

Mathematical analysis of global solutions to the
Boltzmann equation

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Chapter 1

Introduction to the Boltzmann equation

Kinetic theory of gases is an important branch of physics. It is a study of behaviour of dilute gases, described as a set of a large number of particles (atoms or molecules for example). When a system is “dilute”, molecules in the system are so sparsely distributed that it cannot be modelled as a continuum. It is well-known that usual fluids like flows of a river, the ocean, and the atmosphere are treated as continua and modelled by the famous Navier-Stokes equations (or its variants). However, these equations are not suitable to describe the dilute gas, because its statistic mechanical effects are not negligible. A typical region analysed via the kinetic theory is outer space, more than 100 km away from a surface of the earth. Therefore, the study is not a purely theoretical subject to mathematicians and physicists, but also important for astronautic engineers who construct, such as, a space shuttle and a space station.

In 1872, an Austrian physicist Ludwig Boltzmann founded one of the most fundamental equation in the kinetic theory named after him, taking into account of collisions of particles in a dilute gas. Although there are some forms of the equation corresponding to certain physical models, we will focus on the most commonly investigated model. The Boltzmann equation describing a monatomic gas interacting via an inverse-power force, under a contemporary notation, reads:

$$\begin{cases} \partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = Q(f, f)(t, x, v), & t > 0, (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3, \\ f(0, x, v) = f_0(x, v), \end{cases} \quad (1.1)$$

where the quadratic operator Q is defined as

$$Q(f, g)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \sigma) (f(v'_*)g(v') - f(v_*)g(v)) d\sigma dv_*, \quad (1.2)$$

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \sigma \in \mathbb{S}^2, \quad (1.3)$$

$$B(|v - v_*|, \sigma) = |v - v_*|^\gamma b(\cos \theta), \quad \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma. \quad (1.4)$$

Here, $t \geq 0$ is time, x and v are position and velocity of a particle respectively, and $f = f(t, x, v)$ is a distribution function of particles whose position and velocity are (x, v) at time t . ∂_t stands for the differential operator $\partial/\partial t$, and $\partial_t + v \cdot \nabla_x$ is called the transport operator, expressing a macroscopic flow of the system. Q is called the collision operator. Roughly speaking, Q “counts” a number of particles with velocity v , when a distribution of two particles is governed by the tensor product $f \otimes g$. The nonnegative function B in (1.4) is called the Boltzmann collision kernel. As its definition indicates, B depends on the relative velocity $|v - v_*|$ and the deviation angle θ via σ . B defines what kinds of collision are likely (or unlikely) to happen. If it holds

$$\int_{\mathbb{S}^2} b(\cos \theta) d\sigma < \infty,$$

which is known as Grad’s angular cut-off assumption, the right hand side of (1.2) can be naturally divided, and the former term corresponds to increase of particles with velocity of v , whereas the latter term corresponds to decrease. But this is rather naive decomposition. We will see the reason later.

Equations (1.3) are equivalent to conservation laws of momentum and energy, that is,

$$v + v_* = v' + v'_* \quad \text{and} \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

In other words, the collision is elastic. Although there are many results and their applications when the collision is inelastic (energy dissipates after the collision), we will focus on the elastic collision case.

$\gamma > -3$, and $b(\cos \cdot)$ is assumed to be a smooth function on $(0, \pi]$ satisfying

$$b(\cos \theta) \sin \theta \sim C \theta^{-(1+\nu)} \text{ as } \theta \downarrow 0, \quad C > 0, \quad 0 < \nu < 2.$$

A typical case satisfying this assumption is that interaction of particles obey the inverse-power law potential $\phi(r) = 1/r^s$ ($s > 2$ and $r > 0$ is a distance between two particles), and it is known that $\gamma = (s - 5)/(s - 1)$ and $\nu = 2/(s - 1)$ in this case. We can see that there are two singularities; kinetic singularity of $|v - v_*|$ when γ is negative, and angular singularity of $b(\cos \theta)$ in the sense that it is not integrable on $\mathbb{S}^2$. Thus, in general, Q is not able to be decomposed into the aforementioned two terms. Due to this observation, according to value of γ , the equation is customarily classified into three cases:

1. the hard potential case when $0 < \gamma$,
2. the Maxwellian molecules case when $\gamma = 0$, and
3. the soft potential case when $-3 < \gamma < 0$.

It can be easily imagined that behaviour of solutions to the Boltzmann equation significantly varies as γ takes different values. For example, when we consider the Cauchy problem, the most difficult domain to handle is where $|v - v_*|$ is large in the hard potential case, while it is $\{|v - v_*| < 1\}$ in the soft potential case.

There are several topics investigated which is concerned with the Boltzmann equation: existence and uniqueness of a solution, asymptotic behaviour, regularizing effect and smoothness of a solution, H -theorem, hydrodynamic limits, and so on. For definitions of these problems, we refer to [19].

In this thesis, we are particularly interested in global existence of a solution under certain (suitable and conventional) assumptions. To this aim, let us see some known results, corresponding to each chapter. We again refer to [19] for more detailed description and history of each case.

In the second chapter, the spatially homogeneous Boltzmann equation in the Maxwellian molecules case will be considered. It reads:

$$\partial_t f(t, v) = Q(f, f)(t, v), \quad B(|v - v_*|, \sigma) = b(\cos \theta). \quad (1.5)$$

In short, this is the case where the distribution function f is independent of x , and $\gamma = 0$. This setting may seem too simplified, but it is useful to mainly consider action of the collision kernel, since it only acts on the velocity dependence.

We focus on infinite energy measure solutions. All of the following setting can be generalized to any dimensional case, but we restrict ourselves to the three-dimensional case. To formulate a problem, we first define a measure solution to (1.5). Under the current setting, the celebrated Bobylev formula (see [4], or the review in [5]) gives the Fourier transform of Q as follows:

$$\mathcal{Q}(f, g) := \mathcal{F}[Q(f, g)](\xi) = \int_{S^2} (\hat{f}(\xi^-) \hat{g}(\xi^+) - \hat{f}(0) \hat{g}(\xi)) b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) d\sigma, \quad \xi \in \mathbb{R}^3,$$

where $\hat{f}$ stands for the Fourier transform of f and $\xi^\pm = (\xi \pm |\xi|)/2$, respectively. Due to this formula, if $f_t \in \mathcal{P}(\mathbb{R}^3)$ (a set of all probability measures on $\mathbb{R}^3$) solves (1.5) for initial datum $f_0 \in \mathcal{P}(\mathbb{R}^3)$ in some sense, then it is necessary that the characteristic function $\phi(t, \xi) = \int_{\mathbb{R}^3} e^{-ix\xi} df_t(x)$ solves

$$\begin{cases} \partial_t \phi(t, \xi) = \mathcal{Q}(\phi, \phi)(t, \xi), & t > 0, \quad \xi \in \mathbb{R}^3, \\ \phi(0, \xi) = \phi_0(\xi). \end{cases} \quad (1.6)$$

Denoting a set of probability measures on $\mathbb{R}^3$ with finite α -th moment (and the same mean when $\alpha \geq 1$) by $\mathcal{P}_\alpha(\mathbb{R}^3)$, we define an infinite energy solution to (1.5) as $f_t \in \mathcal{P}(\mathbb{R}^3) \setminus \mathcal{P}_\alpha(\mathbb{R}^3)$ for each t which $\phi(t, \xi)$ solves (1.6). We remark

that probabilistic approach to (1.5) via the quadratic Wasserstein distance was originally done by Tanaka [15], proving that the distance between two solutions $f(t)$ and $g(t)$ to (1.5) is equal to or smaller than that of their corresponding initial data f_0 and g_0 . Equality holds only when both f_0 and g_0 are the global Maxwellian with the same mean. The zero-mean global Maxwellian (1.7) is defined later. This probabilistic approach is later followed by Toscani-Villani [16], who invented a Fourier-transform-based distance. This is called the Toscani distance, which reads

$$T(\mu, \nu) = \sup_{\xi \in \mathbb{R}^3} \frac{|\hat{\mu}(\xi) - \hat{\nu}(\xi)|}{|\xi|^2}$$

for $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^3)$, and $\hat{\mu}, \hat{\nu}$ are their characteristic functions. It was proved that convergence on $\mathcal{P}_2(\mathbb{R}^3)$ defined by the Toscani distance is the same as one defined by the quadratic Wasserstein distance. It should be noted that these results have important influence on invention of the following functional spaces.

Bobylev and Cercignani [6] found that infiniteness of energy yielded a self-similar solution to (1.6), which can be defined for $-\infty < t < \infty$. This was the first paper to deduce such a solution. Following [6], Cannone and Karch [8, 9] showed that the self-similar solution did exist in a mathematically rigorous way. They employed the space $\mathcal{K}^\alpha$ ($0 \leq \alpha < 2$) of characteristic functions, defined by

$$\mathcal{K}^\alpha = \{\phi \in \mathcal{K} \mid \|\phi - 1\|_\alpha := \sup_{\xi \in \mathbb{R}^3} \frac{|\phi(\xi) - 1|}{|\xi|^\alpha} < \infty\},$$

where $\mathcal{K} = \mathcal{K}^0$ is a set of characteristic functions. When initial datum is in $\mathcal{K}^\alpha$, they constructed a solution which takes value in $\mathcal{K}^\alpha$ for $0 < \nu < 1$. This result was later improved by Morimoto [12] for $0 < \nu < 2$. One of the most important properties of $\mathcal{K}^\alpha$ is that $\mathcal{F}(\mathcal{P}_\alpha(\mathbb{R}^3)) \subsetneq \mathcal{K}^\alpha$. Thus, as long as we consider the problem on $\mathcal{K}^\alpha$, we cannot ensure that the Fourier inverse of a solution is always in $\mathcal{P}_\alpha(\mathbb{R}^3)$, even though initial datum is taken in it. To overcome this difficulty, Morimoto, Wang, and Yang [13, 14] introduced the new spaces

$$\begin{aligned} \mathcal{M}^\alpha &= \{\phi \in \mathcal{K} \mid \|\phi - 1\|_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^3} \frac{|\phi(\xi) - 1|}{|\xi|^\alpha} d\xi < \infty\}, \\ \tilde{\mathcal{M}}^\alpha &= \{\phi \in \mathcal{K} \mid \|\operatorname{Re} \phi - 1\|_{\mathcal{M}^\alpha} + \|\phi - 1\|_\alpha < \infty\} \end{aligned}$$

for $0 < \alpha < 2$. They first showed $\mathcal{F}(\mathcal{P}_\alpha(\mathbb{R}^3)) = \mathcal{M}^\alpha$ for $\alpha \neq 1$ in [13], and then proved $\mathcal{F}(\mathcal{P}_\alpha(\mathbb{R}^3)) = \tilde{\mathcal{M}}^\alpha$ for any $\alpha \in (0, 2)$ in [14]. They constructed solutions to (1.6) taking value in these spaces, so now we can see that if we take initial datum in $\mathcal{P}_\alpha(\mathbb{R}^3)$, the solutions should also belong to this space.

Motivated by invention of these spaces of characteristic functions, Cho [7] considered the following problem:

$$\begin{cases} \frac{\partial f}{\partial t}(t, v) + \delta(-\Delta_v)^{p/2} f(t, v) = Q(f, f)(t, v), \\ f(0, v) = f_0(v) \end{cases}$$

for $0 < p \leq 2$, or its equivalent Fourier-transformed version

$$\begin{cases} \frac{\partial \phi}{\partial t}(t, \xi) + \delta |\xi|^p \phi(t, \xi) = \mathcal{Q}(\phi, \phi)(t, \xi), \\ \phi(0, \xi) = \phi_0(v). \end{cases}$$

He proved unique existence of a solution taking value in $\mathcal{K}^\alpha$ for $0 \leq \alpha \leq p$. It should be noted that it is the fractional Laplacian term, which in general assures better regularity for solutions to other partial differential equations, that prevents existence of a solution for better initial data. The main result of the second chapter is an improvement of this result, constructing a solution taking value in $\mathcal{M}^\alpha$ for $0 < \alpha < p$.

The third chapter is dedicated to the perturbation theory of the Boltzmann equation without angular cut-off in a Chemin-Lerner space. The perturbation theory of the equation is formulated as follows: since the zero-mean global Maxwellian distribution

$$\mu(v) = \frac{1}{(2\pi)^{3/2}} e^{-|v|^2} \quad (1.7)$$

is a stationary solution (indeed, it is easy to check $Q(\mu, \mu) = 0$ by (1.3)), inserting $f = \mu + \sqrt{\mu}g$ yields the equation for g , that is,

$$\partial_t g + v \cdot \nabla_x g + \mathcal{L}g = \Gamma(g, g), \quad (1.8)$$

where

$$\Gamma(g, h) = \mu^{-1/2} Q(\mu^{1/2} g, \mu^{1/2} h), \quad \mathcal{L}g = -\Gamma(\mu^{1/2}, g) - \Gamma(g, \mu^{1/2}).$$

For this equation, the pioneering works were made by Ukai [17, 18] in the seventies, followed by many papers, some of which are referred in the third chapter.

Among these results, the main result in the chapter is mainly based on a series of works by Alexandre, Morimoto, Ukai, Yang, and Xu (usually abbreviated to AMUXY) [1, 2, 3] and a work by Duan, Liu, and Xu [11]. One of the most notable results by AMUXY is that they showed existence and uniqueness of a solution to (1.8) without angular cut-off in a weighted Sobolev space, presumably minimal-regularity one under the perturbation setting. On the other hand, [11] was the first paper to consider the Cauchy problem of (1.8) in the Chemin-Lerner space, originally used by [10]. It is a variant of a Besov space by involving integration with respect to variables irrelevant to the Littlewood-Paley decomposition. In our case, we will employ the decomposition with respect to x -variable, and integrate cut-off functions on $[0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$ for a given time $T > 0$. [11] proved unique existence of a solution in a certain Chemin-Lerner space (better than those employed by AMUXY) and its positivity, but under strong assumptions of angular cut-off (strictly speaking, $|b(\cos \theta)|$ is bounded above) and $0 \leq \gamma \leq 1$ (the hard potential case). In order to remove these restrictions as possible, we will employ many strategies and techniques established by AMUXY. The second main result is that we have achieved integrating the two results, showing unique existence of a solution in the same Chemin-Lerner space employed by [11] under the non cut-off assumption. Positivity of the solution is also shown.

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Chapter 2

Some remarks on the homogeneous Boltzmann equation with the fractional Laplacian term

Abstract

We study the homogeneous Boltzmann equation with the fractional Laplacian term. Working on the Fourier side we solve the resulting integral equation, and improve a previous result by Y.-K. Cho. We replace the initial data space with a certain space $\mathcal{M}^\alpha$ introduced by Morimoto, Wang, and Yang. This space precisely captures the Fourier image of probability measures with bounded fractional moments, providing a more natural initial condition. We show existence of a unique global solution, in addition to the expected maximal growth estimates and stability estimates. As a consequence we obtain a continuous density solution of the original equation.

2.1 Introduction

We consider the spatially homogeneous Boltzmann equation with a diffusion term, that is,

$$\begin{cases} \partial_t f(v, t) + \delta_p(-\Delta)^{p/2} f(v, t) = Q(f, f)(v, t), & (v, t) \in \mathbb{R}^3 \times (0, \infty), \\ f(v, 0) = f_0(v), \end{cases} \quad (2.1.1)$$

where $0 < p \leq 2$ and $\delta_p \geq 0$ are constants. This equation is a variant of the homogeneous Boltzmann equation

$$\begin{cases} \partial_t f(v, t) = Q(f, f)(v, t), & (v, t) \in \mathbb{R}^3 \times (0, \infty), \\ f(v, 0) = f_0(v). \end{cases} \quad (2.1.2)$$

Equation (2.1.1) was recently studied by Cho [4]. He discussed the Fourier transform of (2.1.1) and showed that if we take an initial datum from a certain space (which we shall define later), then there exists a unique global solution on the Fourier side and this solution satisfies a maximal stability estimate. The estimate is a consequence of the diffusion term, so such an estimate is likely not valid for solutions of the Fourier transform of (2.1.2). It should be noted that the initial data space used by Cho [4] may contain “undesirable” data for this problem, that is, data which prevents us from inverting the Fourier transformed problem back into the original one (see, for example, Remark 1(ii) of [4]). In order to avoid this obstruction, in this paper we instead use a more suitable space introduced by Morimoto-Wang-Yang [7], as an initial data space. Using that space, we obtain the existence of a time-continuous unique global solution of (2.1.1). Readers may refer to [4, 2, 3, 6, 7] and the references therein for historical progress and recent research on these spaces.

We now discuss (2.1.1) in greater detail. $f(v, t)$ is the density distribution of particles in rarefied gas with velocity $v \in \mathbb{R}^3$ and time $t > 0$. The right-hand side of (2.1.1) is defined by the Boltzmann collision operator corresponding to the the Maxwellian molecule type cross section,

$$Q(f, g)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\mathbf{k} \cdot \sigma) (f'_* g' - f_* g) d\sigma dv_*, \quad (2.1.3)$$

where $f'_* = f(v'_*)$, $g' = g(v')$, $f_* = f(v_*)$, $g = g(v)$, $\mathbf{k} = (v - v_*)/(|v - v_*|)$, and

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.$$

The Maxwellian molecule type cross section $b(\mathbf{k} \cdot \sigma)$ in (2.1.3) is a non-negative function depending only on the deviation angle $\theta = \cos^{-1}(\mathbf{k} \cdot \sigma)$. As in Villani [10], θ is customarily restricted to the range $[0, \pi/2]$ by replacing $b(\mathbf{k} \cdot \sigma) = b(\cos \theta)$ with its “symmetrized version”

$$\left(b(\cos \theta) + b(\cos(\pi - \theta)) \right) \mathbf{1}_{\{0 \leq \theta \leq \pi/2\}}.$$

Since the physical model $b(\cos \theta)$ coming from the inverse power laws has a singularity near $\theta = 0$, that is,

$$b(\cos \theta) \sim K \theta^{-2-2s} \quad (0 < s < 1, K > 0), \quad (2.1.4)$$

we shall throughout this note assume the following weak integrability condition

$$\exists \alpha_0 \in (0, 2] \text{ such that } \int_0^{\pi/2} b(\cos \theta) \sin \theta \sin^{\alpha_0} \left(\frac{\theta}{2} \right) d\theta < \infty. \quad (2.1.5)$$

We note that (2.1.5) is satisfied for any cross section b with property (2.1.4) as long as $\alpha_0 > 2s$, although $b(\cos \theta) \sin \theta$ is not integrable on $[0, \pi/2]$. These constraints coming from the physical model will be used throughout the paper.

Next, we reformulate equation (2.1.1). By taking into account the Bobylev formula, we apply the Fourier transform to (2.1.1) and obtain

$$\begin{cases} (\partial_t + \delta_p |\xi|^p) \phi(\xi, t) = \mathcal{B}(\phi)(\xi, t), & (\xi, t) \in \mathbb{R}^3 \times (0, \infty), \\ \phi(\xi, 0) = \phi_0(\xi), \end{cases} \quad (2.1.6)$$

where $\phi(\xi, t) = \mathcal{F}[f(\cdot, t)](\xi)$,

$$\mathcal{B}(\phi)(\xi) = \mathcal{F}[Q(f, f)](\xi) = \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) [\phi(\xi^+) \phi(\xi^-) - \phi(0) \phi(\xi)] d\sigma, \quad (2.1.7)$$

and $\xi^\pm = (\xi \pm |\xi| \sigma)/2$. (Thus, in (2.1.6) we have $\mathcal{B}(\phi)(\xi, t) = \mathcal{B}(\phi(\cdot, t))(\xi)$ and we will use similar abuse of notation in the sequel without further comment.) The corresponding integral equation is

$$\phi(\xi, t) = e^{-\delta_p |\xi|^p t} \phi_0(\xi) + \int_0^t e^{-\delta_p |\xi|^p (t-\tau)} \mathcal{B}(\phi)(\xi, \tau) d\tau. \quad (2.1.8)$$

We assume that $f(\cdot, t)$ is a probability measure on $\mathbb{R}^3$ for every $t > 0$ at least, thus $\phi(\xi, t)$ is equal to $\int_{\mathbb{R}^3} e^{-iv \cdot \xi} df(v, t)$ and $\phi(0, t) = 1$.

Before we discuss known results concerning (2.1.6), consider first the Fourier transform of (2.1.2), namely,

$$\begin{cases} \partial_t \phi(\xi, t) = \mathcal{B}(\phi)(\xi, t), & (\xi, t) \in \mathbb{R}^3 \times (0, \infty), \\ \phi(\xi, 0) = \phi_0(\xi). \end{cases} \quad (2.1.9)$$

We first introduce some functional spaces. Although only the three-dimensional case is introduced, we remark that the following spaces can be defined for any dimensional case. Define $P_\alpha(\mathbb{R}^3)$ ($\alpha \geq 0$) as the set of all probability measures on $\mathbb{R}^3$ with finite moment of order α . In other words, if $\mu \in P_\alpha(\mathbb{R}^3)$ then μ satisfies

$$\int_{\mathbb{R}^3} d\mu(v) = 1, \quad \int_{\mathbb{R}^3} |v|^\alpha d\mu(v) < \infty.$$

When $\alpha \geq 1$, we add on the condition

$$\int_{\mathbb{R}^3} v_j d\mu(v) = 0 \quad (j = 1, 2, 3)$$

to the definition of $P_\alpha(\mathbb{R}^3)$. Since we expect a solution of (2.1.2) to conserve its mass, momentum, and energy, we thus normalize so that solutions are probability measures with zero mean and finite moment of order 2 (or, finite variance).

In order to establish a solution with infinite energy, Cannone and Karch [2] invented spaces of characteristic functions, namely

$$\mathcal{K} = \left\{ \phi : \mathbb{R}^3 \rightarrow \mathbb{C} \mid \phi(\xi) = \int_{\mathbb{R}^3} e^{-iv \cdot \xi} df(v), f \in P_0(\mathbb{R}^3) \right\},$$

$$\mathcal{K}^\alpha = \left\{ \phi \in \mathcal{K} \mid \|\phi - 1\|_\alpha = \sup_{\xi \in \mathbb{R}^3} \frac{|\phi(\xi) - 1|}{|\xi|^\alpha} < \infty \right\},$$

and analysed the homogeneous Boltzmann equation with the Maxwellian collision kernel. $\mathcal{K}^\alpha$ is a complete metric space endowed with the norm

$$\|\phi - \psi\|_\alpha = \sup_{\xi \in \mathbb{R}^3} \frac{|\phi(\xi) - \psi(\xi)|}{|\xi|^\alpha} \quad (\phi, \psi \in \mathcal{K}^\alpha).$$

Since

$$\{1\} \subset \mathcal{K}^2 \subset \mathcal{K}^{\alpha_1} \subset \mathcal{K}^{\alpha_2} \subset \mathcal{K}^0 = \mathcal{K} \quad (0 \leq \alpha_2 \leq \alpha_1 \leq 2), \quad (2.1.10)$$

the $\mathcal{K}^\alpha$ -valued solutions they established have infinite energy. This result was modified by Morimoto [6] by using finer calculations, and by Morimoto-Wang-Yang [7] by introducing the space

$$\mathcal{M}^\alpha = \left\{ \phi \in \mathcal{K} \mid \|\phi - 1\|_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^3} \frac{|\phi(\xi) - 1|}{|\xi|^{3+\alpha}} d\xi < \infty \right\}.$$

When $0 < \beta \leq \alpha < 2$, this is a complete metric space endowed with

$$\text{dis}_{\alpha,\beta}(\phi, \psi) = \|\phi - \psi\|_{\mathcal{M}^\alpha} + \|\phi - \psi\|_\beta,$$

$$\|\phi - \psi\|_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^3} \frac{|\phi(\xi) - \psi(\xi)|}{|\xi|^{3+\alpha}} d\xi \quad (\phi, \psi \in \mathcal{M}^\alpha).$$

For any $\alpha \in (0, 2)$, $\mathcal{M}^\alpha \subset \mathcal{F}(P_\alpha(\mathbb{R}^3)) (\subsetneq \mathcal{K}^\alpha)$. Moreover, if $\alpha \neq 1$, then $\mathcal{M}^\alpha = \mathcal{F}(P_\alpha(\mathbb{R}^3))$. This is the reason why $\mathcal{M}^\alpha$ is preferable to $\mathcal{K}^\alpha$ for our problem, because this space has a simple interpretation: when we consider the original Boltzmann equation we can take initial data from $P_\alpha(\mathbb{R}^3)$ and analyse the equation on the Fourier side. The result is Theorem 1.4 in [7]. This theorem gives the existence of a unique global measure valued solution of (2.1.2) in $C([0, \infty); P_\alpha(\mathbb{R}^3))$ with a $P_\alpha(\mathbb{R}^3)$ -valued initial datum.

