEP* **05** (2021) 166 [[1912.00004](#)].
- [87] Y. Gu, A. Kitaev and P. Zhang, *A two-way approach to out-of-time-order correlators*, *JHEP* **03** (2022) 133 [[2111.12007](#)].
- [88] E. Witten, *Multitrace operators, boundary conditions, and AdS / CFT correspondence*, [hep-th/0112258](#).
- [89] P. Gao, D. L. Jafferis and A. C. Wall, *Traversable Wormholes via a Double Trace Deformation*, *JHEP* **12** (2017) 151 [[1608.05687](#)].
- [90] J. Maldacena and X.-L. Qi, *Eternal traversable wormhole*, [1804.00491](#).
- [91] O. Aharony, A. B. Clark and A. Karch, *The CFT/AdS correspondence, massive gravitons and a connectivity index conjecture*, *Phys. Rev. D* **74** (2006) 086006 [[hep-th/0608089](#)].
- [92] E. Kiritsis, *Product CFTs, gravitational cloning, massive gravitons and the space of gravitational duals*, *JHEP* **11** (2006) 049 [[hep-th/0608088](#)].
- [93] M. Porrati, *Higgs phenomenon for 4-D gravity in anti-de Sitter space*, *JHEP* **04** (2002) 058 [[hep-th/0112166](#)].
- [94] A. B. Zamolodchikov, *Expectation value of composite field T anti- T in two-dimensional quantum field theory*, [hep-th/0401146](#).
- [95] F. A. Smirnov and A. B. Zamolodchikov, *On space of integrable quantum field theories*, *Nucl. Phys. B* **915** (2017) 363 [[1608.05499](#)].
- [96] A. Cavaglià, S. Negro, I. M. Szécsényi and R. Tateo, *$T\bar{T}$ -deformed 2D Quantum Field Theories*, *JHEP* **10** (2016) 112 [[1608.05534](#)].
- [97] L. McGough, M. Mezei and H. Verlinde, *Moving the CFT into the bulk with $T\bar{T}$* , *JHEP* **04** (2018) 010 [[1611.03470](#)].
- [98] M. Guica and R. Monten, *$T\bar{T}$ and the mirage of a bulk cutoff*, *SciPost Phys.* **10** (2021) 024 [[1906.11251](#)].
- [99] M. Guica, “ $T\bar{T}$ deformations and holography.”
https://indico.cern.ch/event/857396/contributions/3706292/attachments/2036750/3410352/ttbar_cern_v1s.pdf.

- [100] J. Cardy, *The $T\bar{T}$ deformation of quantum field theory as random geometry*, *JHEP* **10** (2018) 186 [[1801.06895](#)].
- [101] P. Caputa, P. Caputa, S. Datta, S. Datta, Y. Jiang, Y. Jiang et al., *Geometrizing $T\bar{T}$* , *JHEP* **03** (2021) 140 [[2011.04664](#)].
- [102] T. Biswas, A. Conroy, A. S. Koshelev and A. Mazumdar, *Generalized ghost-free quadratic curvature gravity*, *Class. Quant. Grav.* **31** (2014) 015022 [[1308.2319](#)].
- [103] A. B. Zamolodchikov, *Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory*, *JETP Lett.* **43** (1986) 730.
- [104] J. Polchinski, *Scale and Conformal Invariance in Quantum Field Theory*, *Nucl. Phys. B* **303** (1988) 226.
- [105] Y. Nakayama, *Scale invariance vs conformal invariance*, *Phys. Rept.* **569** (2015) 1 [[1302.0884](#)].
- [106] S. R. Coleman, *Why There Is Nothing Rather Than Something: A Theory of the Cosmological Constant*, *Nucl. Phys. B* **310** (1988) 643.
- [107] J. Preskill, *Wormholes in Space-time and the Constants of Nature*, *Nucl. Phys. B* **323** (1989) 141.
- [108] S. R. Coleman and K.-M. Lee, *WORMHOLES MADE WITHOUT MASSLESS MATTER FIELDS*, *Nucl. Phys. B* **329** (1990) 387.
- [109] A. Hebecker, T. Mikhail and P. Soler, *Euclidean wormholes, baby universes, and their impact on particle physics and cosmology*, *Front. Astron. Space Sci.* **5** (2018) 35 [[1807.00824](#)].
- [110] L. Apolo, P.-X. Hao, W.-X. Lai and W. Song, *Glue-on AdS holography for $T\bar{T}$ -deformed CFTs*, *JHEP* **06** (2023) 117 [[2303.04836](#)].
- [111] T. Takayanagi, *Holographic dual of a boundary conformal field theory*, *Physical Review Letters* **107** (2011) .
- [112] M. Fujita, T. Takayanagi and E. Tonni, *Aspects of AdS/BCFT*, *JHEP* **11** (2011) 043 [[1108.5152](#)].
- [113] P. Horava and E. Witten, *Heterotic and type I string dynamics from eleven-dimensions*, *Nucl. Phys. B* **460** (1996) 506 [[hep-th/9510209](#)].

- [114] I. Akal, Y. Kusuki, N. Shiba, T. Takayanagi and Z. Wei, *Holographic moving mirrors*, *Class. Quant. Grav.* **38** (2021) 224001 [[2106.11179](#)].
- [115] D. Bak, M. Gutperle and S. Hirano, *Three dimensional Janus and time-dependent black holes*, *JHEP* **02** (2007) 068 [[hep-th/0701108](#)].
- [116] D. Bak, M. Gutperle and A. Karch, *Time dependent black holes and thermal equilibration*, *JHEP* **12** (2007) 034 [[0708.3691](#)].
- [117] L. Randall and R. Sundrum, *An Alternative to compactification*, *Phys. Rev. Lett.* **83** (1999) 4690 [[hep-th/9906064](#)].
- [118] S. B. Giddings, E. Katz and L. Randall, *Linearized gravity in brane backgrounds*, *JHEP* **03** (2000) 023 [[hep-th/0002091](#)].
- [119] T. Shiromizu, T. Torii and D. Ida, *Brane world and holography*, *JHEP* **03** (2002) 007 [[hep-th/0105256](#)].
- [120] S. Goldstein, J. L. Lebowitz, C. Mastrodonato, R. Tumulka and N. Zanghi, *Approach to thermal equilibrium of macroscopic quantum systems*, *Phys. Rev. E* **81** (2010) 011109.
- [121] S. Goldstein, J. L. Lebowitz, R. Tumulka and N. Zanghi, *Long-time behavior of macroscopic quantum systems*, *The European Physical Journal H* **35** (2010) 173.
- [122] Y. Ogata, *Approximating macroscopic observables in quantum spin systems with commuting matrices*, 2011.
- [123] H. Tasaki, *Typicality of thermal equilibrium and thermalization in isolated macroscopic quantum systems*, *Journal of Statistical Physics* **163** (2016) 937.
- [124] S. Popescu, A. J. Short and A. Winter, *The foundations of statistical mechanics from entanglement: Individual states vs. averages*, .
- [125] S. Popescu, A. J. Short and A. Winter, *Entanglement and the foundations of statistical mechanics*, *Nature Physics* **2** (2006) 754.
- [126] S. Goldstein, J. L. Lebowitz, R. Tumulka and N. Zanghi, *Canonical typicality*, *Physical Review Letters* **96** (2006) .
- [127] A. Sugita, *On the basis of quantum statistical mechanics*, 2006.
- [128] N. Lashkari, A. Dymarsky and H. Liu, *Eigenstate thermalization hypothesis in conformal field theory*, .

- [129] S. Goldstein, D. A. Huse, J. L. Lebowitz and R. Tumulka, *Macroscopic and microscopic thermal equilibrium*, *Annalen der Physik* **529** (2017) 1600301.
- [130] A. Sugita, “Rims kokyuroku 1507, 147.” <https://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/1507-15.pdf>, 2006.
- [131] A. Sugita, *On the basis of quantum statistical mechanics*, 2006.