Now let us recall some known results concerning (2.1.6). Using techniques from [2] and [6] (in which (2.1.9) is studied) Cho [4] proved that (2.1.6) has a solution in $\mathcal{S}^\alpha(\mathbb{R}^3 \times [0, \infty))$. A stability estimate and a maximum growth estimate were also shown. Here, for all $T > 0$,

$$\mathcal{S}^\alpha(\mathbb{R}^3 \times [0, T]) = \{ \phi \in C([0, T]; \mathcal{K}^\alpha) \mid \phi(\xi, \cdot) \in C([0, T]) ,$$

$$\partial_t \phi(\xi, \cdot) \in C((0, T)) \text{ for } \forall \xi \in \mathbb{R}^3 \},$$

$$\mathcal{S}^\alpha(\mathbb{R}^3 \times [0, \infty)) = \bigcup_{T>0} \mathcal{S}^\alpha(\mathbb{R}^3 \times [0, T]).$$

We shall consider (2.1.8) with initial data ϕ_0 in $\mathcal{M}^\alpha$ ($\alpha \in (0, 2)$). We apply the technique developed in [7] to discuss (2.1.8). Our main theorems are the following.

Theorem 2.1.1. *Let $\alpha_0 \leq \alpha < p$. Then for any $\phi_0 \in \mathcal{M}^\alpha$, (2.1.8) has a classical solution ϕ in*

$$\mathcal{T}^\alpha(\mathbb{R}^3 \times [0, \infty)) = \{\phi \in C([0, \infty); \mathcal{M}^\alpha) \mid \phi(\xi, \cdot) \in C([0, \infty)), \\ \partial_t \phi(\xi, \cdot) \in C((0, \infty)) \text{ for } \forall \xi \in \mathbb{R}^3\}.$$

This solution ϕ satisfies the a priori estimate

$$\sup_{\xi \in \mathbb{R}^3} e^{\delta_p |\xi|^p t} |\phi(\xi, t)| \leq 1 \quad \text{for each } t \geq 0. \quad (2.1.11)$$

Since ϕ in Theorem 2.1.1 satisfies $\partial_t \phi(\xi, \cdot) \in C((0, \infty))$, ϕ is also a solution of (2.1.6). (2.1.11) is a consequence of adding the diffusion term $\delta_p(-\Delta)^{p/2}$. When $\delta_p = 0$, (2.1.11) just gives the obvious statement $|\phi(\xi, t)| \leq 1$ for any ξ and t . This makes difference between (2.1.1) and (2.1.2).

In order to state our stability result, we introduce the notation $\mathcal{T}_p^\alpha(\mathbb{R}^3 \times [0, \infty))$ for all elements of $\mathcal{T}^\alpha(\mathbb{R}^3 \times [0, \infty))$ which satisfy (2.1.11). The following constants will also be needed in the sequel. For $2s < \alpha_0 \leq \alpha < p \leq 2$, we define

$$\begin{aligned} \gamma_\alpha &= 2\pi \int_0^{\pi/2} b(\cos \theta) \sin \theta \left[\cos^\alpha \left(\frac{\theta}{2} \right) + \sin^\alpha \left(\frac{\theta}{2} \right) \right] d\theta, \\ \lambda_\alpha &= 2\pi \int_0^{\pi/2} b(\cos \theta) \sin \theta \left[\cos^\alpha \left(\frac{\theta}{2} \right) + \sin^\alpha \left(\frac{\theta}{2} \right) - 1 \right] d\theta, \\ \mu_\alpha &= 2\pi \int_0^{\pi/2} b(\cos \theta) \sin \theta \sin^\alpha \left(\frac{\theta}{2} \right) d\theta, \\ C_{p,\alpha} &= \int_{\mathbb{R}^3} \frac{1 - e^{-|\xi|^p}}{|\xi|^{3+\alpha}} d\xi = \frac{4\pi}{\alpha} \Gamma \left(1 - \frac{\alpha}{p} \right), \end{aligned}$$

where Γ is the gamma function. γ_α is used to discuss (2.1.8) under the cutoff assumption, that is, when $b \in L^1(\mathbb{S}^2)$. Easily we see that $\gamma_2 \leq \gamma_\alpha \leq 2\gamma_2$. When we discuss the equation under the non-cutoff assumption, we use λ_α and μ_α . In this case, $0 \leq \lambda_\alpha < \infty$ (see [2]) and $0 \leq \mu_\alpha < \infty$ (due to (2.1.5)). The second identity for $C_{p,\alpha}$, which was mentioned to us by Cho, shows that $C_{p,\alpha}$ is finite if and only if $0 < \alpha < p$. The identity is easy to prove using a change of variables followed by integration by parts.

Theorem 2.1.2. *Under the assumptions of Theorem 2.1.1, let $\phi, \psi \in \mathcal{T}_p^\alpha(\mathbb{R}^3 \times [0, \infty))$ be solutions of (2.1.8) with initial data $\phi_0, \psi_0 \in \mathcal{M}^\alpha$ respectively. Then the following stability estimate holds:*

$$\left\| e^{\delta_p |\cdot|^p t} (\phi(\cdot, t) - \psi(\cdot, t)) \right\|_{\mathcal{M}^\alpha} \leq e^{\lambda_\alpha t} \|\phi_0 - \psi_0\|_{\mathcal{M}^\alpha}.$$

In particular, (2.1.8) has at most one solution in $\mathcal{T}_p^\alpha(\mathbb{R}^3 \times [0, \infty))$.

Interpreting Theorem 2.1.2 in terms of the original equation yields the following result.

Theorem 2.1.3. *Let $\alpha_0 \leq \alpha < p$, $\alpha \neq 1$, and $f_0 \in P_\alpha(\mathbb{R}^3)$. Let ϕ be the unique global solution of (2.1.6) with initial datum $\phi_0 = \mathcal{F}[f_0] \in \mathcal{M}^\alpha$ given by Theorems 2.1.1 and 2.1.2. Then $f(v, t) = \mathcal{F}^{-1}[\phi(\cdot, t)](v)$ is a unique global solution of (2.1.1) with initial datum f_0 . This inverse Fourier transform is well-defined. $f(v, t)$ is a smooth probability density of v and tends to 0 as $|v| \rightarrow \infty$ for each $t > 0$. Moreover, $f \in C([0, \infty); P_\alpha(\mathbb{R}^3))$, where time continuity of f is interpreted in the following sense: for any $t_0 > 0$ and $\psi \in C(\mathbb{R}^3)$ satisfying $|\psi(v)| \leq C(1 + |v|^2)^{\alpha/2}$ for some positive C ,*

$$\lim_{t \rightarrow t_0} \int_{\mathbb{R}^3} \psi(v) f(v, t) dv = \int_{\mathbb{R}^3} \psi(v) f(v, t_0) dv. \quad (2.1.12)$$

Note that a slightly stronger assumption is needed to establish a solution with the same properties when $\alpha = 1$ (see [7, Remark 1.5]).

The outline of this paper is as follows. We cite some lemmas in Section 2. We modify some of them so that they are applicable to our problem. In Section 3 we prove that under the cutoff assumption we have a global unique solution of (2.1.8), and deduce an a priori estimate and a stability estimate. These results are used to discuss (2.1.8) under the non-cutoff assumption in Section 4. The proof of main theorems are shown in this section. In Section 5 we discuss non-existence of solutions in the case $p \leq \alpha < 2$.

2.2 Preliminaries

Under the cutoff assumption, Q is split into a gain term Q^+ and a loss term Q^- in a self-evident way. We denote the Fourier transform of the gain term Q^+ by

$$\mathcal{G}(\phi)(\xi) = \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \phi(\xi^+) \phi(\xi^-) d\sigma.$$

First, we cite two lemmas from Morimoto-Wang-Yang [7].

Lemma 2.2.1 (Morimoto-Wang-Yang). *Assume $b \in L^1(\mathbb{S}^2)$ ($\Leftrightarrow \gamma_2 < \infty$). For all ϕ and $\psi \in \mathcal{M}^\alpha$,*

$$\|\mathcal{G}(\phi) - \mathcal{G}(\psi)\|_{\mathcal{M}^\alpha} \leq \gamma_\alpha \|\phi - \psi\|_{\mathcal{M}^\alpha}.$$

Lemma 2.2.2 (Morimoto-Wang-Yang). *There exists a constant $C_0 > 0$ independent of $\phi \in \mathcal{M}^\alpha$, such that*

$$\int_{\mathbb{R}^3} \frac{|\mathcal{B}(\phi)(\xi)|}{|\xi|^{3+\alpha}} d\xi \leq C_0 \mu_\alpha \|\phi - 1\|_{\mathcal{M}^\alpha}.$$

Then, we show a lemma concerning $e^{-|\xi|^p}$, which is a classical result in probability theory.

Lemma 2.2.3. $e^{-|\xi|^p t}$ ($\xi \in \mathbb{R}^3$, $t \geq 0$) is a positive definite function of ξ if and only if $0 < p \leq 2$. For every $t > 0$, $f_p(v, t)$, which is defined as

$$f_p(v, t) = \frac{1}{2\pi^2} \int_0^\infty e^{-r^p t} r^2 \frac{\sin r|v|}{r|v|} dr,$$

is a probability density on $\mathbb{R}^3$, and $\mathcal{F}[f_p(\cdot, t)](\xi) = e^{-|\xi|^p t}$. Furthermore,

$$\int_{\mathbb{R}^3} |v|^\alpha f_p(v, t) dv < \infty \quad (0 \leq \alpha < p) \quad \text{and} \quad \int_{\mathbb{R}^3} |v|^p f_p(v, t) dv = \infty.$$

Proof. Positive definiteness is proved in Schoenberg [8]. For each $t > 0$, $e^{-|\xi|^p t}$ is a rapidly decreasing radial function defined on $\mathbb{R}^3$. Its inverse Fourier transform is

$$\mathcal{F}^{-1}[e^{-|\cdot|^p t}](v) = \frac{1}{(2\pi)^3} (2\pi)^{\frac{3}{2}} |v|^{-\frac{1}{2}} \int_0^\infty J_{\frac{1}{2}}(r|v|) e^{-r^p t} r^{\frac{3}{2}} dr, \quad (2.2.13)$$

where $J_\nu(z)$ ($\nu \in \mathbb{C}$, $\text{Re}(\nu) > -1/2$) is the Bessel function

$$J_\nu(z) = \left[\Gamma\left(\frac{1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right) \right]^{-1} \left(\frac{z}{2}\right)^\nu \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} e^{izt} dt$$

(see Taylor [9] for instance). A straightforward calculation of $J_{\frac{1}{2}}(r|v|)$ together with (2.2.13) gives

$$\mathcal{F}^{-1}[e^{-|\cdot|^p t}](v) = \frac{1}{2\pi^2} \int_0^\infty e^{-r^p t} r^2 \frac{\sin r|v|}{r|v|} dr = f_p(v, t). \quad (2.2.14)$$

Hence, $\mathcal{F}[f_p(\cdot, t)](\xi) = e^{-|\xi|^p t}$.

We now turn to the moment estimates. A simple calculation shows that $f_p(v, t) = t^{-3/p} f_p(t^{-1/p} v, 1)$, so it suffices to consider the case $t = 1$. From [1, Theorem 2.1] we have

$$\lim_{|v| \rightarrow \infty} |v|^{3+p} f_p(v, 1) = \frac{p 2^{p-1}}{\pi^{\frac{5}{2}}} \sin\left(\frac{p\pi}{2}\right) \Gamma\left(\frac{3+p}{2}\right) \Gamma\left(\frac{p}{2}\right),$$

and this leads to the desired estimates. $\square$

Cho [4, Section 3] discusses conditions for which $\mathcal{G}(\phi)$ and $\mathcal{B}(\phi)$ make sense. Since $\mathcal{M}^\alpha \subset \mathcal{K}^\alpha \subset \mathcal{K}$ ($0 < \alpha < 2$), it follows that Lemma 3.1 and Lemma 3.3 in [4] are still valid when we take ϕ from $\mathcal{M}^\alpha$. We state these facts as Lemmas 2.2.4 and 2.2.5.

Lemma 2.2.4. Assume $b \in L^1(\mathbb{S}^2)$ and $T > 0$. Then $\mathcal{G}(\phi)(\xi)$ is a continuous positive definite function of ξ for each $\phi \in \mathcal{M}^\alpha$.

Moreover, if $\phi(\cdot, t) \in \mathcal{M}^\alpha$ for each $t \in [0, T]$ and $\phi(\xi, \cdot) \in C([0, T])$ for each $\xi \in \mathbb{R}^3$, then $\mathcal{G}(\phi)(\xi, t) \in C(\mathbb{R}^3 \times [0, T])$.

Lemma 2.2.5. Assume $\mu_{\alpha_0} < \infty$ and $T > 0$. If $\phi \in C([0, T]; \mathcal{M}^\alpha)$ and $\phi(\xi, \cdot) \in C([0, T])$ for each $\xi \in \mathbb{R}^3$, then $\mathcal{B}(\phi)(\xi, \cdot) \in C([0, T])$ for each $\xi \in \mathbb{R}^3$.

2.3 Existence and behaviour of a solution under the cutoff assumption

In order to consider the non-cutoff case in Section 2.4, we first prove that under the cutoff assumption ($\gamma_2 < \infty$), (2.1.8) has a unique solution in $\mathcal{T}^\alpha(\mathbb{R}^3 \times [0, \infty))$. We also provide a stability estimate and an a priori estimate for this solution.

2.3.1 Existence

Assume $b \in L^1(\mathbb{S}^2)$. Under the cutoff assumption, (2.1.8) is written as

$$\phi(\xi, t) = e^{-(\gamma_2 + \delta_p |\xi|^p)t} \phi_0(\xi) + \int_0^t e^{-(\gamma_2 + \delta_p |\xi|^p)(t-\tau)} \mathcal{G}(\phi)(\xi, \tau) d\tau. \quad (2.3.15)$$

In this subsection we will prove the following theorem.

Theorem 2.3.1. *Let $\alpha_0 \leq \alpha < p$, $b \in L^1(\mathbb{S}^2)$ and $\phi_0 \in \mathcal{M}^\alpha$. Then (2.3.15) has a solution $\phi \in \mathcal{T}^\alpha(\mathbb{R}^3 \times [0, \infty))$, which satisfies*

$$\sup_{\xi \in \mathbb{R}^3} e^{\delta_p |\xi|^p t} |\phi(\xi, t)| \leq 1 \text{ for each } t \geq 0.$$

For a fixed $T > 0$, define

$$\Omega_T = \{\phi \in C([0, T]; \mathcal{M}^\alpha) \mid \phi(\xi, \cdot) \in C([0, T]) \text{ for each } \xi \in \mathbb{R}^3\}.$$

Since $\mathcal{M}^\alpha$ is a complete metric space endowed with $\text{dis}_\alpha(\phi, \psi) = \|\phi - \psi\|_{\mathcal{M}^\alpha} + \|\phi - \psi\|_\alpha$ (see [7]), Ω_T is also a complete metric space endowed with

$$D_T(\phi, \psi) = \max_{t \in [0, T]} \text{dis}_\alpha(\phi(t), \psi(t)).$$

Let us think of the right-hand side of (2.3.15) as the image $\Phi(\phi)(t)$ of an operator Φ . By Lemma 2.2.4, Φ is well-defined on Ω_T . We will prove that Φ is a contraction on Ω_T provided that T is sufficiently small. Since $\mathcal{M}^\alpha$ is endowed with dis_α and the contraction estimate for $\|\cdot\|_\alpha$ was obtained in [4], it is enough to consider the contraction estimate for $\|\cdot\|_{\mathcal{M}^\alpha}$.

Lemma 2.3.2. *If $\alpha_0 \leq \alpha < p$, $T > 0$, and $\phi \in \Omega_T$, then*

$$\|\Phi(\phi)(t) - 1\|_{\mathcal{M}^\alpha} \leq C_{p, \alpha} (\delta_p t)^{\alpha/p} + 2 \max_{\tau \in [0, T]} \|\phi(\tau) - 1\|_{\mathcal{M}^\alpha},$$

$$\|\Phi(\phi)(t) - \Phi(\phi)(s)\|_{\mathcal{M}^\alpha} \leq C(\phi, T) |t - s|^{\alpha/p},$$

where $s, t \in [0, T]$ and

$$C(\phi, T) = 2C_{p, \alpha} \delta_p^{\alpha/p} (2 + \gamma_2 T) + 3\gamma_\alpha T^{1-\alpha/p} \max_{t \in [0, T]} \|\phi(t) - 1\|_{\mathcal{M}^\alpha}.$$

Proof. When $t \geq 0$, we have $\mathcal{G}(\phi)(0, t) = \gamma_2$, so

$$\Phi(\phi)(t) - 1 = I_1(t) + I_2(t) + I_3(t),$$

where

$$\begin{aligned} I_1(t) &= e^{-\gamma_2 t} \left[e^{-\delta_p |\xi|^p t} \phi_0(\xi) - 1 \right], \\ I_2(t) &= \int_0^t e^{-(\gamma_2 + \delta_p |\xi|^p)(t-\tau)} [\mathcal{G}(\phi)(\xi, \tau) - \mathcal{G}(\phi)(0, \tau)] d\tau, \\ I_3(t) &= \gamma_2 \int_0^t e^{-\gamma_2(t-\tau)} \left[e^{-\delta_p |\xi|^p(t-\tau)} - 1 \right] d\tau. \end{aligned}$$

Note that for any $a > 0$,

$$\int_{\mathbb{R}^3} \frac{1 - e^{-a|\xi|^p}}{|\xi|^{3+\alpha}} d\xi = C_{p,\alpha} a^{\alpha/p}.$$

Since

$$e^{-\delta_p |\xi|^p t} \phi_0(\xi) - 1 = e^{-\delta_p |\xi|^p t} (\phi_0(\xi) - 1) - (1 - e^{-\delta_p |\xi|^p t}),$$

this gives

$$\int_{\mathbb{R}^3} \frac{|I_1(t)|}{|\xi|^{3+\alpha}} d\xi \leq e^{-\gamma_2 t} \left[\|\phi_0 - 1\|_{\mathcal{M}^\alpha} + C_{p,\alpha} (\delta_p t)^{\alpha/p} \right].$$

Next, by Lemma 2.2.1,

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|I_2(t)|}{|\xi|^{3+\alpha}} d\xi &\leq \gamma_\alpha \int_0^t e^{-\gamma_2(t-\tau)} \|\phi(\tau) - 1\|_{\mathcal{M}^\alpha} d\tau \\ &\leq \gamma_\alpha \left(\frac{1 - e^{-\gamma_2 t}}{\gamma_2} \right) \max_{\tau \in [0, T]} \|\phi(\tau) - 1\|_{\mathcal{M}^\alpha}. \end{aligned}$$

Finally, I_3 is estimated in a similar way as I_1 , yielding

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|I_3(t)|}{|\xi|^{3+\alpha}} d\xi &\leq \gamma_2 \int_0^t e^{-\gamma_2(t-\tau)} C_{p,\alpha} [\delta_p(t-\tau)]^{\alpha/p} d\tau \\ &\leq C_{p,\alpha} (\delta_p t)^{\alpha/p} (1 - e^{-\gamma_2 t}). \end{aligned}$$

Combining these estimates we obtain the first inequality of the lemma.

To obtain the second inequality of the lemma, we estimate

$$\Phi(\phi)(t) - \Phi(\phi)(s) = \sum_{i=1}^3 (I_i(t) - I_i(s))$$

with $0 \leq s < t \leq T$ for simplicity. Since

$$\begin{aligned} I_1(t) - I_1(s) &= -e^{-\gamma_2 s} \left[e^{-\delta_p |\xi|^p s} (1 - e^{-\delta_p |\xi|^p(t-s)}) \phi_0(\xi) \right. \\ &\quad \left. + (1 - e^{-\gamma_2(t-s)}) (e^{-\delta_p |\xi|^p t} \phi_0(\xi) - 1) \right], \end{aligned}$$

we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{|I_1(t) - I_1(s)|}{|\xi|^{3+\alpha}} d\xi &\leq C_{p,\alpha} [\delta_p(t-s)]^{\alpha/p} \\
&\quad + \gamma_2(t-s) \left[\|\phi_0 - 1\|_{\mathcal{M}^\alpha} + C_{p,\alpha} (\delta_p t)^{\alpha/p} \right] \\
&\leq C_{p,\alpha} \delta_p^{\alpha/p} (t-s)^{\alpha/p} \\
&\quad + \gamma_2(t-s)^{\alpha/p} \left[T^{1-\alpha/p} \|\phi_0 - 1\|_{\mathcal{M}^\alpha} + C_{p,\alpha} \delta_p^{\alpha/p} T \right].
\end{aligned}$$

Next,

$$\begin{aligned}
I_2(t) - I_2(s) &= \int_s^t e^{-(\gamma_2 + \delta_p |\xi|^p)(t-\tau)} [\mathcal{G}(\phi)(\xi, \tau) - \mathcal{G}(\phi)(0, \tau)] d\tau \\
&\quad - (1 - e^{-\gamma_2(t-s)}) \int_0^s e^{-(\gamma_2 + \delta_p |\xi|^p)(s-\tau)} [\mathcal{G}(\phi)(\xi, \tau) - \mathcal{G}(\phi)(0, \tau)] d\tau \\
&\quad - (1 - e^{-\delta_p |\xi|^p(t-s)}) \int_0^s e^{-\gamma_2(t-\tau) - \delta_p |\xi|^p(s-\tau)} [\mathcal{G}(\phi)(\xi, \tau) - \mathcal{G}(\phi)(0, \tau)] d\tau.
\end{aligned}$$

Since $b \in L^1(\mathbb{S}^2)$, we have $|\mathcal{G}(\phi)(\xi, \tau) - \mathcal{G}(\phi)(0, \tau)| \leq 2\gamma_2$, which together with Lemma 2.2.1 gives

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{|I_2(t) - I_2(s)|}{|\xi|^{3+\alpha}} d\xi \\
&\leq \gamma_\alpha \max_{\tau \in [0, T]} \|\phi(\tau) - 1\|_{\mathcal{M}^\alpha} \left[\int_s^t e^{-\gamma_2(t-\tau)} d\tau + \gamma_2(t-s) \int_0^s e^{-\gamma_2(s-\tau)} d\tau \right] \\
&\quad + C_{p,\alpha} [\delta_p(t-s)]^{\alpha/p} 2\gamma_2 \int_0^s e^{-\gamma_2(t-\tau)} d\tau \\
&\leq (t-s)^{\alpha/p} \left[2\gamma_\alpha T^{1-\alpha/p} \max_{\tau \in [0, T]} \|\phi(\tau) - 1\|_{\mathcal{M}^\alpha} + 2C_{p,\alpha} \delta_p^{\alpha/p} \right].
\end{aligned}$$

Finally,

$$\begin{aligned}
I_3(t) - I_3(s) &= -\gamma_2 \int_s^t e^{-\gamma_2(t-\tau)} (1 - e^{-\delta_p |\xi|^p(t-\tau)}) d\tau \\
&\quad - \gamma_2 (1 - e^{-\delta_p |\xi|^p(t-s)}) \int_0^s e^{-\gamma_2(t-\tau) - \delta_p |\xi|^p(s-\tau)} d\tau \\
&\quad + \gamma_2 (1 - e^{-\gamma_2(t-s)}) \int_0^s e^{-\gamma_2(s-\tau)} (1 - e^{-\delta_p |\xi|^p(s-\tau)}) d\tau,
\end{aligned}$$

so we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{|I_3(t) - I_3(s)|}{|\xi|^{3+\alpha}} d\xi &\leq \gamma_2 \int_s^t e^{-\gamma_2(t-\tau)} C_{p,\alpha} [\delta_p(t-\tau)]^{\alpha/p} d\tau \\
&\quad + \gamma_2 C_{p,\alpha} [\delta_p(t-s)]^{\alpha/p} \int_0^s e^{-\gamma_2(t-\tau)} d\tau \\
&\quad + \gamma_2^2 (t-s) \int_0^s e^{-\gamma_2(s-\tau)} C_{p,\alpha} [\delta_p(s-\tau)]^{\alpha/p} d\tau \\
&\leq C_{p,\alpha} \delta_p^{\alpha/p} (t-s)^{\alpha/p} (1 + \gamma_2 T).
\end{aligned}$$

Together, these estimates yield the second inequality of the lemma. $\square$

Proof of Theorem 2.3.1. To see that Φ is a contraction, note that Φ is a mapping from Ω_T to Ω_T for each $T > 0$ and

$$\max_{t \in [0, T]} \|\Phi(\phi)(t) - \Phi(\psi)(t)\|_{\mathcal{M}^\alpha} \leq 2(1 - e^{-\gamma_2 T}) \max_{t \in [0, T]} \|\phi(t) - \psi(t)\|_{\mathcal{M}^\alpha}. \quad (2.3.16)$$

This follows in the same way as in Cho [4] (replace $\mathcal{K}^\alpha$ with $\mathcal{M}^\alpha$ and $\|\cdot\|_\alpha$ with $\|\cdot\|_{\mathcal{M}^\alpha}$, then this is obvious). Taking $0 < T_0 < \frac{\log 2}{\gamma_2}$ and combining (2.3.16) with the $\|\cdot\|_\alpha$ -estimates in [4], we obtain

$$D_{T_0}(\Phi(\phi), \Phi(\psi)) < D_{T_0}(\phi, \psi).$$

Therefore Φ is a contraction on Ω_{T_0} . By using the Banach fixed point theorem, we obtain a solution of (2.3.15). $\square$

2.3.2 An a priori estimate and a stability estimate

We observe that since $C([0, \infty); \mathcal{M}^\alpha) \subset C([0, \infty); \mathcal{K}^\alpha)$, the a priori estimate deduced in [4] also holds for a solution of (2.3.15):

Lemma 2.3.3. *Let $\alpha_0 \leq \alpha < p$, $b \in L^1(\mathbb{S}^2)$, and $\phi_0 \in \mathcal{M}^\alpha$. If (2.3.15) has a solution ϕ which satisfies $\phi \in C([0, \infty); \mathcal{M}^\alpha)$ and $\phi(\xi, \cdot) \in C([0, \infty))$ for every $\xi \in \mathbb{R}^3$, then the following estimate holds for every $t \geq 0$*

$$\sup_{\xi \in \mathbb{R}^3} e^{\delta_p |\xi|^p t} |\phi(\xi, t)| \leq 1. \quad (2.3.17)$$

Coupled with the existence of a local solution, estimate (2.3.17) gives the existence of a global solution of (2.3.15). We will now show a stability estimate for the uniqueness result.