- [132] P. Reimann, *Typicality for generalized microcanonical ensembles*, *Phys. Rev. Lett.* **99** (2007) 160404.
- [133] H. Tasaki, *On the local equivalence between the canonical and the microcanonical ensembles for quantum spin systems*, *Journal of Statistical Physics* **172** (2018) 905–926.
- [134] W. Beugeling, R. Moessner and M. Haque, *Finite-size scaling of eigenstate thermalization*, *Phys. Rev. E* **89** (2014) 042112.
- [135] T. Mori, *Macrostate equivalence of two general ensembles and specific relative entropies*, *Phys. Rev. E* **94** (2016) 020101.
- [136] N. Linden, S. Popescu, A. J. Short and A. Winter, *Quantum mechanical evolution towards thermal equilibrium*, *Phys. Rev. E* **79** (2009) 061103 [0812.2385].
- [137] T. Mori, *Weak eigenstate thermalization with large deviation bound*, 2016. 10.48550/ARXIV.1609.09776.
- [138] G. Biroli, C. Kollath and A. M. Läuchli, *Effect of rare fluctuations on the thermalization of isolated quantum systems*, *Physical Review Letters* **105** (2010) .
- [139] E. Iyoda, K. Kaneko and T. Sagawa, *Fluctuation theorem for many-body pure quantum states*, *Phys. Rev. Lett.* **119** (2017) 100601.
- [140] T. Yoshizawa, E. Iyoda and T. Sagawa, *Numerical large deviation analysis of the eigenstate thermalization hypothesis*, *Phys. Rev. Lett.* **120** (2018) 200604.
- [141] H. Bernien et al., *Probing many-body dynamics on a 51-atom quantum simulator*, *Nature* **551** (2017) 579 [1707.04344].
- [142] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn and Z. Papić, *Quantum scarred eigenstates in a rydberg atom chain: Entanglement, breakdown of thermalization, and stability to perturbations*, *Physical Review B* **98** (2018) .

- [143] Y. OGATA and L. REY-BELLET, *Ruelle–lanford functions and large deviations for asymptotically decoupled quantum systems*, *Reviews in Mathematical Physics* **23** (2011) 211 [<https://doi.org/10.1142/S0129055X11004291>].
- [144] Y. Ogata, *Approximating macroscopic observables in quantum spin systems with commuting matrices*, 2011. 10.48550/ARXIV.1111.5933.
- [145] S. Popescu, A. J. Short and A. Winter, *The foundations of statistical mechanics from entanglement: Individual states vs. averages*, .
- [146] S. Goldstein, J. L. Lebowitz, R. Tumulka and N. Zanghì, *Canonical typicality*, *Phys. Rev. Lett.* **96** (2006) 050403.
- [147] D. N. Page, *Average entropy of a subsystem*, *Phys. Rev. Lett.* **71** (1993) 1291 [[gr-qc/9305007](https://arxiv.org/abs/gr-qc/9305007)].
- [148] S. Goldstein, D. A. Huse, J. L. Lebowitz and R. Tumulka, *Thermal equilibrium of a macroscopic quantum system in a pure state*, *Phys. Rev. Lett.* **115** (2015) 100402.
- [149] T. Mori, T. N. Ikeda, E. Kaminishi and M. Ueda, *Thermalization and prethermalization in isolated quantum systems: a theoretical overview*, *Journal of Physics B: Atomic, Molecular and Optical Physics* **51** (2018) 112001.
- [150] A. Dymarsky, N. Lashkari and H. Liu, *Subsystem ETH*, *Phys. Rev. E* **97** (2018) 012140 [[1611.08764](https://arxiv.org/abs/1611.08764)].
- [151] H. Liu and S. J. Suh, *Entanglement growth during thermalization in holographic systems*, *Physical Review D* **89** (2014) .
- [152] L. D'Alessio, Y. Kafri, A. Polkovnikov and M. Rigol, *From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics*, *Advances in Physics* **65** (2016) 239.
- [153] A. Peres, *Ergodicity and mixing in quantum theory. i*, *Phys. Rev. A* **30** (1984) 504.