Theorem 2.3.4. *Let $\gamma_2 < \infty$ and $\phi, \psi \in \mathcal{T}^\alpha(\mathbb{R}^3 \times [0, \infty))$ be solutions of (2.3.15), corresponding to initial data $\phi_0, \psi_0 \in \mathcal{M}^\alpha$ respectively. Then for all $t \geq 0$,*

$$\left\| e^{\delta_p |\cdot|^p t} (\phi(\cdot, t) - \psi(\cdot, t)) \right\|_{\mathcal{M}^\alpha} \leq e^{\lambda_\alpha t} \|\phi_0 - \psi_0\|_{\mathcal{M}^\alpha}.$$

Proof. Define

$$U(\xi, t) = e^{(\gamma_2 + \delta_p |\xi|^p)t} \left(\frac{\phi(\xi, t) - \psi(\xi, t)}{|\xi|^{3+\alpha}} \right)$$

for $\xi \neq 0$ and $U(0, t) = 0$. We have

$$U(\xi, t) = U(\xi, 0) + \int_0^t e^{(\gamma_2 + \delta_p |\xi|^p)\tau} \left(\frac{\mathcal{G}(\phi)(\xi, \tau) - \mathcal{G}(\psi)(\xi, \tau)}{|\xi|^{3+\alpha}} \right) d\tau. \quad (2.3.18)$$

Since $|\xi|^p \leq |\xi^+|^p + |\xi^-|^p$ for $0 \leq p \leq 2$, (2.3.17) gives

$$\begin{aligned} & \int_{\mathbb{R}^3} e^{\delta_p |\xi|^p \tau} \frac{|\mathcal{G}(\phi)(\xi, \tau) - \mathcal{G}(\psi)(\xi, \tau)|}{|\xi|^{3+\alpha}} d\xi \\ & \leq \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) e^{\delta_p (|\xi^+|^p + |\xi^-|^p) \tau}}{|\xi|^{3+\alpha}} (|\phi^+| |\phi^- - \psi^-| + |\psi^-| |\psi^+ - \phi^+|) d\sigma d\xi \\ & \leq \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{b\left(\frac{\xi \cdot \sigma}{|\xi|}\right)}{|\xi|^{3+\alpha}} \left(e^{\delta_p |\xi^+|^p \tau} |\phi^+ - \psi^+| + e^{\delta_p |\xi^-|^p \tau} |\phi^- - \psi^-| \right) d\sigma d\xi \\ & = J_1 + J_2. \end{aligned}$$

Here we used the simplified notation $\phi(\xi^+, \tau) = \phi^+$, with ϕ^- and $\psi^\pm$ similarly defined. By using the change of variables used in the proof of [5, Lemma 8.1] (see also [7, Lemma 2.1]), J_1 and J_2 are calculated as

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \frac{e^{\delta_p |\xi|^p \tau}}{|\xi|^{3+\alpha}} |\phi(\xi, \tau) - \psi(\xi, \tau)| \cos^\alpha\left(\frac{\theta}{2}\right) d\sigma d\xi, \\ J_2 &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \frac{e^{\delta_p |\xi|^p \tau}}{|\xi|^{3+\alpha}} |\phi(\xi, \tau) - \psi(\xi, \tau)| \sin^\alpha\left(\frac{\theta}{2}\right) d\sigma d\xi, \end{aligned}$$

from which we conclude that

$$\left\| e^{\delta_p |\cdot|^p \tau} (\mathcal{G}(\phi)(\tau) - \mathcal{G}(\psi)(\tau)) \right\|_{\mathcal{M}^\alpha} \leq \gamma_\alpha \left\| e^{\delta_p |\cdot|^p \tau} (\phi(\tau) - \psi(\tau)) \right\|_{\mathcal{M}^\alpha}.$$

For simplicity we denote $\left\| e^{\delta_p |\cdot|^p t} (\phi(\cdot, t) - \psi(\cdot, t)) \right\|_{\mathcal{M}^\alpha}$ by $V(t)$. Integrating (2.3.18) with respect to ξ , we obtain

$$e^{\gamma_2 t} V(t) \leq V(0) + \gamma_\alpha \int_0^t e^{\gamma_2 \tau} V(\tau) d\tau.$$

The Gronwall inequality yields the result. $\square$

2.4 Global existence of a solution under the non-cutoff condition

In this section, we will construct a solution of (2.1.8) without the cutoff assumption by using the results of the last section. For this purpose, first we prove two lemmas.

Lemma 2.4.1. *Let $\alpha_0 \leq \alpha < p \leq 2$, $\mu_\alpha < \infty$, and $\phi_0 \in \mathcal{M}^\alpha$. If (2.1.8) has a solution $\phi \in \mathcal{T}^\alpha(\mathbb{R}^3 \times [0, \infty))$ then ϕ satisfies*

$$\|\phi(t) - 1\|_{\mathcal{M}^\alpha} \leq e^{C_0 \mu_\alpha t} \left[\|\phi_0 - 1\|_{\mathcal{M}^\alpha} + C_{p,\alpha} (\delta_p t)^{\alpha/p} \right],$$

where C_0 is the same constant as in Lemma 2.2.2.

Proof. It is easy to see that

$$\frac{\phi(\xi, t) - 1}{|\xi|^{3+\alpha}} = \frac{e^{-\delta_p |\xi|^p t} \phi_0(\xi) - 1}{|\xi|^{3+\alpha}} + \int_0^t e^{-\delta_p |\xi|^p (t-\tau)} \frac{\mathcal{B}(\phi)(\xi, \tau)}{|\xi|^{3+\alpha}} d\tau.$$

By Lemma 2.2.2 we obtain

$$\|\phi(t) - 1\|_{\mathcal{M}^\alpha} \leq \|\phi_0 - 1\|_{\mathcal{M}^\alpha} + C_{p,\alpha} (\delta_p t)^{\alpha/p} + C_0 \mu_\alpha \int_0^t \|\phi(\tau) - 1\|_{\mathcal{M}^\alpha} d\tau.$$

The estimate now follows by applying the Gronwall inequality. $\square$

For brevity, we will denote $C_0 \mu_\alpha e^{C_0 \mu_\alpha T} [\|\phi_0 - 1\|_{\mathcal{M}^\alpha} + C_{p,\alpha} (\delta_p T)^{\alpha/p}]$ by $\tilde{C}_0$.

Lemma 2.4.2. *Fix $T > 0$. Under the assumptions of Lemma 2.4.1 we have for all $s, t \in [0, T]$*

$$\|\phi(t) - \phi(s)\|_{\mathcal{M}^\alpha} \leq C'(\phi_0, T) |t - s|^{\alpha/p},$$

where

$$C'(\phi_0, T) = C_{p,\alpha} \delta_p^{\frac{\alpha}{p}} + \left(1 + \frac{1}{1 - \alpha/p}\right) \tilde{C}_0 T^{1-\alpha/p}.$$

Proof. Set $0 \leq s < t \leq T$ and define K_1, K_2, K_3 by

$$\begin{aligned} \phi(\xi, t) - \phi(\xi, s) &= \left(e^{-\delta_p |\xi|^p t} - e^{-\delta_p |\xi|^p s} \right) \phi_0(\xi) + \int_s^t e^{-\delta_p |\xi|^p (t-\tau)} \mathcal{B}(\phi)(\xi, \tau) d\tau \\ &\quad + \int_0^s \left(e^{-\delta_p |\xi|^p (t-\tau)} - e^{-\delta_p |\xi|^p (s-\tau)} \right) \mathcal{B}(\phi)(\xi, \tau) d\tau \\ &= K_1 + K_2 + K_3. \end{aligned}$$

Obviously

$$\int_{\mathbb{R}^3} \frac{|K_1|}{|\xi|^{3+\alpha}} d\xi \leq \int_{\mathbb{R}^3} \frac{e^{-\delta_p |\xi|^p s} (1 - e^{-\delta_p |\xi|^p (t-s)})}{|\xi|^{3+\alpha}} d\xi \leq C_{p,\alpha} [\delta_p (t-s)]^{\alpha/p}.$$

By Lemma 2.2.2 and Lemma 2.4.1,

$$\int_{\mathbb{R}^3} \frac{|K_2|}{|\xi|^{3+\alpha}} d\xi \leq C_0 \mu_\alpha \int_s^t \|\phi(\tau) - 1\|_{\mathcal{M}^\alpha} d\tau \leq \tilde{C}_0 (t-s).$$

For a fixed $\tau \in [0, s]$, $f(r) = e^{-(s-\tau)r} - e^{-(t-\tau)r}$ ($r \geq 0$) is a nonnegative function, satisfying

$$\max_{r \geq 0} f(r) = \frac{t-s}{t-\tau} \left(\frac{s-\tau}{t-\tau} \right)^{\frac{s-\tau}{t-s}}.$$

Furthermore, we have

$$\left(\frac{s-\tau}{t-\tau} \right)^{\frac{s-\tau}{t-s}} \leq \left(\frac{t-\tau}{t-s} \right)^{1-\alpha/p} \quad (2.4.19)$$

because the left-hand side is smaller than 1 and the right-hand side is larger than 1. Thus

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|K_3|}{|\xi|^{3+\alpha}} d\xi &\leq \int_{\mathbb{R}^3} \int_0^s \left(\frac{t-s}{t-\tau} \right)^{\alpha/p} \cdot \frac{|\mathcal{B}(\phi)(\xi, \tau)|}{|\xi|^{3+\alpha}} d\xi d\tau \\ &\leq (t-s)^{\alpha/p} \cdot \tilde{C}_0 \left[-\frac{1}{1-\alpha/p} (t-\tau)^{1-\alpha/p} \right]_{\tau=0}^s \\ &\leq (t-s)^{\alpha/p} \tilde{C}_0 \frac{t^{1-\alpha/p}}{1-\alpha/p} \leq \frac{T^{1-\alpha/p}}{1-\alpha/p} \tilde{C}_0 (t-s)^{\alpha/p}. \end{aligned}$$

Combining these estimates, the result follows. $\square$

Proof of Theorems 2.1.1-2.1.3. We follow [7]. For each $n \in \mathbb{N}$, define $b_n(\cos \theta) = \min\{b(\cos \theta), n\}$ and ϕ_n as a solution of (2.3.15) with b replaced by b_n . Define $\lambda_{\alpha, n}$ in the same manner. Then we have

$$\|\phi_n(t) - 1\|_{\mathcal{M}^\alpha} \leq e^{\lambda_{\alpha, n} t} \|\phi_0 - 1\|_{\mathcal{M}^\alpha} \leq e^{\lambda_\alpha t} \|\phi_0 - 1\|_{\mathcal{M}^\alpha}.$$

In [4], the Ascoli-Arzelà theorem was used to extract a subsequence of the series of functions $\{\phi_n\}$, which converges to a solution ϕ of (2.1.8) in $C([0, \infty); \mathcal{K}^\alpha)$. Since ϕ is a unique solution in $\mathcal{S}_p^\alpha(\mathbb{R}^3 \times [0, \infty))$, we can extract such a subsequence. We denote this subsequence also by $\{\phi_n\}$. For all $\delta > 0$, we have

$$\int_{\delta < |\xi| < \delta^{-1}} \frac{|\phi(\xi, t) - 1|}{|\xi|^{3+\alpha}} d\xi = \lim_{n \rightarrow \infty} \int_{\delta < |\xi| < \delta^{-1}} \frac{|\phi_n(\xi, t) - 1|}{|\xi|^{3+\alpha}} d\xi \leq e^{\lambda_\alpha t} \|\phi_0 - 1\|_{\mathcal{M}^\alpha}.$$

The Lebesgue convergence theorem therefore implies that $\phi(\cdot, t) \in \mathcal{M}^\alpha$. That $\phi \in C([0, \infty); \mathcal{M}^\alpha)$ follows from Lemma 2.4.2. Finally, by differentiating (2.1.8) with respect to t , we obtain $\phi \in \mathcal{T}^\alpha(\mathbb{R}^3 \times [0, \infty))$. For the proof of the maximum growth estimate and the stability estimate of a solution, we take two sequences $\{\phi_n\}, \{\psi_n\} \subset \mathcal{S}^\alpha(\mathbb{R}^3 \times [0, \infty))$ which converge to solutions $\phi, \psi \in \mathcal{S}^\alpha(\mathbb{R}^3 \times [0, \infty))$ with initial data $\phi_0, \psi_0 \in \mathcal{M}^\alpha$, respectively. Then for any $t \geq 0$

$$\sup_{\xi \in \mathbb{R}^3} e^{\delta_p |\xi|^p t} |\phi_n(\xi, t)| \leq 1, \quad \left\| e^{\delta_p |\cdot|^p t} (\phi_n(\cdot, t) - \psi_n(\cdot, t)) \right\|_{\mathcal{M}^\alpha} \leq e^{\lambda_\alpha t} \|\phi_0 - \psi_0\|_{\mathcal{M}^\alpha}. \quad (2.4.20)$$

Letting $n \rightarrow \infty$ in these inequalities yields Theorem 2.1.2. Once (2.1.11) is proved, we immediately conclude that $\phi(\xi, t)$ in Theorem 2.1.1 belongs to L^1 and to any polynomially weighted L^2 space with respect to ξ . The Fourier inversion formula implies that $\phi(\xi, t)$ has a unique probability density $f(v, t)$ vanishing at infinity, and the Sobolev embedding theorem shows that $f(v, t)$ is a smooth function of v for every positive t . The proof is completed by repeating the arguments of [7, Section 3.3], the details of which are omitted. $\square$

2.5 Non-existence of a solution in the case $p \leq \alpha < 2$

In [4], a solution of (2.1.8) was found in $\mathcal{S}^\alpha$ for initial data in $\mathcal{K}^\alpha$ when $0 < \alpha \leq p \leq 2$ and non-existence of solutions was proved when $p < \alpha \leq 2$. However, in our case the existence of a solution in $\mathcal{T}^\alpha$ for initial data in $\mathcal{M}^\alpha$ fails also when $\alpha = p$. Here we recall Lemma 2.2.3. Since $\mathcal{M}^\alpha = \mathcal{F}(P_\alpha(\mathbb{R}^3))$ when $0 < \alpha < 2$ and $\alpha \neq 1$, the following theorem precisely reflects the moment estimates of $f_p(v, t)$.

Theorem 2.5.1. *Let $p \leq \alpha < 2$. Then for any $T > 0$ and $\phi_0 \in \mathcal{M}^\alpha$, there is no solution of (2.1.8) which belongs to $\mathcal{T}^\alpha(\mathbb{R}^3 \times [0, T])$.*

Proof. Assume to reach a contradiction that $\phi \in \mathcal{T}^\alpha(\mathbb{R}^3 \times [0, T])$ is a solution of (2.1.8). It is easy to check that

$$1 - e^{-\delta_p |\xi|^p t} = e^{-\delta_p |\xi|^p t} (\phi_0(t) - 1) + 1 - \phi(\xi, t) + \int_0^t e^{-\delta_p |\xi|^p (t-\tau)} \mathcal{B}(\phi)(\xi, \tau) d\tau.$$

Clearly,

$$\int_{\mathbb{R}^3} \frac{1 - e^{-\delta_p |\xi|^p t}}{|\xi|^{3+\alpha}} d\xi \leq 2 \max_{t \in [0, T]} \|\phi(t) - 1\|_{\mathcal{M}^\alpha} + \int_0^t C_0 \mu_\alpha \|\phi(\tau) - 1\|_{\mathcal{M}^\alpha} d\tau < \infty,$$

which contradicts the fact that $\int_{\mathbb{R}^3} \frac{1 - e^{-\delta_p |\xi|^p t}}{|\xi|^{3+\alpha}} d\xi$ diverges. $\square$

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Chapter 3

Global solutions in the critical Besov space for the non cutoff Boltzmann equation

Abstract

The Boltzmann equation is studied without the cutoff assumption. Under a perturbation setting, a unique global solution of the Cauchy problem of the equation is established in a critical Chemin-Lerner space. In order to analyse the collisional term of the equation, a Chemin-Lerner norm is combined with a non-isotropic norm with respect to a velocity variable, which yields an a priori estimate for an energy estimate. Together with local existence following from commutator estimates and the Hahn-Banach extension theorem, the desired solution is obtained. Also, the non-negativity of the solution is rigorously shown. This work is based on a joint work with Yoshinori Morimoto (Kyoto University, Professor Emeritus).

3.1 Introduction

We consider the Boltzmann equation in $\mathbb{R}^3$,

$$\begin{cases} \partial_t f(x, v, t) + v \cdot \nabla_x f(x, v, t) = Q(f, f)(x, v, t), \\ f(x, v, 0) = f_0(x, v), \end{cases} \quad (3.1.1)$$

where $f = f(x, v, t)$ is a density distribution of particles in a rarefied gas with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time $t \geq 0$, and f_0 is an initial datum.

The aim of this paper is to show the global existence and uniqueness of solution to (3.1.1) in an appropriate Besov space under a perturbative framework. There are many references concerning well-posedness of solutions to the Boltzmann equation around the global Maxwellian equilibrium in different spaces. Recently, Duan, Liu, and Xu [13] proved that the Boltzmann equation has a unique global strong solution near Maxwellian in the Chemin-Lerner space $\tilde{L}_T^\infty \tilde{L}_v^2(B_{2,1}^{3/2})$ (defined later: this space is a space-velocity-time Besov space) under the Grad's angular cutoff assumption. This was the first result which applied Besov spaces to the Cauchy problem of the Boltzmann equation around the equilibrium. In this paper, we consider (3.1.1) without angular cutoff by using the triple norm, introduced by Alexandre, Morimoto, Ukai, Xu, and Yang (AMUXY in what follows) [4, 7]. The triple norm was originally adopted to discuss the equation in $L_t^\infty(0, \infty; H_l^k(\mathbb{R}_{x,v}^6))$ for suitable $l, k \in \mathbb{N}$. We combine both ideas [13] and [7] (see also [5, 6, 8]) to discuss the Cauchy problem without angular cutoff. For this purpose, we later introduce a Chemin-Lerner type triple norm, which enables us to take full advantage of the two ideas.

We discuss (3.1.1) in greater detail. The bilinear operator $Q(f, g)$ is called the Boltzmann collision operator and is defined by

$$Q(f, g)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) (f'_* g' - f_* g) d\sigma dv_*,$$

where $f'_* = f(v'_*)$, $g' = g(v')$, $f_* = f(v_*)$, $g = g(v)$, and

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma \quad (\sigma \in \mathbb{S}^2).$$

The operator Q acts only on v and $Q(f, g)(x, v, t) := Q(f(x, \cdot, t), g(x, \cdot, t))(v)$. A collision kernel B is correspondingly defined to reflect various physical phenomena. The first assumption is

$$B(v - v_*, \sigma) = b \left(\frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \Phi_\gamma(|v - v_*|) \geq 0,$$

where $\Phi_\gamma(|v - v_*|) = |v - v_*|^\gamma$ for $\gamma > -3$. This model is physically derived when the particle interaction obeys the inverse power law $\phi(r) = r^{-(p-1)}$ ($2 < p < \infty$), where r is a distance between two interacting particles. Before mentioning to the second assumption, we should recall that b is a function of the deviation angle $\theta \in [0, \pi]$, defined by $\cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma$, so we write $b \left(\frac{v - v_*}{|v - v_*|} \cdot \sigma \right) = b(\cos \theta)$. As an usual convention, $b(\cos \theta)$ ($\theta \in [0, \pi]$) is replaced by

$$[b(\cos \theta) + b(\cos(\pi - \theta))] \mathbf{1}_{\{0 \leq \theta \leq \pi/2\}}$$

so that a domain of b is $[0, \pi/2]$. The second one is called the non-cutoff assumption: for some $\nu \in (0, 2)$ and $K > 0$,

$$b \left(\frac{v - v_*}{|v - v_*|} \cdot \sigma \right) = b(\cos \theta) \sim K \theta^{-2-\nu} \quad (\theta \downarrow 0),$$

which implies that b is not integrable near $\theta = 0$. In the physical model of the inverse power law potential, relations among γ , ν , and p are given by $\gamma = (p-5)/(p-1)$ and $\nu = 2/(p-1)$. In this paper, we particularly consider the case when

$$\gamma > \max\{-3, -3/2 - \nu\}.$$

The latter component of this below estimate is due to a technical reason which will appear when we derive various estimates for the collision operator and a non-linear term derived from Q . However, it is easily verified that $-1 < \gamma + \nu < 1$ as long as $\gamma = (p-5)/(p-1)$ and $\nu = 2/(p-1)$, that is, all the physically possible combinations of γ and ν are contained in our restriction.

We consider the Boltzmann equation around the normalized Maxwellian

$$\mu = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2},$$

thus we set $f = \mu + \mu^{1/2}g$, so that (3.1.1) is equivalent to the equation

$$\partial_t g + v \cdot \nabla_x g + \mathcal{L}g = \Gamma(g, g), \quad (3.1.2)$$

with initial datum g_0 defined from $f_0 = \mu + \mu^{1/2}g_0$, where

$$\Gamma(g, h) = \mu^{-1/2} Q(\mu^{1/2}g, \mu^{1/2}h), \quad \mathcal{L}g = \mathcal{L}_1g + \mathcal{L}_2g := -\Gamma(\mu^{1/2}, g) - \Gamma(g, \mu^{1/2}).$$

These are the non-linear and the linear term of the Boltzmann equation, respectively. It is known that $\mathcal{L}$ is nonnegative-definite on $L_v^2(\mathbb{R}^3)$ and

$$\ker \mathcal{L} = \text{span}\{\mu^{1/2}, \mu^{1/2}v_i \ (i = 1, 2, 3), \mu^{1/2}|v|\}.$$

The projection operator on $\ker \mathcal{L}$ is denoted by $\mathbf{P}$, therefore for each g ,

$$\mathbf{P}g(x, v, t) = [a(x, t) + v \cdot b(x, t) + |v|^2 c(x, t)] \mu^{1/2}(v).$$

The decomposition $g = \mathbf{P}g + (\mathbf{I} - \mathbf{P})g$ is called the macro-micro decomposition, where $\mathbf{I}$ is the identity map.

The energy term and the dissipation term are respectively defined by

$$\mathcal{E}_T(g) \sim \|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}$$

and

$$\mathcal{D}_T(g) = \|\nabla_x(a, b, c)\|_{\tilde{L}_T^2(B_x^{1/2})} + \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^{3/2}}.$$

Here and hereafter, $B_x^s := B_{2,1}^s(\mathbb{R}_x^3)$. We have just showed hidden indices, and the above norms will be rigorously defined in Section 2.

The main theorem of this paper is the following.

Theorem 3.1.1. *Let $0 < \nu < 2$ and $\gamma > \max\{-3, -\nu - 3/2\}$. There are constants $\varepsilon_0 > 0$ and $C > 0$ such that if*

$$\|g_0\|_{\tilde{L}_v^2(B_x^{3/2})} \leq \varepsilon_0,$$

then there exists a unique global solution $g(x, v, t)$ of (3.1.2) with initial datum $g_0(x, v)$. This solution satisfies

$$\mathcal{E}_T(g) + \mathcal{D}_T(g) \leq C \|g_0\|_{\tilde{L}_v^2(B_x^{3/2})} \quad (3.1.3)$$

for any $T > 0$. Moreover if $f_0(x, v) = \mu + \sqrt{\mu}g_0(x, v) \geq 0$ then $f(x, v, t) = \mu + \sqrt{\mu}g(x, v, t) \geq 0$.

Let us make some comments on the theorem. We note that Besov embedding $B_{2,1}^{3/2}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ ensures that $B_{2,1}^{3/2}(\mathbb{R}^3)$ is a Banach algebra, that is, if f and g are elements of $B_{2,1}^{3/2}(\mathbb{R}^3)$, then so is the product fg (see Corollary 2.86 of [11] for instance). This is the reason why we employed this space with respect to x , combined with $L_T^\infty L_v^2$. In this sense, we used the word *critical* in the title; compare with the well-known fact that we need $s > 3/2$ for $H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. We also remark that the Chemin-Lerner space $\tilde{L}_T^\infty \tilde{L}_v^2(B_{2,1}^{3/2})$ defines stronger topology than $L_T^\infty L_v^2(B_{2,1}^{3/2})$. Thus we shall adopt the Chemin-Lerner norm to the problem.

We review some known results, confining ourselves to research on the Cauchy problem around the global Maxwellian. The first fruitful work was done by Ukai [22], [23], based on spectral analysis of the collision operator and the bootstrap argument. This is the first result obtaining a global mild solution to the equation. Notable works following this perspective are, for example, Nishida and Imai [19], and Ukai and Asano [24]. Most of these results were secured in the 70's and the 80's. They adopted L^∞ -norm to estimate a solution with respect to v . Recent study based on L_v^2 -framework has started from similar but independent results by both Guo[15] and Liu and Yu[18], Liu, Yang, and Yu[17]. They applied the macro-micro decomposition, which clarifies dissipative and equilibrating effects of the equation. This allowed us to utilize the energy method. They succeeded in constructing a classical solution in a x - and v -Sobolev space.

There are two independent results concerning global existence around the equilibrium for the non-cutoff Boltzmann equation. To establish a global solution to the non cut-off Boltzmann equation in the whole space, AMUXY [4, 5, 7] followed Guo's framework with intensive pseudo-differential calculus on the collisional term, in both general hard potential case $\gamma + \nu > 0$ and general soft potential case $\gamma + \nu \leq 0$. More precisely, when $\gamma + \nu > 0$, [5] constructed global solutions in $L_t^\infty(0, \infty; H_l^k(\mathbb{R}_{x,v}^6))$ with $k \geq 6$ and $l > 3/2 + \gamma + \nu$. Here, $H_l^k(\mathbb{R}_{x,v}^6) = \{f \in \mathcal{S}'(\mathbb{R}_{x,v}^6); \langle v \rangle^l f \in H^k(\mathbb{R}_{x,v}^6)\}$. For the general soft potential case under an additional condition $\gamma > \max\{-3, -\nu - 3/2\}$, AMUXY [7] found a solution in $L_t^\infty(0, \infty; L_v^2(\mathbb{R}^3; H_x^N(\mathbb{R}^3)))$ with $\mathbb{N} \ni N \geq 4$ and another solution in $L_t^\infty(0, \infty; \tilde{\mathcal{H}}_\ell^k(\mathbb{R}_{x,v}^6))$ with $4 \leq k \in \mathbb{N}, \ell \geq k$, where

$$\tilde{\mathcal{H}}_\ell^k(\mathbb{R}_{x,v}^6) = \{f \in \mathcal{S}'(\mathbb{R}_{x,v}^6); \sup_{|\alpha+\beta| \leq k} \|\langle v \rangle^{(\ell-|\beta|)|\gamma+\nu|} \partial_x^\alpha \partial_v^\beta f\|_{L^2(\mathbb{R}^6)}^2 < \infty\}.$$

AMUXY [6] also studied for qualitative properties including smoothing effect, uniqueness, non-negativity, and convergence rate towards the equilibrium. When

the space variable moves in a torus, the global existence was shown by Gressman and Strain [14] with convergence rates. They employed an anisotropic metric on the “lifted” paraboloid $d(v, v') = \sqrt{|v - v'|^2 + (|v|^2 - |v'|^2)^2/4}$ to capture changes in the power of the weight. For the general hard potential case, their result is sharper than [5] and covers the global well-posedness in non-weighted and non-velocity derivative energy functional spaces $L_t^\infty(0, \infty; L_v^N(\mathbb{R}^3; H_x^2(\mathbb{T}^3)))$ with $N \geq 2$. It should be mentioned that the soft potential case was also studied by [14] in the space $L_t^\infty(0, \infty; \tilde{\mathcal{H}}_\ell^k(\mathbb{T}_x^3 \times \mathbb{R}_v^3))$ with $4 \leq k \in \mathbb{N}, \ell \geq 0$, without additional condition $\gamma > \max\{-3, -\nu - 3/2\}$. The main theorem of the present paper improves the results in non-weighted and non-velocity derivative energy functional spaces, given in [7] for the general soft potential case and in [14] for the general hard potential case, respectively.

Our work is motivated by Duan, Liu, and Xu [13]. Under the cut-off assumption, they studied the hard potential case $\gamma > 0$ by using the Chemin-Lerner space $\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})$, which was originally invented in [12] to investigate the existence of the flow for solutions of the Navier-Stokes equations. One advantage of this time-velocity-space Besov space over a usual space $L_T^\infty L_v^2 B_x^{3/2}$ can be explained as follows: in general, it is easier to bound dyadic blocks of the Littlewood-Paley decomposition in $L_T^\infty L_v^2 L_x^2$ rather than to estimate directly the solution for the equation in $L_T^\infty L_v^2 B_x^{3/2}$. Since the global well-posed theory in $\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})$ was established by [13] for the cutoff case, we extend their methods to the non cut-off case by using upper bound estimates of the non-linear term and calculation of commutators which were devised in a series of AMUXY’s works.

There are some other papers applying the Besov space to analysis of the Boltzmann equation. We will simply introduce some of them concerning the Cauchy problem. Recently, Tang and Liu [21] followed [13] under the cut-off assumption and improved the result by replacing $B_{2,1}^{3/2}$ with narrower spaces $B_{2,r}^s$ ($(s, r) \in (3/2, \infty) \times [1, 2]$ or $= (3/2, 1)$). For other research, see Alexandre [1], Arsénio and Masmoudi [10], Sohinger and Strain [20].

This paper is organized as follows. In Section 2, we review definitions of the Littlewood-Paley decomposition, Besov spaces, and Chemin-Lerner spaces. Taking into account the triple norm defined in [4, 7], we also define a Chemin-Lerner type triple norm. In Section 3, we deduce Besov-type tri-linear estimates. We have to derive three different estimates of Besov-type for the non-linear term Γ , which lead us to the global and local existence of solutions. Since σ -integration on the unit sphere is significant in the non cutoff case, the non-linear term can not be estimated by separating the so-called gain and loss terms, different from the cut-off case in [13]. To get estimates analogous to those in Lemma 3.1 of [13], the σ -integration should be dealt with by the triple norm, involving two velocity variables v, v' . Section 4 is devoted to deduce an a priori estimate for the global existence. The microscopic part $(\mathbf{I} - \mathbf{P})g$ is handled by the estimates established in Section 3, however, we cannot adopt them for the macroscopic part $\mathbf{P}g$. To overcome this difficulty, we will bring in a fluid-type system of (a, b, c) . Applying the energy method to it, we will acquire an

estimate of the macroscopic dissipation term. Local existence of a solution will be shown in Section 5. We want to establish a solution by the duality argument and the Hahn-Banach extension theorem, but a problem arises since, as far as we know, the dual space of $\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})$ is unknown. Therefore, we first find a solution to a linearized equation in a wider space $L^\infty(0, T; L^2(\mathbb{R}_{x,v}^6))$ and construct an approximate sequence. Then we show that a limit of the sequence is a solution to the equation, and it belongs to a suitable solution space indeed. This is our strategy, however, we have to employ very delicate commutator estimates involving the non-linear term Γ and cut-off functions with respect to both x and v . Proofs of these estimates are fairly technical, so in this section we postpone proving them and focus on solving the equation assuming that the necessary lemmas are evidenced. Such lemmas are collected in Appendix with proofs. Non-negativity of the solution obtained will be given in Section 6. Known inequalities of Besov spaces and the triple norm used in this paper are also collected in Appendix.

3.2 Preliminaries

In this section, we define some function spaces for later use. Readers may consult [11] for this topic. First, we introduce a dyadic partition of unity, also known as the Littlewood-Paley decomposition. Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^3 \mid 3/4 \leq |\xi| \leq 8/3\}$ and $\mathcal{B}$ be the ball $B(0, 4/3)$. Then there exist radial functions $\chi \in C_0^\infty(\mathcal{B})$ and $\phi \in C_0^\infty(\mathcal{C})$ satisfying the following conditions:

$$\begin{aligned} 0 &\leq \chi(\xi), \phi(\xi) \leq 1 \quad \text{for any } \xi \in \mathbb{R}^3, \\ \chi(\xi) + \sum_{q \geq 0} \phi(2^{-q}\xi) &= 1 \quad \text{for any } \xi \in \mathbb{R}^3, \\ \sum_{q \in \mathbb{Z}} \phi(2^{-q}\xi) &= 1 \quad \text{for any } \xi \in \mathbb{R}^3 \setminus \{0\}, \\ |q - q'| \geq 2 &\Rightarrow \text{supp } \phi(2^{-q}\cdot) \cap \text{supp } \phi(2^{-q'}\cdot) = \emptyset, \\ q \geq 1 &\Rightarrow \text{supp } \chi \cap \text{supp } \phi(2^{-q}\cdot) = \emptyset, \\ |q - q'| \geq 5 &\Rightarrow 2^{q'}\tilde{\mathcal{C}} \cap 2^q\mathcal{C} = \emptyset, \end{aligned}$$

where $\tilde{\mathcal{C}} := B(0, 2/3) + \mathcal{C}$. We fix these functions and write $h := \mathcal{F}^{-1}\phi$ and $\tilde{h} := \mathcal{F}^{-1}\chi$. For each $f \in \mathcal{S}'(\mathbb{R}_x^3)$, the non-homogeneous dyadic blocks Δ_q are defined by

$$\begin{aligned} \Delta_{-1}f &:= \chi(D)f = \int_{\mathbb{R}^3} \tilde{h}(y)f(x-y)dy, \\ \Delta_qf &:= \phi(2^{-q}D)f = 2^{3q} \int_{\mathbb{R}^3} h(2^qy)f(x-y)dy \quad (q \in \mathbb{N} \cup \{0\}) \end{aligned}$$

and $\Delta_q f := 0$ if $q \leq -2$. The non-homogeneous low-frequency cutoff operator S_q is defined by

$$S_q f = \sum_{q' \leq q-1} \Delta_{q'} f.$$

We denote the set of all polynomials on $\mathbb{R}^3$ by $\mathcal{P}$. Regarding any polynomial P as a distribution, we notice that $\dot{\Delta}_q P = 0$. Therefore, the homogeneous dyadic blocks $\dot{\Delta}_q$ are defined by

$$\dot{\Delta}_q f := \phi(2^{-q} D) f := 2^{3q} \int_{\mathbb{R}^3} h(2^q y) f(x - y) dy$$

for any $f \in \mathcal{S}'(\mathbb{R}_x^3)/\mathcal{P}$ and $q \in \mathbb{Z}$.

We now give the definition of non-homogeneous Besov spaces as follows.

Definition 3.2.1. *Let $1 \leq p, r \leq \infty$ and $s \in \mathbb{R}$. The non-homogeneous Besov space B_{pr}^s is defined by*

$$B_{pr}^s := \left\{ f \in \mathcal{S}'(\mathbb{R}_x^3) \mid \|f\|_{B_{pr}^s} := \left\| (2^{qs} \|\Delta_q f\|_{L_x^p})_{q \geq -1} \right\|_{\ell^r} < \infty \right\}.$$

When $r = \infty$, we set $\|f\|_{B_{p\infty}^s} := \sup_{q \geq -1} 2^{qs} \|\Delta_q f\|_{L_x^p}$.

Here $L_x^p := L^p(\mathbb{R}_x^3)$ and this kind of abuse of notation will be used throughout the paper.

The definition of homogeneous Besov spaces is as follows:

Definition 3.2.2. *Let $1 \leq p, r \leq \infty$ and $s \in \mathbb{R}$. The homogeneous Besov space $\dot{B}_{pr}^s$ is defined by*

$$\dot{B}_{pr}^s := \left\{ f \in \mathcal{S}'(\mathbb{R}_x^3)/\mathcal{P} \mid \|f\|_{\dot{B}_{pr}^s} := \left\| (2^{qs} \|\dot{\Delta}_q f\|_{L_x^p})_{q \in \mathbb{Z}} \right\|_{\ell^r} < \infty \right\}.$$

When $r = \infty$, we set $\|f\|_{\dot{B}_{pr}^s} := \sup_{q \in \mathbb{Z}} 2^{qs} \|\dot{\Delta}_q f\|_{L_x^p}$.

For simplicity of notations, we will write $B_{2,1}^s =: B_x^s$ and $\dot{B}_{2,1}^s =: \dot{B}_x^s$, as stated in the introduction.

Next, we define Chemin-Lerner spaces, which is a generalization of the Besov space.

Definition 3.2.3. *Let $1 \leq p, r, \alpha, \beta \leq \infty$ and $s \in \mathbb{R}$. For $T \in [0, \infty)$, the Chemin-Lerner space $\tilde{L}_T^\alpha \tilde{L}_v^\beta(B_{pr}^s)$ is defined by*

$$\tilde{L}_T^\alpha \tilde{L}_v^\beta(B_{pr}^s) := \left\{ f(\cdot, v, t) \in \mathcal{S}' \mid \|f\|_{\tilde{L}_T^\alpha \tilde{L}_v^\beta(B_{pr}^s)} < \infty \right\},$$

where

$$\begin{aligned} \|f\|_{\tilde{L}_T^\alpha \tilde{L}_v^\beta(B_{pr}^s)} &:= \left\| (2^{qs} \|\Delta_q f\|_{L_T^\alpha L_v^\beta L_x^p})_{q \geq -1} \right\|_{\ell^r}, \\ \|\Delta_q f\|_{L_T^\alpha L_v^\beta L_x^p} &:= \left(\int_0^T \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |\Delta_q f(x, v, t)|^p dx \right)^{\beta/p} dv \right)^{\alpha/\beta} dt \right)^{1/\alpha} \end{aligned}$$

with the usual convention when at least one of p, r, α, β is equal to ∞ . We also define $\tilde{L}_T^\alpha \tilde{L}_v^\beta(\dot{B}_{pr}^s)$ similarly.

We denote $\tilde{L}_T^\alpha \tilde{L}_v^\beta(B_{2,1}^s)$ by $\tilde{L}_T^\alpha \tilde{L}_v^\beta(B_x^s)$, and $\tilde{L}_T^\alpha \tilde{L}_v^\beta(\dot{B}_{2,1}^s)$ by $\tilde{L}_T^\alpha \tilde{L}_v^\beta(\dot{B}_x^s)$. Finally, we give the definition of the non-isotropic norm $\|\cdot\|$ and the space $\mathcal{T}_{Tpr}^s$ and $\dot{\mathcal{T}}_{Tpr}^s$, which are endowed with the ‘‘Chemin-Lerner type triple norm’’.

Definition 3.2.4. Let $1 \leq p, r \leq \infty$, $T > 0$ and $s \in \mathbb{R}$. $\|\cdot\|$ is defined by

$$\begin{aligned} \|\cdot\|^2 &:= \iiint B(v - v_*, \sigma) \mu_*(f' - f)^2 dv dv_* d\sigma \\ &\quad + \iiint B(v - v_*, \sigma) f_*^2 (\sqrt{\mu'} - \sqrt{\mu})^2 dv dv_* d\sigma \\ &= J_1^{\Phi_\gamma}(g) + J_2^{\Phi_\gamma}(g), \end{aligned}$$

and the space $\mathcal{T}_{Tpr}^s$ is defined by

$$\mathcal{T}_{Tpr}^s := \left\{ f \mid \|f\|_{\mathcal{T}_{Tpr}^s} = \left\| (2^{qs} \|\cdot\|_{L_T^\alpha L_x^p})_{q \geq -1} \right\|_{\ell^1} < \infty \right\}.$$

$\dot{\mathcal{T}}_{Tpr}^s$ is defined in the same manner.

$\|\cdot\|$ is called the triple norm, and it is known that this norm is estimated from both above and below by weighted Sobolev norms (see [7, Proposition 2.2]):

$$\|f\|_{H_{\gamma/2}^{\nu/2}}^2 + \|f\|_{L_{(\nu+\gamma)/2}^2}^2 \lesssim \|\cdot\|^2 \lesssim \|f\|_{H_{(\nu+\gamma)/2}^{\nu/2}}^2. \quad (3.2.1)$$

If p or r is an explicit number, $\mathcal{T}_{Tpr}^s$ is denoted by $\mathcal{T}_{T,p,r}^s$ to avoid ambiguity.

In order to deduce Chemin-Lerner estimates in the following sections, we will use some properties of the above spaces. See Appendix for these properties. It must be emphasized again that $\|\cdot\|_{\tilde{L}_T^\alpha \tilde{L}_v^\beta(B_{pr}^s)}$ is usually easier to handle than $\|\cdot\|_{L_T^\alpha L_v^\beta(B_{pr}^s)}$.

3.3 Chemin-Lerner type tri-linear estimates

We will show an important estimate of the non-linear term Γ in the suitable Chemin-Lerner space. This estimate will be used many times throughout the paper. We denote the usual $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ and $L^2(\mathbb{R}_v^3)$ inner product by $(\cdot, \cdot)_{x,v}$ and $(\cdot, \cdot)_v$, respectively.

Lemma 3.3.1. *Let $0 < s \leq \frac{3}{2}$ and $0 < T \leq \infty$. Define*

$$A_T(f, g, h) := \sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2}.$$

Then the following tri-linear estimates hold:

$$A_T(f, g, h) \lesssim \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,2,2}^{3/2}}^{1/2} \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2}, \quad (3.3.1)$$

$$A_T(f, g, h) \lesssim \left(\|f\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,\infty,2}^s}^{1/2} + \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^s)}^{1/2} \|g\|_{\mathcal{T}_{T,2,2}^{3/2}}^{1/2} \right) \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2}. \quad (3.3.2)$$

Moreover, if $\gamma + \nu \leq 0$, we have

$$\begin{aligned} A_T(f, g, h) &\lesssim \left(\|\mu^{1/10} f\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,\infty,2}^s}^{1/2} \right. \\ &\quad \left. + \|\mu^{1/10} f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^s)}^{1/2} \|g\|_{\mathcal{T}_{T,2,2}^{3/2}}^{1/2} \right) \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \\ &\quad + \left(\|f\|_{L_T^2 L_{v,(\nu+\gamma)/2}^2(\dot{B}_x^{3/2})}^{1/2} \|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^s)}^{1/2} \right. \\ &\quad \left. + \|f\|_{\tilde{L}_T^\infty \tilde{L}_{v,(\nu+\gamma)/2}^2(B_x^s)}^{1/2} \|g\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}^{1/2} \right) \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2}. \end{aligned} \quad (3.3.3)$$

Proof. Before starting the proof of this lemma, we recall the Bony decomposition. Since $\sum_j \Delta_j = \text{Id}$, we have, at least formally, $fg = \sum_{j,j'} \Delta_j f \Delta_{j'} g$. Dividing this summation according to the frequencies, we obtain the following Bony decomposition:

$$fg = \mathcal{T}_f g + \mathcal{T}_g f + \mathcal{R}(f, g),$$

where

$$\begin{aligned} \mathcal{T}_f g &:= \sum_j S_{j-1} f \Delta_j g, \quad \mathcal{T}_g f := \sum_j \Delta_j f S_{j-1} g, \\ \mathcal{R}(f, g) &:= \sum_j \left(\sum_{|j-j'| \leq 1} \Delta_{j'} f \Delta_j g \right). \end{aligned}$$

Using this decomposition, we divide the inner product into three parts:

$$(\Delta_q \Gamma(f, g), \Delta_q h) = (\Delta_q (\Gamma^1(f, g) + \Gamma^2(f, g) + \Gamma^3(f, g)), \Delta_q h),$$

where $\Gamma^1(f, g) := \sum_j \Gamma(S_{j-1} f, \Delta_j g) = \sum_j \Gamma_j^1(f, g)$, and $\Gamma^2(f, g)$ and $\Gamma^3(f, g)$ are similarly defined.