- [154] S. Sugimoto, R. Hamazaki and M. Ueda, *Test of the Eigenstate Thermalization Hypothesis Based on Local Random Matrix Theory*, *Phys. Rev. Lett.* **126** (2021) 120602 [[2005.06379](https://arxiv.org/abs/2005.06379)].

- [155] S. Sugimoto, J. Henheik, V. Riabov and L. Erdős, *Eigenstate Thermalisation Hypothesis for Translation Invariant Spin Systems*, *J. Statist. Phys.* **190** (2023) 128 [2304.04213].
- [156] M. Srednicki, *The approach to thermal equilibrium in quantized chaotic systems*, *Journal of Physics A: Mathematical and General* **32** (1999) 1163.
- [157] S. Pappalardi, L. Foini and J. Kurchan, *Eigenstate Thermalization Hypothesis and Free Probability*, *Phys. Rev. Lett.* **129** (2022) 170603 [2204.11679].
- [158] K. Kaneko, E. Iyoda and T. Sagawa, *Characterizing complexity of many-body quantum dynamics by higher-order eigenstate thermalization*, *Phys. Rev. A* **101** (2020) 042126 [1911.10755].
- [159] M. Fava, J. Kurchan and S. Pappalardi, *Designs via Free Probability*, *Phys. Rev. X* **15** (2025) 011031 [2308.06200].
- [160] “free probability, between maths and physics”.
<https://irs.math.cnrs.fr/2023/free-web-titles.html>, 2023.
- [161] T. Hasebe, “free probability theory”. <https://www.math.sci.hokudai.ac.jp/~thasebe/HasebeICMSAA2014.pdf>, 2014.
- [162] E. Vallini and S. Pappalardi, *Long-time Freeness in the Kicked Top*, **2411.12050**.
- [163] R. Speicher, “Free Probability Theory, Lecture notes.”
<https://rolandspeicher.com/wp-content/uploads/2019/08/free-probability.pdf>, 2018.
- [164] J. Wang, *Beyond islands: a free probabilistic approach*, *JHEP* **10** (2023) 040 [2209.10546].
- [165] Udon, “Free Probability Theory.”
<https://www.youtube.com/watch?v=ZIH5YLsr07s>, 2024.
- [166] N. Dowling, J. De Nardis, M. Heinrich, X. Turkeshi and S. Pappalardi, *Free Independence and Unitary Design from Random Matrix Product Unitaries*, **2508.00051**.
- [167] F. Haake, S. Gnutzmann and M. Kuś, *Quantum Signatures of Chaos*, Springer Series in Synergetics. Springer Cham, 4 ed., 2018, 10.1007/978-3-319-97580-1.

- [168] J. Liu, *Spectral form factors and late time quantum chaos*, *Phys. Rev. D* **98** (2018) 086026 [[1806.05316](#)].
- [169] M. Khramtsov and E. Lanina, *Spectral form factor in the double-scaled SYK model*, *JHEP* **03** (2021) 031 [[2011.01906](#)].
- [170] M. Winer and B. Swingle, *Hydrodynamic Theory of the Connected Spectral form Factor*, *Phys. Rev. X* **12** (2022) 021009 [[2012.01436](#)].
- [171] O. Bouverot-Dupuis, S. Pappalardi, J. Kurchan, A. Polkovnikov and L. Foini, *Random matrix universality in dynamical correlation functions at late times*, *SciPost Phys.* **19** (2025) 050 [[2407.12103](#)].
- [172] D. A. Roberts and B. Yoshida, *Chaos and complexity by design*, *JHEP* **04** (2017) 121 [[1610.04903](#)].
- [173] S. Boucheron, G. Lugosi and P. Massart, *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press, 02, 2013, [10.1093/acprof:oso/9780199535255.001.0001](#).
- [174] R. Vershynin, *High-Dimensional Probability: An Introduction with Applications in Data Science*, Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2018.
- [175] <https://dlmf.nist.gov/8.11>.