First, we treat $\Gamma^1(f, g)$. From shapes of supports of χ and $\phi(2^{-j}\cdot)$, we notice

$$\Delta_q \sum_j (S_{j-1} f \Delta_j g) = \Delta_q \sum_{|j-q| \leq 4} (S_{j-1} f \Delta_j g),$$

that is, when Δ_q is applied to, the above summation with respect to j becomes finite. Hence we have

$$\begin{aligned} \left(\Delta_q \sum_j \Gamma_j^1(f, g), \Delta_q h \right)_{x,v} &= \sum_{|j-q| \leq 4} (\Delta_q \Gamma_j^1(f, g), \Delta_q h)_{x,v} \\ &= \sum_{|j-q| \leq 4} (\Gamma_j^1(f, g), \Delta_q^2 h)_{x,v} \\ &\lesssim \sum_{|j-q| \leq 4} \int_{\mathbb{R}^3} \|S_{j-1} f\|_{L_v^2} \|\Delta_j g\| \|\Delta_q^2 h\| dx \\ &\lesssim \sum_{|j-q| \leq 4} \|f\|_{L_v^2 L_x^\infty} \|\Delta_j g\|_{L_x^2} \|\Delta_q h\|_{L_x^2}, \end{aligned}$$

where we used Corollary 3.7.2 and Lemma 3.7.5. We also used the inequality

$$\|\Delta_q^2 h\|_{L_x^2} \lesssim \|\Delta_q h\|_{L_x^2},$$

which is verified by direct calculation. Hence, we have so far

$$\begin{aligned} &\sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma^1(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2} \\ &\lesssim \sum_{q \geq -1} 2^{qs} \left(\int_0^T \sum_{|j-q| \leq 4} \|f\|_{L_v^2 L_x^\infty} \|\Delta_j g\|_{L_x^2} \|\Delta_q h\|_{L_x^2} dt \right)^{1/2}. \end{aligned} \quad (3.3.4)$$

Starting from (3.3.4), we shall estimate $\sum 2^{qs} \left(\int_0^T |(\Delta_q \Gamma^1(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2}$ in two different ways. Since appearing terms will be lengthy, we set

$$X_j = \left(\int_0^T \|\Delta_j g\|_{L_x^2}^2 dt \right)^{1/2}, \quad Y_q = \left(\int_0^T \|\Delta_q h\|_{L_x^2}^2 dt \right)^{1/2}.$$

Using the Cauchy-Schwarz inequality, embedding $B_x^{3/2} \hookrightarrow L_x^\infty$ and Lemma

3.7.8, we have

$$\begin{aligned}
& \sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma^1(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2} \\
& \lesssim \sum_{q \geq -1} 2^{qs} \|f\|_{L_T^\infty L_v^2 L_x^\infty}^{1/2} \left(\sum_{|j-q| \leq 4} (X_j)^2 \right)^{1/4} (Y_q)^{1/2} \\
& \lesssim \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \left(\sum_{q \geq -1} 2^{qs} \left(\sum_{|j-q| \leq 4} X_j \right)^{1/2} \right)^{1/2} \left(\sum_{q \geq -1} 2^{qs} Y_q \right)^{1/2} \\
& \leq \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \left(\sum_{q \geq -1} 2^{qs} \sum_{|j-q| \leq 4} 2^{-js} c_j \right)^{1/2} \|g\|_{\mathcal{T}_{T,2,2}^s}^{1/2}. \quad (3.3.5)
\end{aligned}$$

Here, we defined a ℓ^1 -sequence $\{c_j\}$ by $c_j := 2^{js} X_j / \|g\|_{\mathcal{T}_{T,2,2}^s}$. We will use abuse of notation c_j in the sequel to express similar sequences, defined by other appropriate $\mathcal{T}_{T,pr}^s$ -norms in each inequality. Note that all of them are in ℓ^1 . Now, the double summation is finite because by Fubini's theorem and Young's inequality, we have

$$\begin{aligned}
\sum_{q \geq -1} 2^{qs} \sum_{|j-q| \leq 4} 2^{-js} c_j &= \sum_{q \geq -1} [(\mathbf{1}_{|j| \leq 4} 2^{js}) * c_j](q) \\
&\leq \sum_j \mathbf{1}_{|j| \leq 4} 2^{js} \cdot \sum_j c_j < \infty.
\end{aligned}$$

Therefore, we have

$$\sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma^1(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2} \lesssim \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2}.$$

The Γ^1 part of the second estimate (3.3.2) is as follows. In this case, we apply embedding $\dot{B}_x^{3/2} \hookrightarrow L_x^\infty$ to (3.3.4).

$$\begin{aligned}
& \sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma^1(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2} \\
& \lesssim \|f\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}^{1/2} \sum_{q \geq -1} 2^{qs} \left(\sum_{|j-q| \leq 4} \|\Delta_j g\|_{L_T^\infty L_x^2} \right)^{1/2} \left(\int_0^T \|\Delta_q h\|_{L_x^2}^2 dt \right)^{1/4} \\
& \leq \|f\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,\infty,2}^s}^{1/2} \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \left(\sum_{q \geq -1} 2^{qs} \sum_{|j-q| \leq 4} 2^{-js} c_j \right)^{1/2} \\
& \lesssim \|f\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,\infty,2}^s}^{1/2} \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2}.
\end{aligned}$$

Next, we estimate $\sum_q 2^{qs} \left(\int_0^T |(\Delta_q \Gamma^2(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2}$. Since Γ^1 and Γ^2 have symmetry, we can similarly calculate it and obtain

$$\begin{aligned}
& \sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma^2(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2} \\
& \lesssim \sum_{q \geq -1} 2^{qs} \left(\sum_{|j-q| \leq 4} \int_0^T \|S_{j-1} g\|_{L_x^\infty} \|\Delta_j f\|_{L_{x,v}^2} \| \Delta_q h \|_{L_x^2} dt \right)^{1/2} \\
& \leq \sum_{q \geq -1} 2^{qs} \left(\sum_{|j-q| \leq 4} \|\Delta_j f\|_{L_T^\infty L_{x,v}^2} \|S_{j-1} g\|_{L_T^2 L_x^\infty} \| \Delta_q h \|_{L_T^2 L_x^2} \right)^{1/2} \\
& \lesssim \sum_{q \geq -1} 2^{qs} \left(\sum_{|j-q| \leq 4} \|\Delta_j f\|_{L_T^\infty L_{x,v}^2} \|g\|_{\dot{T}_{T,2,2}^{3/2}} \right)^{1/2} \| \Delta_q h \|_{L_T^2 L_x^2}^{1/2} \\
& \leq \|g\|_{\dot{T}_{T,2,2}^{3/2}}^{1/2} \left(\sum_{q \geq -1} 2^{qs} \sum_{|j-q| \leq 4} \|\Delta_j f\|_{L_T^\infty L_{x,v}^2} \right)^{1/2} \left(\sum_{q \geq -1} 2^{qs} \| \Delta_q h \|_{L_T^2 L_x^2} \right)^{1/2} \\
& \lesssim \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^s)}^{1/2} \|g\|_{\dot{T}_{T,2,2}^{3/2}}^{1/2} \|h\|_{\dot{T}_{T,2,2}^{1/2}}^{1/2},
\end{aligned}$$

where we used Lemma 3.7.9. Since $\|g\|_{\dot{T}_{T,2,2}^{3/2}} \lesssim \|g\|_{\mathcal{T}_{T,2,2}^{3/2}}$ and $B_{pr}^{s_1} \hookrightarrow B_{pr}^{s_2}$ when $s_2 \leq s_1$, we can deduce that

$$\sum_q 2^{qs} \left(\int_0^T |(\Delta_q \Gamma^2(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2} \lesssim \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,2,2}^{3/2}}^{1/2} \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2}.$$

Therefore the estimates of the second part Γ^2 for (3.3.2) and (3.3.1) are completed.

Finally, we calculate the third part Γ^3 . Recalling the size of supports of $\mathcal{F}[\Delta_{j'} f]$ and $\mathcal{F}[\Delta_j g]$, first we have

$$\Delta_q \left(\sum_j \sum_{|j-j'| \leq 1} (\Delta_{j'} f, \Delta_j g) \right) = \Delta_q \left(\sum_{\max\{j,j'\} \geq q-2} \sum_{|j-j'| \leq 1} (\Delta_{j'} f, \Delta_j g) \right).$$

We notice that the summation concerning j is not finite at this time. However, double summation appearing later can be shown finite. Similar to Γ^1 , an

estimate of Γ^3 with respect to x and v is

$$\begin{aligned} & \sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma^3(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2} \\ & \lesssim \sum_{q \geq -1} 2^{qs} \left(\sum_{j \geq q-3} \int_0^T \|f\|_{L_v^2 L_x^\infty} \|\Delta_j g\|_{L_x^2} \|\Delta_q h\|_{L_x^2} dt \right)^{1/2}. \end{aligned} \quad (3.3.6)$$

Using the Cauchy-Schwarz inequality with respect to t -integration, we have

$$\begin{aligned} & \sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma^3(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2} \\ & \lesssim \sum_{q \geq -1} 2^{qs} \|f\|_{L_T^\infty L_v^2(B_x^{3/2})}^{1/2} \left(\sum_{j \geq q-3} \|\Delta_j g\|_{L_{T,x}^2} \right)^{1/2} \left(\|\Delta_q h\|_{L_{T,x}^2} \right)^{1/2} \\ & \leq \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \left(\sum_{q \geq -1} \sum_{j \geq q-3} 2^{qs} \|\Delta_j g\|_{L_{T,x}^2} \right)^{1/2} \left(\sum_{q \geq -1} 2^{qs} \|\Delta_q h\|_{L_{T,x}^2} \right)^{1/2} \\ & \lesssim \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \left(\sum_{q \geq -1} 2^{qs} \sum_{j \geq q-3} 2^{-js} c_j \right)^{1/2}. \end{aligned}$$

The last factor of the previous line is finite because

$$\sum_{q \geq -1} 2^{qs} \sum_{j \geq q-3} 2^{-js} c_j = \sum_{j \geq -4} c_j \sum_{j-q \geq -3} 2^{-(j-q)s} < \infty.$$

Here we applied Fubini's theorem and Young's inequality again. This is an estimate corresponding to (3.3.1). Moreover, we calculate that

$$\begin{aligned} & \sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma^3(f, g), \Delta_q h)_{x,v}| dt \right)^{1/2} \\ & \lesssim \sum_{q \geq -1} 2^{qs} \|f\|_{L_T^2 L_v^2 L_x^\infty}^{1/2} \left(\sum_{j \geq q-3} \|\Delta_j g\|_{L_T^\infty L_x^2} \right)^{1/2} \|\Delta_q h\|_{L_{T,x}^2}^{1/2} \\ & \lesssim \|f\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}^{1/2} \left(\sum_{q \geq -1} \sum_{j \geq q-3} 2^{qs} \|\Delta_j g\|_{L_T^\infty L_x^2} \right)^{1/2} \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \\ & \lesssim \|f\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,\infty,2}^s}^{1/2} \|h\|_{\mathcal{T}_{T,2,2}^s}^{1/2}. \end{aligned}$$

Combining the above estimates properly, we obtain the second estimate (3.3.2). For the third estimate (3.3.3), we apply Corollary 3.7.3 instead of Corollary

3.7.2. Then the completely same argument of (3.3.2) yields (3.3.3). Notice that all we have to do is to replace f with $\mu^{1/10}f$ in the first term on the right hand side of (3.3.3), and to replace $\mathcal{T}_{T,\infty,2}^s$ and $\dot{\mathcal{T}}_{T,2,2}^{3/2}$ with $\tilde{L}_T^\infty \tilde{L}_v^2(B_x^s)$ and $L_T^2 L_v^2(\dot{B}_x^{3/2})$ respectively in the second term. $\square$

3.4 Estimate on nonlinear term and a priori estimate

Inserting certain functions into the inequalities of Lemma 3.3.1, we first estimate some non-linear terms by the energy and the dissipation term. After that, we will derive an a priori estimate. In this section, we decompose f into $\mu + \mu^{1/2}g$, and all Lemmas are statements on this g .

Lemma 3.4.1. *Assume $0 < s \leq \frac{3}{2}$. Then we have*

$$\sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma(g, g), \Delta_q(\mathbf{I} - \mathbf{P})g)| dt \right)^{1/2} \lesssim \sqrt{\mathcal{E}_T(g)} \mathcal{D}_T(g).$$

Proof. We divide $\Gamma(g, g)$ into $\Gamma(g, \mathbf{P}g)$ and $\Gamma(g, (\mathbf{I} - \mathbf{P})g)$, and estimate the respective terms. Using (3.3.1) of Lemma 3.3.1, we obtain

$$\begin{aligned} \sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma(g, (\mathbf{I} - \mathbf{P})g), \Delta_q(\mathbf{I} - \mathbf{P})g)| dt \right)^{1/2} \\ \lesssim \|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^s} \lesssim \sqrt{\mathcal{E}_T(g)} \mathcal{D}_T(g). \end{aligned}$$

When $\gamma + \nu > 0$, (3.3.2) of Lemma 3.3.1 yields

$$\begin{aligned} \sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma(g, \mathbf{P}g), \Delta_q(\mathbf{I} - \mathbf{P})g)| dt \right)^{1/2} \\ \lesssim \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \left(\|g\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}^{1/2} \|\mathbf{P}g\|_{\mathcal{T}_{T,\infty,2}^s}^{1/2} + \|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^s)}^{1/2} \|\mathbf{P}g\|_{\dot{\mathcal{T}}_{T,2,2}^{3/2}}^{1/2} \right) \\ \lesssim \sqrt{\mathcal{E}_T(g)} \mathcal{D}_T(g), \end{aligned}$$

where we used the following inequalities:

$$\begin{aligned} \|g\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})} &\lesssim \|g\|_{\dot{\mathcal{T}}_{T,2,2}^{3/2}} \lesssim \|\mathbf{P}g\|_{\dot{\mathcal{T}}_{T,2,2}^{3/2}} + \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^{3/2}} \sim \mathcal{D}_T(g), \\ \|\mathbf{P}g\|_{\mathcal{T}_{T,\infty,2}^s} &\lesssim \|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^s)}. \end{aligned}$$

The first inequality is deduced by Lemmas 3.7.7-3.7.8, and the fact that $\|\cdot\|_{L_v^2} \leq \|\cdot\|_{L_{v,(\gamma+\nu)/2}^2} \lesssim \|\cdot\|$ when $\gamma + \nu > 0$. We recall $\|\mathbf{P}f\| \lesssim \|f\|_{L_v^2}$ and Lemma 3.7.8 for the second one.

When $\gamma + \nu \leq 0$, (3.3.3) of Lemma 3.3.1 similarly yields

$$\begin{aligned}
& \sum_{q \geq -1} 2^{qs} \left(\int_0^T |(\Delta_q \Gamma(g, \mathbf{P}g), \Delta_q (\mathbf{I} - \mathbf{P})g)| dt \right)^{1/2} \\
& \lesssim \|\mu^{1/10} g\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}^{1/2} \|\mathbf{P}g\|_{\mathcal{T}_{T,\infty,2}^s}^{1/2} \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \\
& + \|\mu^{1/10} g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^s)}^{1/2} \|\mathbf{P}g\|_{\mathcal{T}_{T,2,2}^{3/2}}^{1/2} \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \\
& + \|g\|_{L_T^2 L_{v,(\nu+\gamma)/2}^2(\dot{B}_x^{3/2})}^{1/2} \|\mathbf{P}g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^s)}^{1/2} \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \\
& + \|g\|_{\tilde{L}_T^\infty \tilde{L}_{v,(\nu+\gamma)/2}^2(B_x^s)}^{1/2} \|\mathbf{P}g\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}^{1/2} \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^s}^{1/2} \\
& \lesssim \sqrt{\mathcal{E}_T(g)} \mathcal{D}_T(g).
\end{aligned}$$

Owing to the fact $\|\mathbf{P}g\| \lesssim \|g\|_{L_v^2}$, $\|\mathbf{P}g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^s)}$ and $\|\mathbf{P}g\|_{\mathcal{T}_{T,\infty,2}^s}$ are similarly estimated. So are $\|\mathbf{P}g\|_{\mathcal{T}_{T,2,2}^{3/2}}$ and $\|\mathbf{P}g\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})}$. $\square$

Lemma 3.4.2. *Let $\phi \in \mathcal{S}(\mathbb{R}_v^3)$ and $0 < s \leq \frac{3}{2}$. Then we have*

$$\sum_{q \geq -1} 2^{qs} \left(\int_0^T \|\Delta_q(\Gamma(g, g), \phi)\|_{L_x^2}^2 dt \right)^{1/2} \lesssim \mathcal{E}_T(g) \mathcal{D}_T(g).$$

Proof. First, we consider the case $\gamma + \nu > 0$. Together with the decomposition $\Gamma(g, g) = \Gamma^1(g, g) + \Gamma^2(g, g) + \Gamma^3(g, g)$, the generalized Minkowski inequality gives

$$\begin{aligned}
& \left(\int_0^T \|\Delta_q(\Gamma(g, g), \phi)\|_{L_x^2}^2 dt \right)^{1/2} \leq \sum_{i=1}^3 \left(\int_0^T \|\Delta_q(\Gamma^i(g, g), \phi)\|_{L_x^2}^2 dt \right)^{1/2} \\
& \leq \sum_{|j-q| \leq 4} \left(\int_0^T \|\Delta_q(\Gamma(S_{j-1}g, \Delta_j g), \phi)\|_{L_x^2}^2 dt \right)^{1/2} \\
& + \sum_{|j-q| \leq 4} \left(\int_0^T \|\Delta_q(\Gamma(\Delta_j g, S_{j-1}g), \phi)\|_{L_x^2}^2 dt \right)^{1/2} \\
& + \sum_{\max\{j, j'\} \geq q-2} \sum_{|j-j'| \leq 1} \left(\int_0^T \|\Delta_q(\Gamma(\Delta_{j'}g, \Delta_j g), \phi)\|_{L_x^2}^2 dt \right)^{1/2} \\
& =: I_q^1 + I_q^2 + I_q^3.
\end{aligned}$$

By using Corollary 3.7.2 and Lemma 3.7.5, each term is calculated as follows: We use the macro-micro decomposition $g = \mathbf{P}g + (\mathbf{I} - \mathbf{P})g$ to estimate the first

and third term properly. Then we have

$$\begin{aligned}
\sum_{q \geq -1} 2^{qs} I_q^1 &\leq \sum_{q \geq -1} 2^{qs} \sum_{|j-q| \leq 4} \left(\int_0^T \int \|S_{j-1}g\|_{\tilde{L}_v^2}^2 \|\Delta_j \mathbf{P}g\|^2 dx dt \right)^{1/2} \\
&+ \sum_{q \geq -1} 2^{qs} \sum_{|j-q| \leq 4} \left(\int_0^T \int \|S_{j-1}g\|_{\tilde{L}_v^2}^2 \|\Delta_j (\mathbf{I} - \mathbf{P})g\|^2 dx dt \right)^{1/2} \\
&\leq \sum_{q \geq -1} 2^{qs} \sum_{|j-q| \leq 4} \left(\int_0^T \|g\|_{\tilde{L}_v^2 L_x^\infty}^2 dt \right)^{1/2} \|\Delta_j \mathbf{P}g\|_{L_T^\infty L_v^2 L_x^2} \\
&+ \sum_{q \geq -1} 2^{qs} \sum_{|j-q| \leq 4} \|g\|_{L_T^\infty L_v^2 L_x^\infty} \left(\int_0^T \|\Delta_j (\mathbf{I} - \mathbf{P})g\|_{L_x^2}^2 dt \right)^{1/2} \\
&\lesssim \|g\|_{L_T^2 L_v^2 (\dot{B}_x^{3/2})} \|\mathbf{P}g\|_{\tilde{L}_T^\infty \tilde{L}_v^2 (B_x^s)} + \|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2 (B_x^{3/2})} \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^s} \\
&\lesssim \mathcal{E}_T(g) \mathcal{D}_T(g),
\end{aligned}$$

$$\begin{aligned}
\sum_{q \geq -1} 2^{qs} I_q^2 &\leq \sum_{q \geq -1} 2^{qs} \sum_{|j-q| \leq 4} \|\Delta_j g\|_{L_T^\infty L_{x,v}^2} \left(\int_0^T \|S_{j-1}g\|_{L_x^2}^2 dt \right)^{1/2} \\
&\lesssim \|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2 (B_x^s)} \|g\|_{\dot{\mathcal{T}}_{T,2,2}^{3/2}} \lesssim \mathcal{E}_T(g) \mathcal{D}_T(g),
\end{aligned}$$

$$\begin{aligned}
\sum_{q \geq -1} 2^{qs} I_q^3 &\leq \sum_{q \geq -1} \sum_{j \geq q-3} 2^{qs} \|\Delta_j g\|_{L_T^\infty L_{x,v}^2} \left(\int_0^T \|\mathbf{P}g\|_{L_x^\infty}^2 dt \right)^{1/2} \\
&+ \sum_{q \geq -1} \sum_{j' \geq q-3} 2^{qs} \|g\|_{L_T^\infty L_v^2 L_x^\infty} \left(\int_0^T \|\Delta_{j'} (\mathbf{I} - \mathbf{P})g\|_{L_x^2}^2 dt \right)^{1/2} \\
&\lesssim \|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2 (B_x^s)} \|\mathbf{P}g\|_{\dot{\mathcal{T}}_{T,2,2}^{3/2}} + \|g\|_{L_T^\infty L_v^2 (B_x^{3/2})} \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^s} \\
&\lesssim \mathcal{E}_T(g) \mathcal{D}_T(g).
\end{aligned}$$

When $\gamma + \nu \leq 0$, apply Corollary 3.7.3 instead of Corollary 3.7.2 for the

estimate of I_q^1 . Then we have

$$\begin{aligned}
\sum_{q \geq -1} 2^{qs} I_q^1 &\leq \sum_{q \geq -1} 2^{qs} \sum_{|j-q| \leq 4} \left(\int_0^T \int \|\langle v \rangle^{(\gamma+\nu)/2} S_{j-1} g\|_{L_v^2}^2 \|\Delta_j \mathbf{P} g\|^2 dx dt \right)^{1/2} \\
&\quad + \sum_{q \geq -1} 2^{qs} \sum_{|j-q| \leq 4} \left(\int_0^T \int \|S_{j-1} g\|_{L_v^2}^2 \|\Delta_j (\mathbf{I} - \mathbf{P}) g\|^2 dx dt \right)^{1/2} \\
&\lesssim \|\langle v \rangle^{(\gamma+\nu)/2} g\|_{L_T^2 L_v^2(\dot{B}_x^{3/2})} \|\mathbf{P} g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^s)} \\
&\quad + \|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})} \|(\mathbf{I} - \mathbf{P}) g\|_{\tau_{T,2,2}^s} \\
&\lesssim \mathcal{E}_T(g) \mathcal{D}_T(g).
\end{aligned}$$

Other parts follow in the same manner. $\square$

As for the upper bound of the linear term $\mathcal{L}$ we have

Lemma 3.4.3. *Let $\phi \in \mathcal{S}(\mathbb{R}_v^3)$ and $s > 0$. Then we have*

$$\sum_{q \geq -1} 2^{qs} \left(\int_0^T \|\Delta_q(\mathcal{L}(\mathbf{I} - \mathbf{P})g, \phi)\|_{L_x^2}^2 dt \right)^{1/2} \lesssim \|(\mathbf{I} - \mathbf{P})g\|_{\tau_{T,2,2}^s}.$$

Proof. Since it follows from Corollaries 3.7.2 and 3.7.3 that

$$\begin{aligned}
&|\Delta_q(\mathcal{L}(\mathbf{I} - \mathbf{P})g, \phi)_{L_v^2}| \\
&\leq \left| (\Gamma(\mu^{1/2}, \Delta_q(\mathbf{I} - \mathbf{P})g), \phi)_{L_v^2} \right| + \left| (\Gamma(\Delta_q(\mathbf{I} - \mathbf{P})g, \mu^{1/2}), \phi)_{L_v^2} \right| \\
&\lesssim \|\Delta_q(\mathbf{I} - \mathbf{P})g\| + \|\Delta_q(\mathbf{I} - \mathbf{P})g\|_{L_{(\gamma+\nu)/2}^2(\mathbb{R}_v^3)},
\end{aligned}$$

we obtain the desired estimate. $\square$

We next estimate derivative of the macroscopic part.

Lemma 3.4.4. *It holds that*

$$\begin{aligned}
&\|\nabla_x(a, b, c)\|_{\tilde{L}_T^2(B_x^{1/2})} \\
&\lesssim \|g_0\|_{\tilde{L}_v^2(B_x^{3/2})} + \mathcal{E}_T(g) + \|(\mathbf{I} - \mathbf{P})g\|_{\tau_{T,2,2}^{3/2}} + \mathcal{E}_T(g) \mathcal{D}_T(g). \quad (3.4.1)
\end{aligned}$$

Proof. We start from (3.1.1). Multiplying the equation by 1, v , and $|v|^2$ and taking v -integration, we have the following local macroscopic balance laws:

$$\begin{aligned}
\partial_t \int_{\mathbb{R}_v^3} f dv + \nabla_x \cdot \int_{\mathbb{R}_v^3} v f dv &= 0, \\
\partial_t \int_{\mathbb{R}_v^3} v f dv + \nabla_x \cdot \int_{\mathbb{R}_v^3} v \otimes v f dv &= 0, \\
\partial_t \int_{\mathbb{R}_v^3} |v|^2 f dv + \nabla_x \cdot \int_{\mathbb{R}_v^3} |v|^2 v f dv &= 0.
\end{aligned}$$

We decompose f into $f = \mu + \mu^{1/2}g$ and further decompose g into $g = \mathbf{P}g + (\mathbf{I} - \mathbf{P})g = g_1 + g_2$. In order to express the above balance laws with the macroscopic functions (a, b, c) , we need to calculate some moments of Maxwellian μ .

$$\begin{aligned}(1, \mu) &= 1, (|v_i|^2, \mu) = 1, (|v|^2, \mu) = 3, \\ (|v_i|^2 |v_j|^2, \mu) &= 1 \ (i \neq j), (|v_i|^4, \mu) = 3, \\ (|v|^2 |v_i|^2, \mu) &= 5, (|v|^4, \mu) = 15, (|v|^4 |v_i|^2 \mu) = 35.\end{aligned}$$

Keeping these values in mind, we compute that

$$\begin{aligned}\int_{\mathbb{R}_v^3} f dv &= 1 + (a + 3c), \\ \int_{\mathbb{R}_v^3} v f dv &= b, \\ \int_{\mathbb{R}_v^3} |v|^2 f dv &= 3 + 3(a + 5c), \\ \int_{\mathbb{R}_v^3} v_i v_j f dv &= \delta_{ij} + (a + 5c)\delta_{ij} + (v_i v_j \mu^{1/2}, g_2), \\ \int_{\mathbb{R}_v^3} |v|^2 v f dv &= 5b + (|v|^2 v \mu^{1/2}, g_2).\end{aligned}$$

Here, δ_{ij} is Kronecker's delta. Inserting these identities into the balance laws, we have

$$\begin{aligned}\partial_t(a + 3c) + \nabla_x \cdot b &= 0, \\ \partial_t b + \nabla_x(a + 5c) + \nabla_x \cdot (v \otimes v \mu^{1/2}, g_2) &= 0, \\ 3\partial_t(a + 5c) + 5\nabla_x \cdot b + \nabla_x \cdot (|v|^2 v \mu^{1/2}, g_2) &= 0.\end{aligned}$$

This is equivalent to the system

$$\begin{aligned}\partial_t a - \frac{1}{2} \nabla_x \cdot (|v|^2 v \mu^{1/2}, g_2) &= 0, \\ \partial_t b + \nabla_x(a + 5c) + \nabla_x \cdot (v \otimes v \mu^{1/2}, g_2) &= 0, \\ \partial_t c + \frac{1}{3} \nabla_x \cdot b + \frac{1}{6} \nabla_x \cdot (|v|^2 v \mu^{1/2}, g_2) &= 0.\end{aligned}$$

Next, we rewrite (3.1.2) as

$$\partial_t g_1 + v \cdot \nabla_x g_1 = -\partial_t g_2 + R_1 + R_2,$$

where $R_1 = -v \cdot \nabla_x g_2$ and $R_2 = -\mathcal{L}g_2 + \Gamma(g, g)$. We rewrite this equation once again so that we express the equation in terms of (a, b, c) .

$$\begin{aligned}\partial_t a \mu^{1/2} + (\partial_t b + \nabla_x a) \cdot v \mu^{1/2} + \sum_i (\partial_t c + \partial_i b_i) |v_i|^2 \mu^{1/2} \\ + \sum_{i < j} (\partial_j b_i + \partial_i b_j) v_i v_j \mu^{1/2} + \nabla_x c \cdot |v|^2 v \mu^{1/2} = -\partial_t g_2 + R_1 + R_2.\end{aligned}\quad (3.4.2)$$

Here, $\partial_i = \partial_{x_i}$. We define the high-order moment functions $A(g) = (A_{ij}(g))_{3 \times 3}$ and $B(g) = (B_i(g))_{1 \times 3}$ by

$$A_{ij}(g) = ((v_i v_j - \delta_{ij})\mu^{1/2}, g), B_i(g) = \frac{1}{10}((|v|^2 - 5)v_i \mu^{1/2}, g).$$

Notice that these functions operate only on v and

$$|A_{ij}(g)|, |B_i(g)| \leq C \|g\|_{L^2_{v, (\gamma+\nu)/2}}$$

since we take inner products of g and rapidly decreasing functions. Applying A_{ij} and B_i to both sides of (3.4.2), we have

$$\begin{aligned} \partial_t(A_{ij}(g_2) + 2c\delta_{ij}) + \partial_j b_i + \partial_i b_j &= A_{ij}(R_1 + R_2), \\ \partial_t B_i(g_2) + \partial_i c &= B_i(R_1 + R_2). \end{aligned}$$

Thus, we obtained the following system:

$$\begin{cases} \partial_t a - \frac{1}{2} \nabla_x \cdot (|v|^2 v \mu^{1/2}, g_2) = 0, \\ \partial_t b + \nabla_x(a + 5c) + \nabla_x \cdot (v \otimes v \mu^{1/2}, g_2) = 0, \\ \partial_t c + \frac{1}{3} \nabla_x \cdot b + \frac{1}{6} \nabla_x \cdot (|v|^2 v \mu^{1/2}, g_2) = 0, \\ \partial_t(A_{ij}(g_2) + 2c\delta_{ij}) + \partial_j b_i + \partial_i b_j = A_{ij}(R_1 + R_2), \\ \partial_t B_i(g_2) + \partial_i c = B_i(R_1 + R_2). \end{cases} \quad (3.4.3)$$

We derive the desired estimate from this system. For later use, we define the energy functional

$$\begin{aligned} E_q(g) &= E_q^1(g) + \delta_2 E_q^2(g) + \delta_3 E_q^3(g), \\ E_q^1(g) &= \sum_i (B_i(\Delta_q g_2), \partial_i \Delta_q c), \\ E_q^2(g) &= \sum_{i,j} (A_{ij}(\Delta_q g_2) + 2\Delta_q c \delta_{ij}, \partial_j \Delta_q b_i + \partial_i \Delta_q b_j), \\ E_q^3(g) &= (\Delta_q b, \nabla_x \Delta_q a). \end{aligned}$$

δ_2 and δ_3 are two small positive numbers, which will be chosen later. First, we apply Δ_q with $q \geq -1$ to (3.4.3). We have

$$\begin{cases} \partial_t \Delta_q a - \frac{1}{2} \nabla_x \cdot (|v|^2 v \mu^{1/2}, \Delta_q g_2) = 0, \\ \partial_t \Delta_q b + \nabla_x \Delta_q(a + 5c) + \nabla_x \cdot (v \otimes v \mu^{1/2}, \Delta_q g_2) = 0, \\ \partial_t \Delta_q c + \frac{1}{3} \nabla_x \cdot \Delta_q b + \frac{1}{6} \nabla_x \cdot (|v|^2 v \mu^{1/2}, \Delta_q g_2) = 0, \\ \partial_t(A_{ij}(\Delta_q g_2) + 2\Delta_q c \delta_{ij}) + \partial_j \Delta_q b_i + \partial_i \Delta_q b_j = A_{ij}(\Delta_q R_1 + \Delta_q R_2), \\ \partial_t B_i(\Delta_q g_2) + \partial_i \Delta_q c = B_i(\Delta_q R_1 + \Delta_q R_2). \end{cases} \quad (3.4.4)$$

In what follows, by $(3.4.4)_n$ we denote the n -th equation of (3.4.4). Multiplying $(3.4.4)_5$ by $\partial_i \Delta_q c$, we have

$$\begin{aligned} \frac{d}{dt} (B_i(\Delta_q g_2), \partial_i \Delta_q c) - (B_i(\Delta_q g_2), \partial_i \Delta_q \partial_t c) + \|\partial_i \Delta_q c\|_{L_x^2}^2 \\ = (B_i(\Delta_q R_1 + \Delta_q R_2), \partial_i \Delta_q c). \end{aligned}$$

By Young's inequality it is obvious that

$$(B_i(\Delta_q R_1), \partial_i \Delta_q c) \leq \varepsilon \|\partial_i \Delta_q c\|_{L_x^2}^2 + \frac{C}{\varepsilon} \|\nabla_x \Delta_q g_2\|_{L_{v,(\gamma+\nu)/2}^2 L_x^2}^2$$

for small positive ε . We note that if $\|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})} < \infty$, the Cauchy-Schwarz inequality gives $(g(x, \cdot, t), \phi) \in \tilde{L}_T^\infty(B_x^{3/2})$ for any $\phi \in \mathcal{S}(\mathbb{R}_v^3)$. Also, that $\|g\|_{\mathcal{T}_{T,2,2}^{3/2}} < \infty$ implies $\sum_q 2^{3/2} \left(\int_0^T \int |(\Delta_q g(x, \cdot, t), \phi)|^2 dx \right)^{1/2} < \infty$ for any $\phi \in \mathcal{S}(\mathbb{R}_v^3)$. In order to estimate $(B_i(\Delta_q g_2), \partial_i \Delta_q \partial_t c)$, we use $(3.4.4)_3$ and partial integral. As long as $f, g \in B_{2,1}^{3/2}$, $(\partial_i \Delta_q f, \Delta_q g) = -(\Delta_q f, \partial_i \Delta_q g)$. Indeed, if $f \in B_{2,1}^{3/2}$, $\Delta_q f$ is also in $B_{2,1}^{3/2} \subset H^1$. Moreover, that $f \in L^\infty$ implies $\Delta_q f \in C_b^\infty$. So both $(\partial_i \Delta_q f, \Delta_q g)$ and $-(\Delta_q f, \partial_i \Delta_q g)$ exist and the same.

$$\begin{aligned} |(B_i(\Delta_q g_2), \partial_i \Delta_q \partial_t c)| &= \left| \left(\partial_i B_i(\Delta_q g_2), \frac{1}{3} \nabla_x \cdot \Delta_q b + \frac{1}{6} \nabla_x \cdot (|v|^2 v \mu^{1/2}, \Delta_q g_2) \right) \right| \\ &\leq \varepsilon \|\nabla_x \Delta_q b\|_{L_x^2}^2 + \frac{C}{\varepsilon} \|\nabla_x \Delta_q g_2\|_{L_{v,(\gamma+\nu)/2}^2 L_x^2}^2. \end{aligned}$$

Here we set $\|\nabla_x \Delta_q b\|_{L^2 x} := \|\nabla_x \otimes \Delta_q b\|_{L^2 x}$ for brevity. From this inequality, we have for small $\lambda' > 0$ and $\varepsilon_1 > 0$,

$$\begin{aligned} \frac{d}{dt} E_q^1(g) + \lambda' \|\nabla_x \Delta_q c\|_{L_x^2}^2 &\leq \varepsilon_1 \|\nabla_x \Delta_q b\|_{L_x^2}^2 \\ &+ \frac{C}{\varepsilon_1} \|\nabla_x \Delta_q g_2\|_{L_{v,(\gamma+\nu)/2}^2 L_x^2}^2 + \frac{C}{\varepsilon_1} \sum_i \|B_i(\Delta_q R_2)\|_{L_x^2}^2. \end{aligned} \quad (3.4.5)$$

Next, we multiply $(3.4.4)_4$ by $\partial_j \Delta_q b_i + \partial_i \Delta_q b_j$ and take summation with i and j . We remark that

$$\begin{aligned} \|\partial_j \Delta_q b_i + \partial_i \Delta_q b_j\|_{L_x^2}^2 &= \|\partial_j \Delta_q b_i\|_{L_x^2}^2 + \|\partial_i \Delta_q b_j\|_{L_x^2}^2 + 2(\partial_j \Delta_q b_i, \partial_i \Delta_q b_j) \\ &= \|\partial_j \Delta_q b_i\|_{L_x^2}^2 + \|\partial_i \Delta_q b_j\|_{L_x^2}^2 + 2(\partial_i \Delta_q b_i, \partial_j \Delta_q b_j). \end{aligned}$$

Thus,

$$\sum_{i,j} \|\partial_j \Delta_q b_i + \partial_i \Delta_q b_j\|_{L_x^2}^2 = 2\|\nabla_x \Delta_q b\|_{L_x^2}^2 + 2\|\nabla_x \cdot \Delta_q b\|_{L_x^2}^2,$$

and this implies

$$\begin{aligned} \frac{d}{dt} E_q^2(g) - \sum_{i,j} (A_{ij}(\Delta_q g_2) + 2\Delta_q c \delta_{ij}, \partial_j \Delta_q \partial_t b_i + \partial_i \Delta_q \partial_t b_j) \\ + 2\|\nabla_x \Delta_q b\|_{L_x^2}^2 + 2\|\nabla_x \cdot \Delta_q b\|_{L_x^2}^2 = (A_{ij}(\Delta_q R_1 + \Delta_q R_2), \partial_j \Delta_q b_i + \partial_i \Delta_q b_j). \end{aligned}$$

Substituting (3.4.4)₂ to eliminate $\partial_t \Delta_q b$, we have

$$\begin{aligned}
& |(A_{ij}(\Delta_q g_2) + 2\Delta_q c \delta_{ij}, \partial_j \Delta_q \partial_t b_i)| \\
&= \left| \left(\partial_j A_{ij}(\Delta_q g_2) + 2\partial_j \Delta_q c \delta_{ij}, \partial_i \Delta_q (a + 5c) + \partial_i \cdot \left(\sum_l v_i v_l \mu^{1/2}, \Delta_q g_2 \right) \right) \right| \\
&\leq \varepsilon \|\nabla_x \Delta_q a\|_{L_x^2}^2 + \frac{C}{\varepsilon} \|\nabla_x \Delta_q c\|_{L_x^2}^2 + \frac{C}{\varepsilon} \|\nabla_x \Delta_q g_2\|_{L_{v,(\gamma+\nu)/2}^2 L_x^2}^2
\end{aligned}$$

and other terms on the left hand side are similarly estimated. Hence, for small $\varepsilon_2 > 0$ we have

$$\begin{aligned}
\frac{d}{dt} \sum_{i,j} E_q^2(g) + \lambda' \|\nabla_x \Delta_q b\|_{L_x^2}^2 &\leq \varepsilon_2 \|\nabla_x \Delta_q a\|_{L_x^2}^2 + \frac{C}{\varepsilon_2} \|\nabla_x \Delta_q c\|_{L_x^2}^2 \\
&+ \frac{C}{\varepsilon_2} \|\nabla_x \Delta_q g_2\|_{L_{v,(\gamma+\nu)/2}^2 L_x^2}^2 + \sum_{i,j} \|A_{ij}(\Delta_q R_2)\|_{L_x^2}^2. \quad (3.4.6)
\end{aligned}$$

Lastly, from (3.4.4)₂ we have

$$\begin{aligned}
\frac{d}{dt} E_q^3(g) - \sum_i (\Delta_q b_i, \partial_i \Delta_q \partial_t a) + \|\nabla_x \Delta_q a\|_{L_x^2}^2 &= -5 \sum_i (\partial_i \Delta_q c, \partial_i \Delta_q a) \\
&- \sum_{i,j} (\partial_j (v_i v_j \mu^{1/2}, \Delta_q g_2), \partial_i \Delta_q a).
\end{aligned}$$

Eliminating $\partial_t \Delta_q a$ by (3.4.4)₁, we have

$$\frac{d}{dt} E_q^3(g) + \lambda' \|\nabla_x \Delta_q a\|_{L_x^2}^2 \leq C \|\nabla_x \Delta_q (b, c)\|_{L_x^2}^2 + C \|\nabla_x \Delta_q g_2\|_{L_{v,(\gamma+\nu)/2}^2 L_x^2}^2. \quad (3.4.7)$$

For sufficiently small δ_2 and δ_3 with $0 < \delta_3 \ll \delta_2 \ll 1$, taking summation (3.4.5) + δ_2 (3.4.6) + δ_3 (3.4.7) and then choosing small ε_1 and ε_2 , we obtain for small $\lambda > 0$,

$$\begin{aligned}
\frac{d}{dt} E_q(g(t)) + \lambda \|\nabla_x \Delta_q (a, b, c)\|_{L_x^2}^2 &\lesssim \|\nabla_x \Delta_q g_2\|_{L_{v,(\gamma+\nu)/2}^2 L_x^2}^2 \\
&+ \sum_{i,j} \|A_{ij}(\Delta_q R_2)\|_{L_x^2}^2 + \sum_i \|B_i(\Delta_q R_2)\|_{L_x^2}^2. \quad (3.4.8)
\end{aligned}$$

Integrating (3.4.8) on $[0, T]$ and taking the square root of the resultant inequality, we have

$$\begin{aligned}
\|\nabla_x \Delta_q (a, b, c)\|_{L_T^2 L_x^2} &\lesssim \sqrt{|E_q(g(0))|} + \sqrt{|E_q(g(T))|} \\
&+ \|\nabla_x \Delta_q g_2\|_{L_T^2 L_{v,(\gamma+\nu)/2}^2 L_x^2} + \sum_{i,j} \|A_{ij}(\Delta_q R_2)\|_{L_T^2 L_x^2} \\
&+ \sum_i \|B_i(\Delta_q R_2)\|_{L_T^2 L_x^2}. \quad (3.4.9)
\end{aligned}$$

Multiplying this inequality by $2^{q/2}$ and taking summation over $q \geq -1$ yield

$$\begin{aligned}
\|\nabla_x(a, b, c)\|_{\tilde{L}_T^2(B_x^{1/2})} &\lesssim \sum_{q \geq -1} 2^{q/2} \sqrt{|E_q(g(0))|} + \sum_{q \geq -1} 2^{q/2} \sqrt{|E_q(g(T))|} \\
&\quad + \|g_2\|_{\tilde{L}_T^2 \tilde{L}_{v, (\gamma+\nu)/2}^2(B_x^{3/2})} + \sum_{i,j} \sum_{q \geq -1} 2^{q/2} \|A_{ij}(\Delta_q R_2)\|_{L_T^2 L_x^2} \\
&\quad + \sum_i \sum_{q \geq -1} 2^{q/2} \|B_i(\Delta_q R_2)\|_{L_T^2 L_x^2}. \tag{3.4.10}
\end{aligned}$$

Here we used Lemma 3.7.7 to estimate the sum corresponding to the third term of the right hand side of (3.4.9) by $\|g_2\|_{\tilde{L}_T^2 \tilde{L}_{v, (\gamma+\nu)/2}^2(B_x^{3/2})}$. This term is governed by $\|g_2\|_{\mathcal{T}_{T,2,2}^{3/2}}$ since $\|\cdot\|_{L_{v, (\gamma+\nu)/2}^2} \lesssim \|\cdot\|$. The other four terms on the right hand side are estimated as follows. The Cauchy-Schwarz inequality yields

$$\begin{aligned}
\sum_{q \geq -1} 2^{q/2} \sqrt{|E_q(g(t))|} &\lesssim \sum_{q \geq -1} 2^{q/2} [\|\nabla_x \Delta_q(a, b, c)(t)\|_{L_x^2} + \|\Delta_q(b, c)(t)\|_{L_x^2} \\
&\quad + \|\nabla_x \Delta_q g_2(t)\|_{L_v^2 L_x^2}].
\end{aligned}$$

Using Lemma 3.7.7 once again, we have

$$\sum_{q \geq -1} 2^{q/2} \sqrt{|E_q(g(0))|} \lesssim \|g_0\|_{\tilde{L}_v^2(B_x^{3/2})}, \quad \sum_{q \geq -1} 2^{q/2} \sqrt{|E_q(g(t))|} \lesssim \mathcal{E}_T(g).$$

Applying Lemmas 3.4.2 and 3.4.3 with $\phi(v) = (v_i v_j - \delta_{ij}) \mu^{1/2}(v)$ and $\phi(v) = \frac{1}{10}(|v|^2 - 5)v_i \mu^{1/2}(v)$ implies

$$\begin{aligned}
\sum_{i,j} \sum_{q \geq -1} 2^{q/2} \|A_{ij}(\Delta_q R_2)\|_{L_T^2 L_x^2} + \sum_i \sum_{q \geq -1} 2^{q/2} \|B_i(\Delta_q R_2)\|_{L_T^2 L_x^2} \\
\lesssim \|g_2\|_{\mathcal{T}_{T,2,2}^{3/2}} + \mathcal{E}_T(g) \mathcal{D}_T(g).
\end{aligned}$$

Therefore, substituting the above four inequalities into (3.4.10), we finally obtained the desired estimate. $\square$

At the end of this section, we derive an a priori estimate of the energy and the dissipation terms.

Lemma 3.4.5. *It holds that*

$$\mathcal{E}_T(g) + \mathcal{D}_T(g) \lesssim \|g_0\|_{\tilde{L}_v^2(B_x^{3/2})} + \left(\mathcal{E}_T(g) + \sqrt{\mathcal{E}_T(g)} \right) \mathcal{D}_T(g).$$

Proof. Applying Δ_q to (3.1.2) and taking the inner product with $\Delta_q g$ over $\mathbb{R}_x^3 \times \mathbb{R}_v^3$, we have

$$(\partial_t \Delta_q g, \Delta_q g) + (v \cdot \Delta_q \nabla_x g, \Delta_q g) + (\mathcal{L}(\Delta_q g), \Delta_q g) = (\Delta_q \Gamma(g, g), \Delta_q g).$$

Since $\Delta_q \nabla_x g = \nabla_x \Delta_q g$, we have $(v \cdot \Delta_q \nabla_x g, \Delta_q g)_{L_x^2} = 0$. Moreover,

$$\begin{aligned}
& (\Delta_q \Gamma(g, g), \Delta_q \mathbf{P}g)_{L_v^2} \\
&= \int_{\mathbb{R}_v^3 \times \mathbb{R}_{v_*}^3 \times \mathbb{S}^2} B \mu_*^{1/2} (\Delta_q(g'_* g') - \Delta_q(g_* g)) \Delta_q \mathbf{P}g dv dv_* d\sigma \\
&= \frac{1}{2} \int_{\mathbb{R}_v^3 \times \mathbb{R}_{v_*}^3 \times \mathbb{S}^2} B \Delta_q(g_* g) \Delta_q((\mathbf{P}g)' \mu_*'^{1/2} + (\mathbf{P}g)'_* \mu'^{1/2} \\
&\quad - \mathbf{P}g \mu_*^{1/2} - (\mathbf{P}g)_* \mu^{1/2}) dv dv_* d\sigma \\
&= 0.
\end{aligned}$$

Therefore, we have

$$\frac{1}{2} \frac{d}{dt} \|\Delta_q g\|_{L_v^2 L_x^2}^2 + 2\lambda_0 \|\Delta_q(\mathbf{I} - \mathbf{P})g\|_{L_x^2}^2 \leq |(\Delta_q \Gamma(g, g), \Delta_q(\mathbf{I} - \mathbf{P})g)|,$$

where λ_0 is taken in Lemma 3.7.4.

Integrate this inequality over $[0, t]$ with $0 \leq t \leq T$, take the square root of the resultant inequality and multiply it by $2^{3q/2}$. Then we have

$$\begin{aligned}
& 2^{3q/2} \|\Delta_q g(t)\|_{L_v^2 L_x^2} + \sqrt{\lambda_0} 2^{3q/2} \left(\int_0^t \|\Delta_q(\mathbf{I} - \mathbf{P})g(\tau)\|_{L_x^2}^2 d\tau \right)^{1/2} \\
& \leq 2^{3q/2} \|\Delta_q g_0\|_{L_v^2 L_x^2} + 2^{3q/2} \left(\int_0^t |(\Delta_q \Gamma(g, g), \Delta_q(\mathbf{I} - \mathbf{P})g)| d\tau \right)^{1/2}.
\end{aligned}$$

We take supremum over $0 \leq t \leq T$ on the left hand side and take the summation over $q \geq -1$. Together with Lemma 3.4.1, we have

$$\|g\|_{\tilde{L}_T^2 \tilde{L}_v^2(B_x^{3/2})} + \sqrt{\lambda_0} \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^{3/2}} \leq \|g_0\|_{\tilde{L}_v^2(B_x^{3/2})} + \sqrt{\mathcal{E}_T(g)} \mathcal{D}_T(g). \quad (3.4.11)$$

We now use Lemma 3.4.4. Taking summation $\delta(3.4.1) + (3.4.11)$ with small positive δ , we have

$$\begin{aligned}
& (1 - \delta) \mathcal{E}_t(g) + (\sqrt{\lambda_0} - \delta) \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{T}_{T,2,2}^{3/2}} + \delta \|\nabla_x(a, b, c)\|_{\tilde{L}_T^2(B_x^{1/2})} \\
& \lesssim \|g_0\|_{\tilde{L}_v^2(B_x^{3/2})} + (\sqrt{\mathcal{E}_T(g)} + \mathcal{E}_T(g)) \mathcal{D}_T(g).
\end{aligned}$$

The proof is complete by choosing sufficiently small δ . $\square$

Since (3.1.3) follows from Lemma 3.4.5, the existence of a unique global solution in Theorem 3.1.1 is a direct consequence of Theorem 3.5.2 in the next section, concerning the existence of a local solution. The non-negativity of solutions to the Cauchy problem (3.1.1) is sent to Section 3.6.

3.5 Local existence

3.5.1 Local existence of linear and non-linear equations

Lemma 3.5.1 (Local existence for a linear equation). *There exist some $C_0 > 1$, $\epsilon_0 > 0$, $T_0 > 0$ such that for all $0 < T \leq T_0$, $g_0 \in \tilde{L}_v^2(B_x^{3/2})$, $f \in \tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})$ satisfying*

$$\|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})} \leq \epsilon_0,$$

the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathcal{L}_1 g = \Gamma(f, g) - \mathcal{L}_2 f, \\ g|_{t=0} = g_0, \end{cases} \quad (3.5.1)$$

admits a weak solution $g \in L^\infty([0, T]; L^2(\mathbb{R}_{x,v}^6))$ satisfying

$$\|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})} + \|g\|_{\mathcal{T}_{T,2,2}^{3/2}} \leq C_0 \left(\|g_0\|_{\tilde{L}_v^2(B_x^{3/2})} + \sqrt{T} \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})} \right) \quad (3.5.2)$$

Proof. Consider

$$\mathcal{Q} = -\partial_t + (v \cdot \nabla_x + \mathcal{L}_1 - \Gamma(f, \cdot))^*,$$

where the adjoint operator $(\cdot)^*$ is taken with respect to the scalar product in $L^2(\mathbb{R}_{x,v}^6)$. Then, for all $h \in C^\infty([0, T], \mathcal{S}(\mathbb{R}_{x,v}^6))$, with $h(T) = 0$ and $0 \leq t \leq T$,

$$\begin{aligned} \operatorname{Re}(h(t), \mathcal{Q}h(t))_{x,v} &= -\frac{1}{2} \frac{d}{dt} (\|h\|_{L_{x,v}^2}^2) \\ &\quad + \operatorname{Re}(v \cdot \nabla_x h, h)_{x,v} + \operatorname{Re}(\mathcal{L}_1 h, h)_{x,v} - \operatorname{Re}(\Gamma(f, h), h)_{x,v} \\ &\geq -\frac{1}{2} \frac{d}{dt} (\|h(t)\|_{L_{x,v}^2}^2) + \frac{1}{C} \|\llbracket h(t) \rrbracket\|_{L_x^2}^2 - C \|h(t)\|_{L_{x,v}^2}^2 \\ &\quad - C \|f\|_{L^\infty([0,T] \times \mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} \|\llbracket h(t) \rrbracket\|_{L_x^2}^2, \end{aligned}$$

because $\mathcal{L}_1$ is a selfadjoint operator and $\operatorname{Re}(v \cdot \nabla_x h, h)_{L_{x,v}^2} = 0$.