- [176] E. Keski-Vakkuri, *Bulk and boundary dynamics in BTZ black holes*, *Phys. Rev. D* **59** (1999) 104001 [[hep-th/9808037](#)].
- [177] J. M. Maldacena, *Eternal black holes in anti-de Sitter*, *JHEP* **04** (2003) 021 [[hep-th/0106112](#)].
- [178] G. Festuccia and H. Liu, *The Arrow of time, black holes, and quantum mixing of large N Yang-Mills theories*, *JHEP* **12** (2007) 027 [[hep-th/0611098](#)].
- [179] D. Ruelle, *Statistical Mechanics: Rigorous Results*. 1999, [10.1142/4090](#).
- [180] R. Lima, *Equivalence of ensembles in quantum lattice systems: States*, *Communications in Mathematical Physics* **24** (1972) 180.
- [181] F. G. S. L. Brandao and M. Cramer, *Equivalence of Statistical Mechanical Ensembles for Non-Critical Quantum Systems*, *arXiv e-prints* (2015) [arXiv:1502.03263](#) [[1502.03263](#)].

- [182] H. Tasaki, *On the Local Equivalence Between the Canonical and the Microcanonical Ensembles for Quantum Spin Systems*, *Journal of Statistical Physics* **172** (2018) 905 [[1609.06983](#)].
- [183] G. T. Horowitz and V. E. Hubeny, *Quasinormal modes of AdS black holes and the approach to thermal equilibrium*, *Phys. Rev. D* **62** (2000) 024027 [[hep-th/9909056](#)].
- [184] D. Birmingham, *Choptuik scaling and quasinormal modes in the AdS / CFT correspondence*, *Phys. Rev. D* **64** (2001) 064024 [[hep-th/0101194](#)].
- [185] D. Birmingham, I. Sachs and S. N. Solodukhin, *Conformal field theory interpretation of black hole quasinormal modes*, *Phys. Rev. Lett.* **88** (2002) 151301 [[hep-th/0112055](#)].
- [186] M. Ammon and J. Erdmenger, *Gauge/gravity duality: Foundations and applications*. Cambridge University Press, Cambridge, 4, 2015, [10.1017/CBO9780511846373](#).
- [187] D. Harlow and D. Stanford, *Operator Dictionaries and Wave Functions in AdS/CFT and dS/CFT*, [1104.2621](#).
- [188] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT(d) / AdS(d+1) correspondence*, *Nucl. Phys. B* **546** (1999) 96 [[hep-th/9804058](#)].
- [189] V. Balasubramanian and S. F. Ross, *Holographic particle detection*, *Phys. Rev. D* **61** (2000) 044007 [[hep-th/9906226](#)].
- [190] J. Louko, D. Marolf and S. F. Ross, *On geodesic propagators and black hole holography*, *Phys. Rev. D* **62** (2000) 044041 [[hep-th/0002111](#)].
- [191] L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker, *The Black hole singularity in AdS / CFT*, *JHEP* **02** (2004) 014 [[hep-th/0306170](#)].
- [192] Z. Yang, *The Quantum Gravity Dynamics of Near Extremal Black Holes*, *JHEP* **05** (2019) 205 [[1809.08647](#)].
- [193] L. Susskind, *Entanglement is not enough*, *Fortsch. Phys.* **64** (2016) 49 [[1411.0690](#)].
- [194] M. Kleban, M. Porrati and R. Rabadan, *Poincare recurrences and topological diversity*, *JHEP* **10** (2004) 030 [[hep-th/0407192](#)].

- [195] J. L. F. Barbon and E. Rabinovici, *Very long time scales and black hole thermal equilibrium*, *JHEP* **11** (2003) 047 [[hep-th/0308063](#)].
- [196] J. Cotler and K. Jensen, *Wormholes and black hole microstates in AdS/CFT*, *JHEP* **09** (2021) 001 [[2104.00601](#)].
- [197] G. Di Ubaldo and E. Perlmutter, *AdS₃/RMT₂ duality*, *JHEP* **12** (2023) 179 [[2307.03707](#)].
- [198] D. L. Jafferis, L. Rozenberg and G. Wong, *3d gravity as a random ensemble*, *JHEP* **02** (2025) 208 [[2407.02649](#)].
- [199] Y. Chen, V. Ivo and J. Maldacena, *Comments on the double cone wormhole*, *JHEP* **04** (2024) 124 [[2310.11617](#)].
- [200] C. Gogolin and J. Eisert, *Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems*, *Rept. Prog. Phys.* **79** (2016) 056001 [[1503.07538](#)].
- [201] P. Calabrese and J. Cardy, *Evolution of entanglement entropy in one-dimensional systems*, *Journal of Statistical Mechanics: Theory and Experiment* **2005** (2005) P04010.
- [202] J. Cardy, *Bulk Renormalization Group Flows and Boundary States in Conformal Field Theories*, *SciPost Phys.* **3** (2017) 011 [[1706.01568](#)].
- [203] V. Balasubramanian, A. Bernamonti, J. de Boer, N. Copland, B. Craps, E. Keski-Vakkuri et al., *Holographic thermalization*, *Physical Review D* **84** (2011) .
- [204] T. Hartman and J. Maldacena, *Time Evolution of Entanglement Entropy from Black Hole Interiors*, *JHEP* **05** (2013) 014 [[1303.1080](#)].
- [205] D. Simmons-Duffin, “TASI Lectures on Conformal Field Theory in Lorentzian Signature.”
<https://gitlab.com/davidsd/lorentzian-cft-notes>, 2019.
- [206] M. Bañados, E. Bianchi, I. Muñoz and K. Skenderis, *Bulk renormalization and the AdS/CFT correspondence*, *Phys. Rev. D* **107** (2023) L021901 [[2208.11539](#)].
- [207] S. A. W. Leutheusser, *Emergent Times in Holographic Duality*, *Phys. Rev. D* **108** (2023) 086020 [[2112.12156](#)].

- [208] M. Knysh, H. Liu and N. Pinzani-Fokeeva, *New horizon symmetries, hydrodynamics, and quantum chaos*, *JHEP* **09** (2024) 162 [[2405.17559](#)].
- [209] S. Pasterski, M. Pate and A.-M. Raclariu, *Celestial Holography*, in *Snowmass 2021*, 11, 2021, [2111.11392](#).
- [210] K. Nguyen, *Lectures on Carrollian Holography*, [2511.10162](#).
- [211] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, *The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole*, *Journal of High Energy Physics* **2019** (2019) .
- [212] G. Penington, S. H. Shenker, D. Stanford and Z. Yang, *Replica wormholes and the black hole interior*, *JHEP* **03** (2022) 205 [[1911.11977](#)].
- [213] H. Geng, A. Karch, C. Perez-Pardavila, S. Raju, L. Randall, M. Riojas et al., *Entanglement phase structure of a holographic BCFT in a black hole background*, *JHEP* **05** (2022) 153 [[2112.09132](#)].
- [214] T. Ishii and D. Takeda, *Lindblad dynamics in holography*, *Phys. Rev. D* **112** (2025) 046020 [[2504.17320](#)].
- [215] D. Bak, M. Gutperle and R. A. Janik, *Janus Black Holes*, *JHEP* **10** (2011) 056 [[1109.2736](#)].
- [216] D. Bak and H. Min, *Multi-faced Black Janus and Entanglement*, *JHEP* **03** (2014) 046 [[1311.5259](#)].
- [217] D. Bak, C. Kim and S.-H. Yi, *Bulk view of teleportation and traversable wormholes*, *JHEP* **08** (2018) 140 [[1805.12349](#)].
- [218] J. Harper, T. Kawamoto, R. Maeda, N. Nakamura and T. Takayanagi, *Non-hermitian density matrices from time-like entanglement and wormholes, upcoming work* (2025) .
- [219] G. T. Horowitz and J. M. Maldacena, *The Black hole final state*, *JHEP* **02** (2004) 008 [[hep-th/0310281](#)].
- [220] D. Gottesman and J. Preskill, *Comment on ‘The Black hole final state’*, *JHEP* **03** (2004) 026 [[hep-th/0311269](#)].