Since $\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2}) \subset L^\infty([0, T] \times \mathbb{R}_x^3; L^2(\mathbb{R}_v^3))$, we have

$$\begin{aligned} -\frac{d}{dt} (e^{2Ct} \|h(t)\|_{L_{x,v}^2}^2) + \frac{1}{C} e^{2Ct} \|\llbracket h(t) \rrbracket\|_{L_x^2}^2 \\ \leq 2e^{2Ct} \|h(t)\|_{L_{x,v}^2} \|\mathcal{Q}h(t)\|_{L_{x,v}^2}, \end{aligned}$$

if ϵ_0 is sufficiently small. Since $h(T) = 0$, for all $t \in [0, T]$ we have

$$\begin{aligned} \|h(t)\|_{L_{x,v}^2}^2 + \frac{1}{C} \|\llbracket h \rrbracket\|_{L^2([t,T] \times \mathbb{R}_x^3)}^2 \\ \leq 2 \int_t^T e^{2C(\tau-t)} \|h(\tau)\|_{L_{x,v}^2} \|\mathcal{Q}h(\tau)\|_{L_{x,v}^2} d\tau \\ \leq 2e^{2CT} \|h\|_{L^\infty([0,T]; L^2(\mathbb{R}_{x,v}^6))} \|\mathcal{Q}h\|_{L^1([0,T], L^2(\mathbb{R}_{x,v}^6))}, \text{ so that} \end{aligned}$$

$$\|h\|_{L^\infty([0,T];L^2(\mathbb{R}_{x,v}^6))} \leq 2e^{2CT} \|\mathcal{Q}h\|_{L^1([0,T];L^2(\mathbb{R}_{x,v}^6))}. \quad (3.5.3)$$

Consider the vector subspace

$$\begin{aligned} \mathbb{W} &= \{w = \mathcal{Q}h : h \in C^\infty([0,T], \mathcal{S}(\mathbb{R}_{x,v}^6)), h(T) = 0\} \\ &\subset L^1([0,T], L^2(\mathbb{R}_{x,v}^6)). \end{aligned}$$

This inclusion holds because it follows from Proposition 3.7.10 that for $g \in L_{x,v}^2$

$$|(\Gamma(f, \cdot)^* h, g)_{L_{x,v}^2}| = |(h, \Gamma(f, g))_{L_{x,v}^2}| \lesssim \|f\|_{L_x^\infty(L_v^2)} \|g\|_{L_{x,v}^2} \|h\|_{L_x^2(H_{\gamma+\nu}^\nu)},$$

and hence, for all $t \in [0, T]$,

$$\|\Gamma(f, \cdot)^* h\|_{L_{x,v}^2} \lesssim \|f\|_{L_x^\infty(L_v^2)} \|h\|_{L_x^2(H_{\gamma+\nu}^\nu)}.$$

Since $g_0 \in L^2(\mathbb{R}_{x,v}^6)$ we define the linear functional

$$\begin{aligned} \mathcal{G} : \quad \mathbb{W} &\longrightarrow \mathbb{C} \\ w = \mathcal{Q}h &\mapsto (g_0, h(0))_{L_{x,v}^2} - (\mathcal{L}_2 f, h)_{L^2([0,T];L_{x,v}^2)}, \end{aligned}$$

where $h \in C^\infty([0,T], \mathcal{S}(\mathbb{R}_{x,v}^6))$, with $h(T) = 0$. According to (3.5.3), the operator $\mathcal{Q}$ is injective. The linear functional $\mathcal{G}$ is therefore well-defined. It follows from (3.5.3) that $\mathcal{G}$ is a continuous linear form on $(\mathbb{W}, \|\cdot\|_{L^1([0,T];L^2(\mathbb{R}_{x,v}^6))})$,

$$\begin{aligned} |\mathcal{G}(w)| &\leq \|g_0\|_{L_{x,v}^2} \|h(0)\|_{L_{x,v}^2} + C_T \|f\|_{L^\infty([0,T];L_{x,v}^2)} \|h\|_{L^1([0,T];L^2(\mathbb{R}_{x,v}^6))} \\ &\leq C'_T \left(\|g_0\|_{L_{x,v}^2} + \|f\|_{L^\infty([0,T];L_{x,v}^2)} \right) \|\mathcal{Q}h\|_{L^1([0,T];L^2(\mathbb{R}_{x,v}^6))} \\ &= C'_T \left(\|g_0\|_{L_{x,v}^2} + \|f\|_{L^\infty([0,T];L_{x,v}^2)} \right) \|w\|_{L^1([0,T];L^2(\mathbb{R}_{x,v}^6))}. \end{aligned}$$

By using the Hahn-Banach theorem, $\mathcal{G}$ may be extended as a continuous linear form on

$$L^1([0,T];L^2(\mathbb{R}_{x,v}^6)),$$

with a norm smaller than $C'_T \left(\|g_0\|_{L_{x,v}^2} + \|f\|_{L^\infty([0,T];L_{x,v}^2)} \right)$. It follows that there exists $g \in L^\infty([0,T];L^2(\mathbb{R}_{x,v}^6))$ satisfying

$$\|g\|_{L^\infty([0,T];L^2(\mathbb{R}_{x,v}^6))} \leq C'_T \left(\|g_0\|_{L_{x,v}^2} + \|f\|_{L^\infty([0,T];L_{x,v}^2)} \right),$$

such that

$$\forall w \in L^1([0,T];L^2(\mathbb{R}_{x,v}^6)), \quad \mathcal{G}(w) = \int_0^T (g(t), w(t))_{L_{x,v}^2} dt.$$

This implies that for all $h \in C_0^\infty((-\infty, T), \mathcal{S}(\mathbb{R}_{x,v}^6))$,

$$\begin{aligned} \mathcal{G}(\mathcal{Q}h) &= \int_0^T (g(t), \mathcal{Q}h(t))_{L_{x,v}^2} dt \\ &= (g_0, h(0))_{L_{x,v}^2} - \int_0^T (\mathcal{L}_2 f(t), h(t))_{L_{x,v}^2} dt. \end{aligned}$$

This shows that $g \in L^\infty([0, T]; L^2(\mathbb{R}_{x,v}^6))$ is a weak solution of the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathcal{L}_1 g = \Gamma(f, g) - \mathcal{L}_2 f, \\ g|_{t=0} = g_0. \end{cases} \quad (3.5.4)$$

It remains to prove that g satisfies (3.5.2). Here we only give a formal proof, temporarily assuming $g \in \tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2}) \cap \mathcal{T}_{T,2,2}^{3/2}$, because too many ingredients are required for its rigorous proof, which will be given at the end of this section. *Formal proof of (3.5.2):* Applying $\Delta_q (q \geq -1)$ to (3.5.4) and taking the inner product $2^{3q} \Delta_q g$ over $\mathbb{R}_{x,v}^6$, we obtain

$$\begin{aligned} & \frac{d}{dt} 2^{3q} \|\Delta_q g\|_{x,v}^2 + \frac{2^{3q}}{C} \|\Delta_q g\|_{L_x^2}^2 \\ & \leq 2^{3q+1} (\Delta_q \Gamma(f, g), \Delta_q g)_{x,v} + 2^{3q} \|\Delta_q f\|_{x,v}^2 + C 2^{3q} \|\Delta_q g\|_{x,v}^2. \end{aligned}$$

Integrate this with respect to the time variable over $[0, t]$ with $0 \leq t \leq T$, take the square root of both sides of the resulting inequality and sum up over $q \geq -1$. Then it follows from (3.3.1) of Lemma 3.3.1 that

$$\begin{aligned} & \|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})} + \frac{1}{\sqrt{C}} \|g\|_{\mathcal{T}_{T,2,2}^{3/2}} \leq C' \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,2,2}^{3/2}} \\ & + 2 \left(\|g_0\|_{\tilde{L}_v^2(B^{3/2})} + \sqrt{T} \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})} + \sqrt{CT} \|g\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})} \right). \end{aligned} \quad (3.5.5)$$

□

Theorem 3.5.2 (Local Existence). *There exist $\varepsilon_1, T > 0$ such that if $g_0 \in \tilde{L}_v^2(B_x^{3/2})$ and*

$$\|g_0\|_{\tilde{L}_v^2(B_x^{3/2})} \leq \varepsilon_1,$$

then the Cauchy problem (3.5.4) admits a unique solution

$$g(x, v, t) \in \tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2}) \cap \mathcal{T}_{T,2,2}^{3/2}.$$

Proof. Consider the sequence of approximate solutions defined by

$$\begin{cases} \partial_t g^{n+1} + v \cdot \nabla_x g^{n+1} + \mathcal{L}_1 g^{n+1} = \Gamma(g^n, g^{n+1}) - \mathcal{L}_2 g^n, \\ g^{n+1}|_{t=0} = g_0, \end{cases} \quad (n = 0, 1, 2, \dots, \quad g^0 = 0). \quad (3.5.6)$$

Use Lemma 3.5.1 with $g = g^{n+1}, f = g^n$ and $T = \min\{T_0, 1/(4C_0^2)\}$. Then we have

$$\|g^n\|_{\tilde{L}_T^\infty \tilde{L}^2(B^{3/2})} + \|g^n\|_{\mathcal{T}_{T,2,2}^{3/2}} \leq \varepsilon_0, \quad (3.5.7)$$

inductively, if ε_1 is taken such that $2C_0\varepsilon_1 \leq \varepsilon_0$. It remains to prove the convergence of the sequence

$$\{g^n, n \in \mathbb{N}\} \subset \tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2}) \cap \mathcal{T}_{T,2,2}^{3/2}.$$

Setting $w^n = g^{n+1} - g^n$, from (3.5.6) we have

$$\partial_t w^n + v \cdot \nabla_x w^n + \mathcal{L}_1 w^n = \Gamma(g^n, w^n) + \Gamma(w^{n-1}, g^n) - \mathcal{L}_2 w^{n-1},$$

with $w^n|_{t=0} = 0$. Similar to the computation for (3.5.5), we obtain

$$\begin{aligned} \|w^n\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})} + \frac{1}{\sqrt{C}} \|w^n\|_{\mathcal{T}_{T,2,2}^{3/2}} &\leq C \left(\|g^n\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})}^{1/2} \|w^n\|_{\mathcal{T}_{T,2,2}^{3/2}} \right. \\ &\quad + \|w^{n-1}\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})}^{1/2} \|g^n\|_{\mathcal{T}_{T,2,2}^{3/2}}^{1/2} \|w^n\|_{\mathcal{T}_{T,2,2}^{3/2}}^{1/2} \\ &\quad \left. + \sqrt{T} \|w^{n-1}\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})}^{1/2} \|w^n\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})}^{1/2} \right). \end{aligned}$$

If ε_0 and T are sufficiently small then we have

$$\begin{aligned} \|w^n\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})} + \frac{1}{\sqrt{C}} \|w^n\|_{\mathcal{T}_{T,2,2}^{3/2}} &\leq \lambda \|w^{n-1}\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})} \\ &\leq \lambda^{n-1} \|w^1\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})} \end{aligned}$$

for some $0 < \lambda < 1$, which concludes that

$$\{g^n\} \text{ is a Cauchy sequence in } \tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2}) \cap \mathcal{T}_{T,2,2}^{3/2},$$

and the limit function g is a desired solution to the Cauchy problem

$$\partial_t g + v \cdot \nabla_x g + \mathcal{L}g = \Gamma(g, g), \quad g|_{t=0} = g_0.$$

□

3.5.2 Rigorous proof of (3.5.2)

The preceding proof of (3.5.2) is formal since we a priori assumed the left hand side of (3.5.5) is finite. The rigorous proof requires more involved procedure. We start by the following lemma.

Lemma 3.5.3. *Let $0 < s \leq \frac{3}{2}$ and $0 < T \leq \infty$. For $M \in \mathbb{N}$ put $f_M = S_M f = \sum_{q \geq -1}^{M-1} \Delta_q f$. If g satisfies*

$$\| \|g\| \|_{L_T^2 L_x^2}^2 = \int_0^T \int_{\mathbb{R}_x^3} \|g\|^2 dx dt < \infty, \quad (3.5.8)$$

then there exists a $C > 0$ independent of M such that for any $\kappa > 0$

$$\begin{aligned} \sum_{q \geq -1} \frac{2^{qs}}{1 + \kappa 2^{2qs}} \left(\int_0^T |(\Delta_q \Gamma(f_M, g), \Delta_q g)_{x,v}| dt \right)^{1/2} \\ \leq C \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,2,2}^{s,\kappa}} \\ + C_M \|f_M\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \| \|g\| \|_{L_T^2 L_x^2}, \end{aligned} \quad (3.5.9)$$

where $C_M > 0$ is a constant depending only on M and

$$\|g\|_{\mathcal{T}_{T,2,2}^{s,\kappa}} = \sum_{q \geq -1}^{\infty} \frac{2^{qs}}{1 + \kappa 2^{2qs}} \|\Delta_q g\|_{L_T^2 L_x^2}.$$

Proof. We notice that $\|g\|_{\mathcal{T}_{T,2,2}^{s,\kappa}} < \infty$ for each $\kappa > 0$ follows from (3.5.8). The proof of the lemma is the almost same procedure as the one for (3.3.1) of Lemma 3.3.1. Indeed, recalling the Bony decomposition, for $\Gamma^1(f_M, g)$ we obtain

$$\begin{aligned} & \sum_{q \geq -1} \frac{2^{qs}}{1 + \kappa 2^{2qs}} \left(\int_0^T |(\Delta_q \Gamma^1(f_M, g), \Delta_q g)_{x,v}| dt \right)^{1/2} \\ & \lesssim \sum_{q \geq -1} \frac{2^{qs}}{1 + \kappa 2^{2qs}} \left(\int_0^T \sum_{|j-q| \leq 4} \|f\|_{L_v^2 L_x^\infty} \|\Delta_j g\|_{L_x^2} \|\Delta_q g\|_{L_x^2} dt \right)^{1/2} \\ & \lesssim \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \left(\sum_{q \geq -1} \frac{2^{qs}}{1 + \kappa 2^{2qs}} \left(\sum_{|j-q| \leq 4} \int_0^T \|\Delta_j g\|_{L_x^2}^2 dt \right)^{1/2} \right)^{1/2} \\ & \quad \times \left(\sum_{q \geq -1} \frac{2^{qs}}{1 + \kappa 2^{2qs}} \left(\int_0^T \|\Delta_q g\|_{L_x^2}^2 dt \right)^{1/2} \right)^{1/2} \\ & \lesssim \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|g\|_{\mathcal{T}_{T,2,2}^{s,\kappa}}, \end{aligned}$$

similar as (3.3.5). As for $\Gamma^2(f_M, g)$, we have

$$\begin{aligned} & \sum_{q \geq -1} \frac{2^{qs}}{1 + \kappa 2^{2qs}} \left(\int_0^T |(\Delta_q \Gamma^2(f_M, g), \Delta_q g)_{x,v}| dt \right)^{1/2} \\ & \lesssim \sum_{q \geq -1} \frac{2^{qs}}{1 + \kappa 2^{2qs}} \left(\sum_{\substack{|j-q| \leq 4 \\ j \leq M}} \int_0^T \|S_{j-1} g\|_{L_x^\infty} \|\Delta_j f_M\|_{L_{x,v}^2} \|\Delta_q g\|_{L_x^2} dt \right)^{1/2} \\ & \lesssim \sum_{q \geq -1}^{M+4} \frac{2^{qs}}{1 + \kappa 2^{2qs}} \left(\|f_M\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|S_{M+4} g\|_{L_T^2 L_x^2} \|\Delta_q g\|_{L_T^2 L_x^2} \right)^{1/2} \\ & \leq C_M \|f_M\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|g\|_{L_T^2 L_x^2}. \end{aligned}$$

Since

$$\begin{aligned} \Delta_q \left(\sum_j \sum_{|j-j'| \leq 1} (\Delta_{j'} f_M, \Delta_j g) \right) &= \Delta_q \left(\sum_{\max\{j,j'\} \geq q-2} \sum_{|j-j'| \leq 1} (\Delta_{j'} f_M, \Delta_j g) \right) \\ &= 0 \text{ if } q \geq M+3, \end{aligned}$$

the term corresponding to Γ^3 is estimated as follows:

$$\begin{aligned}
& \sum_{q \geq -1}^{M+2} \frac{2^{qs}}{1 + \kappa 2^{2qs}} \left(\int_0^T |(\Delta_q \Gamma^3(f_M, g), \Delta_q g)_{x,v}| dt \right)^{1/2} \\
& \lesssim \sum_{q \geq -1}^{M+2} \frac{2^{qs}}{1 + \kappa 2^{2qs}} \left(\sum_{j \leq M+1} \int_0^T \|f_M\|_{L_v^2 L_x^\infty} \|\Delta_j g\|_{L_x^2} \|\Delta_q g\|_{L_x^2} dt \right)^{1/2} \\
& \leq C_M \|f_M\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|g\|_{L_{T,x}^2}.
\end{aligned}$$

Thus the proof is completed. $\square$

For each f_M ($M \in \mathbb{N}$) we consider a weak solution $g_M \in L^\infty([0, T]; L^2(\mathbb{R}_{x,v}^6))$ to the Cauchy problem (3.5.1) with f replaced by f_M . If g_M satisfies (3.5.8), then by the same procedure as in the formal proof of (3.5.2) we obtain

$$\begin{aligned}
& \sum_{q \geq -1} \frac{2^{3q/2}}{1 + \kappa 2^{3q}} \sup_{0 \leq t \leq T} \|\Delta_q g_M(t)\|_{x,v} + \frac{1}{\sqrt{C}} \|g_M\|_{\mathcal{T}_{T,2,2}^{3/2,\kappa}} \\
& \leq C' \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})}^{1/2} \|g_M\|_{\mathcal{T}_{T,2,2}^{3/2,\kappa}} \\
& + 2 \left(\sum_{q \geq -1} \frac{2^{3q/2}}{1 + \kappa 2^{3q}} \|\Delta_q g_0\|_{x,v} + \sqrt{T} \|f\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})} + \sqrt{CT} \right. \\
& \times \left. \sum_{q \geq -1} \frac{2^{3q/2}}{1 + \kappa 2^{3q}} \sup_{0 \leq t \leq T} \|\Delta_q g_M(t)\|_{x,v} \right) + C_M \|f_M\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})}^{1/2} \|g_M\|_{L_T^2 L_x^2}.
\end{aligned}$$

Choose a small $T > 0$ independent of M and κ . Then letting $\kappa \rightarrow 0$, we get

$$\|g_M\|_{\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2})} + \|g_M\|_{\mathcal{T}_{T,2,2}^{3/2}} < \infty,$$

which permits the preceding formal proof and we obtain the energy estimate (3.5.5) for each g_M . Set $w_{M,M'} = g_M - g_{M'}$, $M, M' \in \mathbb{N}$. Then it follows from (3.5.1) that

$$\begin{aligned}
& \partial_t w_{M,M'} + v \cdot \nabla_x w_{M,M'} + \mathcal{L}_1 w_{M,M'} \\
& = \Gamma(f_M - f_{M'}, g_M) + \Gamma(f_{M'}, w_{M,M'}) - \mathcal{L}_2(f_M - f_{M'}).
\end{aligned}$$

Since $\{f_M\}$ is a Cauchy sequence in $\tilde{L}_T^\infty \tilde{L}_v^2(B_x^{3/2})$, it is easy to see that $\{g_M\}$ is a Cauchy sequence in $\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2}) \cap \mathcal{T}_{T,2,2}^{3/2}$, by the similar manipulation as in the proof of Theorem 3.5.2. Since $g = \lim_{M \rightarrow \infty} g_M$ belongs to $\tilde{L}_T^\infty \tilde{L}_v^2(B^{3/2}) \cap \mathcal{T}_{T,2,2}^{3/2}$, the preceding formal proof is justified.

It remains to show (3.5.8) for a weak solution $g_M \in L^\infty([0, T]; L^2(\mathbb{R}_{x,v}^6))$, under the assumption that $\|f_M\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L_2(\mathbb{R}_v^3))}$ is sufficiently small, independent of M , and moreover $\|\nabla_x f_M\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L_2(\mathbb{R}_v^3))} < \infty$. For the brevity, we write g and f instead of g_M and f_M , respectively.

Let us take different $1 > \delta, \delta' > 0$. We use a weight function $W_{\delta'}(v) = \langle \delta' v \rangle^{-N}$ for $N \geq 1$ and mollifiers $M^\delta(D_v), S_\delta(D_x)$ defined in subsections 3.7.4, 3.7.6, respectively. Multiply $W_{\delta'}(v)S_\delta(D_x)\left(M^\delta(D_v)\right)^2 S_\delta(D_x)W_{\delta'}(v)g$ by the equation (3.5.1), and integrate with respect to $t \in [0, T]$ and $(x, v) \in \mathbb{R}^6$. Notice that

$$\begin{aligned}
& \left(\Gamma(f, g), W_{\delta'} S_\delta \left(M^\delta \right)^2 S_\delta W_{\delta'} g \right)_{x,v} - \left(\Gamma(f, M^\delta S_\delta W_{\delta'} g), M^\delta S_\delta W_{\delta'} g \right)_{x,v} \\
&= \left(W_{\delta'} \Gamma(f, g) - \Gamma(f, W_{\delta'} g), S_\delta \left(M^\delta \right)^2 S_\delta W_{\delta'} g \right)_{x,v} \\
&+ \left(S_\delta \Gamma(f, W_{\delta'} g) - \Gamma(f, S_\delta W_{\delta'} g), \left(M^\delta \right)^2 S_\delta W_{\delta'} g \right)_{x,v} \\
&+ \left(\Gamma(f, S_\delta W_{\delta'} g) - Q(\mu^{1/2} f, S_\delta W_{\delta'} g), \left(M^\delta \right)^2 S_\delta W_{\delta'} g \right)_{x,v} \\
&+ \left(M^\delta Q(\mu^{1/2} f, S_\delta W_{\delta'} g) - Q(\mu^{1/2} f, M^\delta S_\delta W_{\delta'} g), M^\delta S_\delta W_{\delta'} g \right)_{x,v} \\
&+ \left(Q(\mu^{1/2} f, M^\delta S_\delta W_{\delta'} g) - \Gamma(f, M^\delta S_\delta W_{\delta'} g), M^\delta S_\delta W_{\delta'} g \right)_{x,v} \\
&= (1) + (2) + (3) + (4) + (5).
\end{aligned}$$

It follows from Proposition 3.7.19 and Lemma 3.7.23 that

$$\begin{aligned}
\int_0^T (1) dt &\lesssim \delta'^{\nu/2} \|f\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L_v^2)} \|\langle v \rangle^{(\gamma+\nu)/2} W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_{x,v}^6)} \\
&\quad \times \|\| M^\delta S_\delta W_{\delta'} g \| \|_{L^2([0, T] \times \mathbb{R}_x^3)} \quad (3.5.10)
\end{aligned}$$

By means of (3.7.29), we have

$$\begin{aligned}
\int_0^T (2) dt &\lesssim \delta^{1-\nu/2} \|\nabla f\|_{L_{T,x}^\infty(L_v^2)} \|\langle v \rangle^{|\gamma|/2+\nu} W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_{x,v}^6)} \\
&\quad \times \|M^\delta S_\delta W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_x^3; H_{\gamma/2}^{\nu/2}(\mathbb{R}_v^3))} \quad (3.5.11)
\end{aligned}$$

because

$$\frac{\delta \langle \xi \rangle^\nu}{(1 + \delta \langle \xi \rangle)^{N_0}} \leq \delta^{1-\nu/2} \langle \xi \rangle^{\nu/2}.$$

It follows from (3.7.11) and Lemma 3.7.23 that

$$\begin{aligned}
\int_0^T (3) + (5) dt &\lesssim \|f\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L_v^2)} \|\langle v \rangle^{(\gamma+\nu)/2} W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_{x,v}^6)} \\
&\quad \times \|\| M^\delta S_\delta W_{\delta'} g \| \|_{L^2([0, T] \times \mathbb{R}_x^3)} \quad (3.5.12)
\end{aligned}$$