- [221] S. Lloyd, L. Maccone, R. Garcia-Patron, V. Giovannetti and Y. Shikano, *Quantum mechanics of time travel through post-selected teleportation*, *Physical Review D* **84** (2011) .

- [222] S. Lloyd, L. Maccone, R. Garcia-Patron, V. Giovannetti, Y. Shikano, S. Pirandola et al., *Closed timelike curves via postselection: Theory and experimental test of consistency*, *Physical Review Letters* **106** (2011) .
- [223] Y. Aharonov, P. G. Bergmann and J. L. Lebowitz, *Time symmetry in the quantum process of measurement*, *Phys. Rev.* **134** (1964) B1410.
- [224] L. Vaidman, *The two-state vector formalism*, 2007.
10.48550/ARXIV.0706.1347.
- [225] Y. Nakata, T. Takayanagi, Y. Taki, K. Tamaoka and Z. Wei, *New holographic generalization of entanglement entropy*, *Phys. Rev. D* **103** (2021) 026005 [[2005.13801](#)].
- [226] A. Mollabashi, N. Shiba, T. Takayanagi, K. Tamaoka and Z. Wei, *Aspects of pseudoentropy in field theories*, *Phys. Rev. Res.* **3** (2021) 033254 [[2106.03118](#)].
- [227] A. Caldeira and A. Leggett, *Path integral approach to quantum brownian motion*, *Physica A: Statistical Mechanics and its Applications* **121** (1983) 587.
- [228] R. Feynman and F. Vernon, *The theory of a general quantum system interacting with a linear dissipative system*, *Annals of Physics* **281** (2000) 547.
- [229] M. Gutperle and A. Strominger, *Space - like branes*, *JHEP* **04** (2002) 018 [[hep-th/0202210](#)].
- [230] T. Azeyanagi, A. Karch, T. Takayanagi and E. G. Thompson, *Holographic calculation of boundary entropy*, *JHEP* **03** (2008) 054 [[0712.1850](#)].
- [231] A. Karch, Z.-X. Luo and H.-Y. Sun, *Universal relations for holographic interfaces*, *JHEP* **09** (2021) 172 [[2107.02165](#)].
- [232] J. Estes, K. Jensen, A. O'Bannon, E. Tsatis and T. Wrase, *On Holographic Defect Entropy*, *JHEP* **05** (2014) 084 [[1403.6475](#)].
- [233] M. Gutperle and A. Trivella, *Note on entanglement entropy and regularization in holographic interface theories*, *Phys. Rev. D* **95** (2017) 066009 [[1611.07595](#)].
- [234] B. Skinner, J. Ruhman and A. Nahum, *Measurement-Induced Phase Transitions in the Dynamics of Entanglement*, *Phys. Rev. X* **9** (2019) 031009 [[1808.05953](#)].

- [235] O. Alberton, M. Buchhold and S. Diehl, *Entanglement Transition in a Monitored Free-Fermion Chain: From Extended Criticality to Area Law*, *Phys. Rev. Lett.* **126** (2021) 170602 [[2005.09722](#)].
- [236] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz*, *Commun. Math. Phys.* **177** (1996) 381 [[hep-th/9412229](#)].
- [237] A. Dymarsky and K. Pavlenko, *Generalized Eigenstate Thermalization Hypothesis in 2D Conformal Field Theories*, *Phys. Rev. Lett.* **123** (2019) 111602 [[1903.03559](#)].
- [238] F. Sato and H. Kodama, *General relativity*, 2016.
- [239] S. de Haro, S. N. Solodukhin and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [[hep-th/0002230](#)].
- [240] A. J. Short and T. C. Farrelly, *Quantum equilibration in finite time*, *New J. Phys.* **14** (2012) 013063 [[1110.5759](#)].
- [241] B. Mukhametzhanov and A. Zhiboedov, *Modular invariance, tauberian theorems and microcanonical entropy*, *JHEP* **10** (2019) 261 [[1904.06359](#)].