Thanks to Proposition 3.7.21 and Proposition 3.7.22, for a $0 < \nu' < \nu$ we have

$$\begin{aligned}
\int_0^T (4) dt &\lesssim \|f\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L_v^2)} \left(\|M^\delta S_\delta W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_x^3; H_{(\nu+\gamma)^+}^{\nu'/2}(\mathbb{R}_v^3))}^2 \right. \\
&\quad \left. + \|\langle v \rangle^{(\gamma+\nu)^+} W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_{x,v}^6)}^2 \right).
\end{aligned}$$

Note that for any $\varepsilon, \kappa, \ell > 0$ there exist $C_{\kappa, \varepsilon, \ell} > 0$ and $N_{\ell, \varepsilon} > 0$ such that

$$\|h\|_{H_\ell^{\nu/2-\varepsilon}} \leq \kappa \|h\|_{H^{\nu/2}} + C_{\kappa, \varepsilon, \ell} \|h\|_{L_{N_{\ell, \varepsilon}}^2},$$

(see for example Lemma 2.4 of [16]). Therefore we obtain

$$\begin{aligned} \int_0^T (4) dt &\lesssim \|f\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L_v^2)} \left(\kappa \|M^\delta S_\delta W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_x^3; H_{\gamma/2}^{\nu/2}(\mathbb{R}_v^3))}^2 \right. \\ &\quad \left. + C_\kappa \|\langle v \rangle^N W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_{x, v}^6)}^2 \right). \end{aligned} \quad (3.5.13)$$

for a sufficiently large $N > 0$. Summing up (3.5.10)-(3.5.13), for any $\kappa > 0$ we obtain

$$\begin{aligned} \int_0^T \left(\Gamma(f, g), W_{\delta'} S_\delta \left(M^\delta \right)^2 S_\delta W_{\delta'} g \right)_{x, v} dt &\leq (C_1 \|f\|_{L_{T, x}^\infty(L_v^2)} + \kappa) \\ &\quad \times \left(\|M^\delta S_\delta W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_x^3)}^2 + C_{\kappa, f} \|\langle v \rangle^N W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_{x, v}^6)}^2 \right), \end{aligned} \quad (3.5.14)$$

by means of (3.7.2). Similarly we obtain

$$\begin{aligned} \int_0^T \left(\mathcal{L}_1 g, W_{\delta'} S_\delta \left(M^\delta \right)^2 S_\delta W_{\delta'} g \right)_{x, v} dt \\ \geq \int_0^T \left(\mathcal{L}_1 M^\delta S_\delta W_{\delta'} g, M^\delta S_\delta W_{\delta'} g \right)_{x, v} dt \\ - \frac{\lambda_0}{2} \|M^\delta S_\delta W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_x^3)}^2 + C \|\langle v \rangle^N W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_{x, v}^6)}^2. \end{aligned} \quad (3.5.15)$$

To handle $v \cdot \nabla_x$ term, we use the similar device as in (3.2.4) of [3]. Indeed,

$$\begin{aligned} &\left(v \cdot \nabla_x g, W_{\delta'} S_\delta \left(M^\delta \right)^2 S_\delta W_{\delta'} g \right)_{x, v} \\ &= \left([M^\delta, v] \cdot \nabla_x S_\delta W_{\delta'} g, M^\delta S_\delta W_{\delta'} g \right)_{x, v} \leq 2 \|M^\delta S_\delta W_{\delta'} g\|_{L^2(\mathbb{R}_{x, v}^6)}^2, \end{aligned} \quad (3.5.16)$$

because $|(\nabla_\xi M^\delta(\xi)) \cdot (\eta S_\delta(\eta))| \leq N_0 M^\delta(\xi) |\delta \eta| S(\delta \eta) \leq 2 M^\delta(\xi) S_\delta(\eta)$. It follows from Lemma 3.7.4 and (3.5.14)-(3.5.16) that we obtain

$$\|M^\delta S_\delta W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_x^3)}^2 \leq C_T \left(\|\langle v \rangle^N W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_{x, v}^6)}^2 + \|g_0\|_{L^2(\mathbb{R}_{x, v}^6)}^2 \right),$$

if $C_1 \|f\|_{L_{T, x}^\infty(L_v^2)} \leq \lambda_0/8$. For arbitrary but fixed $\delta' > 0$, we get

$$\|W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_x^3)}^2 < C_{\delta'}$$

by letting $\delta \rightarrow 0$.

We repeat the same procedure without the factor $\left(M^\delta(D_v)\right)^2$, that is, multiply $\left(W_{\delta'}(v)S_\delta(D_x)\right)^2 g$ by the equation, and integrate with respect to $t \in [0, T]$ and $(x, v) \in \mathbb{R}^6$. By the same way as (3.5.10), we get

$$\begin{aligned} & \int_0^T \left| \left(\Gamma(f, g), \left(W_{\delta'} S_\delta \right)^2 g \right)_{x,v} - \left(\Gamma(f, W_{\delta'} g), (S_\delta)^2 W_{\delta'} g \right)_{x,v} \right| dt \\ & \lesssim \delta'^{\nu/2} \|f\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L_v^2)} \|W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_x^3)}^2. \end{aligned}$$

By using (3.7.2) for $\left(\Gamma(f, W_{\delta'} g), (S_\delta)^2 W_{\delta'} g \right)_{x,v}$, we obtain

$$\int_0^T \left| \left(\Gamma(f, g), \left(W_{\delta'} S_\delta \right)^2 g \right)_{x,v} \right| dt \lesssim \|f\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L_v^2)} \|W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_x^3)}^2.$$

Since $(v \cdot \nabla_x g, (W_{\delta'} S_\delta)^2 g)_{x,v} = 0$, by letting $\delta \rightarrow 0$ we obtain

$$\begin{aligned} & (\lambda_0 - C_1 \|f\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L_v^2)}) \|W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_x^3)}^2 \\ & \lesssim \delta'^\nu \|W_{\delta'} g\|_{L^2([0, T] \times \mathbb{R}_x^3)}^2 + \|g\|_{L^2([0, T] \times \mathbb{R}_{x,v}^6)}^2 + \|g_0\|_{L^2(\mathbb{R}_{x,v}^6)}^2, \end{aligned}$$

because of Lemma 3.7.4 and (3.7.18). Finally we obtain (3.5.8) by letting $\delta' \rightarrow 0$.

3.6 Non-negativity of solutions

The method of the proof is the almost same as the one of Proposition 5.2 in [6]. For the self-containedness, we reproduce it. If $\{g^n\}$ is the sequence of approximate solutions in the proof of Theorem 3.5.2, and if $f^n = \mu + \mu^{1/2} g^n$, then $\{f^n\}$ is constructed successively by the following linear Cauchy problem

$$\begin{cases} \partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} = Q(f^n, f^{n+1}), \\ f^{n+1}|_{t=0} = f_0 = \mu + \mu^{1/2} g_0 \geq 0, \quad (n = 0, 1, 2, \dots, f^0 = \mu). \end{cases} \quad (3.6.1)$$

Hence the non-negativity of the solution to the original Cauchy problem (3.1.1) comes from the following induction argument: Suppose that

$$f^n = \mu + \mu^{1/2} g^n \geq 0, \quad (3.6.2)$$

for some $n \in \mathbb{N}$. Then (3.6.2) is true for $n+1$.

If we put $\tilde{f}^n = \mu^{-1/2} f^n = \mu^{1/2} + g^n$ then $\tilde{f}^n$ satisfies

$$\begin{cases} \partial_t \tilde{f}^{n+1} + v \cdot \nabla_x \tilde{f}^{n+1} = \Gamma(\tilde{f}^n, \tilde{f}^{n+1}), \\ \tilde{f}^{n+1}|_{t=0} = \tilde{f}_0 = \mu^{1/2} + g_0 \geq 0, \quad (n = 0, 1, 2, \dots, \tilde{f}^0 = \mu^{1/2}). \end{cases} \quad (3.6.3)$$

It follows from (3.5.7) and Lemma 3.7.9 that $\int_0^T \|\tilde{f}^n\|_{L_x^\infty}^2 dt < \infty$, and hence, if $\tilde{f}_\pm^n = \pm \max(\pm \tilde{f}^n, 0)$ then we have

$$\int_0^T \|\tilde{f}_+^n\|_{L_x^\infty}^2 dt + \int_0^T \|\tilde{f}_-^n\|_{L_x^\infty}^2 dt < \infty$$

by means of the same argument as in the proof of Theorem 5.2 in [6]. Take the convex function $\beta(s) = \frac{1}{2}(s^-)^2 = \frac{1}{2}s(s^-)$ with $s^- = \min\{s, 0\}$. Let $\varphi(v, x) = (1 + |v|^2 + |x|^2)^{\alpha/2}$ with $\alpha > 3/2$, and notice that

$$\begin{aligned}\beta_s(\tilde{f}^{n+1})\varphi(v, x)^{-1} &:= \left(\frac{d}{ds}\beta\right)(\tilde{f}^{n+1})\varphi(v, x)^{-1} \\ &= \tilde{f}_-^{n+1}\varphi(v, x)^{-1} \in L^\infty([0, T]; L^2(\mathbb{R}_{x,v}^6)).\end{aligned}$$

Multiply the first equation of (3.6.3) by $\beta_s(\tilde{f}^{n+1})\varphi(v, x)^{-2} = \tilde{f}_-^{n+1}\varphi(v, x)^{-2}$ and integrate over $[0, t] \times \mathbb{R}_{x,v}^6$, ($t \in (0, T]$). Then, in view of $\beta(\tilde{f}^{n+1}(0)) = \tilde{f}_{0,-}^2/2 = 0$, we have

$$\begin{aligned}&\int_{\mathbb{R}^6} \beta(\tilde{f}^{n+1}(t))\varphi(v, x)^{-2} dx dv \\ &= \int_0^t \int_{\mathbb{R}^6} \Gamma(\tilde{f}^n(\tau), \tilde{f}^{n+1}(\tau)) \beta_s(\tilde{f}^{n+1}(\tau))\varphi(v, x)^{-2} dx dv d\tau \\ &\quad - \int_0^t \int_{\mathbb{R}^6} v \cdot \nabla_x (\beta(\tilde{f}^{n+1}(\tau))\varphi(v, x)^{-2}) dx dv d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^6} (\varphi(v, x)^2 v \cdot \nabla_x \varphi(v, x)^{-2}) \beta(\tilde{f}^{n+1}(\tau))\varphi(v, x)^{-2} dx dv d\tau,\end{aligned}$$

where the first term on the right hand side is well defined because

$$\int_0^T \|\tilde{f}^{n+1}\|_{L_x^\infty}^2 dt + \int_0^T \|\tilde{f}_-^{n+1}\|_{L_x^\infty}^2 dt < \infty.$$

Since the second term vanishes and $|v \cdot \nabla_x \varphi(v, x)^{-2}| \leq C\varphi(v, x)^{-2}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^6} \beta(\tilde{f}^{n+1}(t))\varphi(v, x)^{-2} dx dv \\ &\leq \int_0^t \left(\int_{\mathbb{R}^6} \Gamma(\tilde{f}^n(\tau), \tilde{f}^{n+1}(\tau)) \beta_s(\tilde{f}^{n+1}(\tau))\varphi(v, x)^{-2} dx dv \right) d\tau \\ &\quad + C \int_0^t \int_{\mathbb{R}^6} \beta(\tilde{f}^{n+1}(\tau))\varphi(v, x)^{-2} dx dv d\tau.\end{aligned}$$

The integrand $(\cdot)$ of the first term on the right hand side is equal to

$$\begin{aligned}&\int_{\mathbb{R}^6} \Gamma(\tilde{f}^n, \tilde{f}_-^{n+1})\tilde{f}_-^{n+1}\varphi(v, x)^{-2} dx dv \\ &\quad + \int B \mu_*^{1/2}(\tilde{f}_*^n)'(\tilde{f}_+^{n+1})' \tilde{f}_-^{n+1}\varphi(v, x)^{-2} dv dv_* d\sigma dx \\ &= A_1 + A_2.\end{aligned}$$

From the induction hypothesis, the second term A_2 is non-positive.

On the other hand, we have

$$\begin{aligned}
A_1 &= \int (\Gamma(\tilde{f}^n, \varphi(v, x)^{-1} \tilde{f}_-^{n+1}), \varphi(v, x)^{-1} \tilde{f}_-^{n+1})_{L^2(\mathbb{R}_v^3)} dx + R \\
&= - \int (\mathcal{L}_1(\varphi(v, x)^{-1} \tilde{f}_-^{n+1}), \varphi(v, x)^{-1} \tilde{f}_-^{n+1})_{L^2(\mathbb{R}_v^3)} dx \\
&\quad + \int (\Gamma(g^n, \varphi(v, x)^{-1} \tilde{f}_-^{n+1}), \varphi(v, x)^{-1} \tilde{f}_-^{n+1})_{L^2(\mathbb{R}_v^3)} dx + R,
\end{aligned}$$

where R is a remainder term. It follows from Corollary 3.7.20 with $f = \tilde{f}^n$ that for any $\kappa > 0$

$$\begin{aligned}
\int_0^t |R| d\tau &\leq \kappa \int_0^t \int_{\mathbb{R}_x^3} \|\varphi(v, x)^{-1} \tilde{f}_-^{n+1}(\tau)\|^2 dx d\tau \\
&\quad + C_\kappa \int_0^t \int_{\mathbb{R}^6} \beta(\tilde{f}^{n+1}(\tau)) \varphi(v, x)^{-2} dx dv d\tau.
\end{aligned}$$

By means of Lemma 3.7.4 and Corollary 3.7.2 with (3.5.7), we obtain

$$\begin{aligned}
\int_0^t A_1 d\tau &\leq -(\lambda_0 - C(\kappa + \varepsilon_0)) \int_0^t \int_{\mathbb{R}_x^3} \|\varphi(v, x)^{-1} \tilde{f}_-^{n+1}(\tau)\|^2 dx d\tau \\
&\quad + C_\kappa \int_0^t \int_{\mathbb{R}^6} \beta(\tilde{f}^{n+1}(\tau)) \varphi(v, x)^{-2} dx dv d\tau.
\end{aligned}$$

Finally we get

$$\int_{\mathbb{R}^6} \beta(\tilde{f}^{n+1}(t)) \varphi(v, x)^{-2} dx dv \lesssim \int_0^t \int_{\mathbb{R}^6} \beta(\tilde{f}^{n+1}(\tau)) \varphi(v, x)^{-2} dx dv d\tau,$$

which implies that $\tilde{f}^{n+1}(t, x, v) \geq 0$ for $(t, x, v) \in [0, T] \times \mathbb{R}^6$.

3.7 Appendix

3.7.1 Fundamental inequalities and Besov embedding theorems

First of all, the following lemma plays the central role in this paper.

Lemma 3.7.1. *Let $\nu \in (0, 2)$ and $\gamma > \max\{-3, -3/2 - \nu\}$. For any $\alpha \geq 0$ and any $\beta \in \mathbb{R}$ there exists a $C = C_{\alpha, \beta} > 0$ such that*

$$\begin{aligned}
|(\Gamma(f, g), h)_{L_v^2}| &\leq C \left(\|\mu^{1/10} f\|_{L_v^2} \|g\| \|h\| \right. \\
&\quad \left. + \|f\|_{L_{-\alpha}^2} \|g\|_{L_{(\gamma+\nu)/2+\alpha-\beta}^2} \|h\|_{L_{(\gamma+\nu)/2+\beta}^2} \right). \quad (3.7.1)
\end{aligned}$$

This lemma will be proved in the next subsection together with another upper bound estimate. If we put $\alpha = \beta = 0$ then we obtain [8, Theorem 1.2], namely,

Corollary 3.7.2. *Let $\nu \in (0, 2)$ and $\gamma > \max\{-3, -3/2 - \nu\}$. Then it holds*

$$|(\Gamma(f, g), h)_{L_v^2}| \lesssim \|f\|_{L_v^2} \|g\| \|h\|. \quad (3.7.2)$$

In the case $\gamma + \nu \leq 0$, we employ the following, by setting $\alpha = -(\gamma + \nu)/2$ and $\beta = (\gamma + \nu)/2$ or 0.

Corollary 3.7.3. *When $\gamma + \nu \leq 0$ we have*

$$\begin{aligned} \left| (\Gamma(f, g), h)_{L_v^2} \right| &\lesssim \|\mu^{1/10} f\|_{L^2} \|g\| \|h\| \\ &+ \|f\|_{L_{(\gamma+\nu)/2}^2} \min \left\{ \|g\|_{L_{(\gamma+\nu)/2}^2} \|h\|_{L^2}, \|g\|_{L^2} \|h\|_{L_{(\gamma+\nu)/2}^2} \right\}. \end{aligned} \quad (3.7.3)$$

We also need the lower estimate of $(\mathcal{L}_1 f, f)$ and the upper estimate of $(\mathcal{L}_2 f, f)$, respectively.

Lemma 3.7.4. *Let $\nu \in (0, 2)$ and $\gamma > -3$. Then there exists a constant $\lambda_0 > 0$ such that*

$$\begin{aligned} (\mathcal{L}_1 f, f)_{L_v^2} &\geq \frac{1}{2} (\mathcal{L} f, f)_{L_v^2} \geq \lambda_0 \|(\mathbf{I} - \mathbf{P})f\|^2, \\ |(\mathcal{L}_2 f, g)_{L_v^2}| &\lesssim \|\mu^{10^{-3}} f\|_{L_v^2} \|\mu^{10^{-3}} g\|_{L_v^2}. \end{aligned}$$

The first and second inequalities of the above lemma are from [7, Proposition 2.1] and [7, Lemma 2.15], respectively.

We catalogue a couple of lemmas which are frequently used in this paper.

Lemma 3.7.5. *Let $1 \leq p \leq \infty$ and $f \in L_x^p$, then there exists a constant $C > 0$ independent of p, q and f such that*

$$\|\Delta_q f\|_{L_x^p} \leq C \|f\|_{L_x^p}, \quad \|S_q f\|_{L_x^p} \leq C \|f\|_{L_x^p}.$$

In short, Δ_q and S_q are bounded operators on L_x^p .

Lemma 3.7.6. *Let $1 \leq p, r \leq \infty$. Then*

1. $B_{pr}^{s_1} \hookrightarrow B_{pr}^{s_2}$ when $s_2 \leq s_1$. This inclusion does not hold for the homogeneous Besov space.
2. $B_{p,1}^{3/p} \hookrightarrow L^\infty$ and $\dot{B}_{p,1}^{3/p} \hookrightarrow L^\infty$ when $1 \leq p < \infty$.

Lemma 3.7.7. *Let $1 \leq p, q, r \leq \infty$ and $s > 0$. Then we have*

$$\|\nabla_x \cdot\|_{\tilde{L}_T^q(\dot{B}_{pr}^s)} \sim \|\cdot\|_{\tilde{L}_T^q(\dot{B}_{pr}^{s+1})}, \quad \|\cdot\|_{\tilde{L}_T^q(\dot{B}_{pr}^s)} \lesssim \|\cdot\|_{\tilde{L}_T^q(B_{pr}^s)}.$$

Lemma 3.7.8. *Let $1 \leq p, \alpha, \beta, r \leq \infty$ and $s > 0$. If $r \leq \min\{\alpha, \beta\}$ then*

$$\|f\|_{L_T^\alpha L_v^\beta(B_{pr}^s)} \leq \|f\|_{\tilde{L}_T^\alpha \tilde{L}_v^\beta(B_{pr}^s)} \text{ and } \|f\|_{L_T^\alpha L_v^\beta(\dot{B}_{pr}^s)} \leq \|f\|_{\tilde{L}_T^\alpha \tilde{L}_v^\beta(\dot{B}_{pr}^s)}.$$

For the proof of Lemma 3.7.5, Lemma 3.7.6 - Lemma 3.7.7 and Lemma 3.7.8, readers may refer to [11], [26] and [13], respectively.

We prove a useful lemma to deal with terms consisting of $\|f\|_{L_x^\infty}$.

Lemma 3.7.9. *For each $T > 0$, we have*

$$\left(\int_0^T \|f\|_{L_x^\infty}^2 dt \right)^{1/2} \lesssim \|f\|_{\dot{J}_{T,2,2}^{3/2}}.$$

Proof. By definition of $\|f\|$ and using both generalized Minkowski's inequality and Besov embedding $\dot{B}_x^{3/2} \hookrightarrow L_x^\infty$, we calculate that

$$\begin{aligned} & \left(\int_0^T \|f\|_{L_x^\infty}^2 dt \right)^{1/2} \\ & \leq \left[\int_0^T \int B \left\{ \mu_* \sup_x (f' - f)^2 + (\mu^{1/2} - \mu'^{1/2}) \sup_x f_*^2 \right\} dv dv_* d\sigma dt \right]^{1/2} \\ & \lesssim \left[\int_0^T \int B \left\{ \mu_* \left(\sum_l 2^{\frac{3}{2}l} \left(\int |\dot{\Delta}_l(f' - f)|^2 dx \right)^{1/2} \right)^2 \right. \right. \\ & \quad \left. \left. + (\mu^{1/2} - \mu'^{1/2}) \left(\sum_l 2^{\frac{3}{2}l} \left(\int |\dot{\Delta}_l f_*|^2 dx \right)^{1/2} \right)^2 \right\} dv dv_* d\sigma dt \right]^{1/2} \\ & \leq \sum_l \left[\int_0^T \int B \left\{ \mu_* 2^{3l} |\dot{\Delta}_l(f' - f)|^2 \right. \right. \\ & \quad \left. \left. + (\mu^{1/2} - \mu'^{1/2}) 2^{3l} |\dot{\Delta}_l f_*|^2 \right\} dx dv dv_* d\sigma dt \right]^{1/2} = \|f\|_{\dot{J}_{T,2,2}^{3/2}}. \end{aligned}$$

□

3.7.2 Proof of Lemma 3.7.1

In this subsection we also give another upper bound for Γ , which is a supplementary variant of [6, Lemma 4.7] concerning the range of the index of the Sobolev space.

Proposition 3.7.10. *Let $0 < \nu < 2$, $\gamma > \max\{-3, -\nu - 3/2\}$. For any $\ell \in \mathbb{R}$ and $m \in [-\nu/2, \nu/2]$ we have*

$$\left| \left(\Gamma(f, g), h \right)_{L^2} \right| \lesssim \|f\|_{L^2} \|g\|_{H_{(\ell+\gamma+\nu)^+}^{\nu/2+m}} \|h\|_{H_{-\ell}^{\nu/2-m}}. \quad (3.7.4)$$

We decompose the kinetic factor of the cross-section into two parts,

$$\Phi(|z|) = |z|^\gamma = |z|^\gamma \varphi_0(|z|) + |z|^\gamma (1 - \varphi_0(|z|)) = \Phi_c(|z|) + \Phi_{\bar{c}}(|z|),$$

where $\varphi_0 \in C_0^\infty(\mathbb{R})$, $\text{Supp } \varphi_0 \subset [-1, 1]$; $\varphi(t) = 1$ for $|t| \leq 1/2$, and put

$$B_c = \Phi_c(|v - v_*|) b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right), \quad B_{\bar{c}} = \Phi_{\bar{c}}(|v - v_*|) b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right).$$

Accordingly, we write

$$Q(f, g) = Q_c(f, g) + Q_{\bar{c}}(f, g),$$

and

$$\Gamma(f, g) = \Gamma_c(f, g) + \Gamma_{\bar{c}}(f, g).$$

Proposition 3.7.11. *Let $0 < \nu < 2, \gamma > \max\{-3, -\nu - 3/2\}$. If $m \in [-\nu/2, \nu/2]$ then we have*

$$|(Q_c(f, g), h)| \lesssim \|f\|_{L^2} \|g\|_{H^{\nu/2+m}} \|h\|_{H^{\nu/2-m}}.$$

Remark 3.7.12. *For any $q \in [1, 2)$, let $\gamma > \max\{-3, -\nu - 3 + 3/q\}$ and $-\nu/2 \leq m \leq \min\{\nu/2, 3/2 - \nu/2\}$. Then we have*

$$|(Q_c(f, g), h)| \lesssim \|f\|_{L^q} \|g\|_{H^{\nu/2+m}} \|h\|_{H^{\nu/2-m}}.$$

This proposition is an improvement of [6, Proposition 2.1], concerning the lower bound of m , where it was assumed $m \geq \nu/2 - 1$. For the self-containedness and the convenience of readers, we repeat the detail proof, including the general case pointed out at the above remark.

For the proof of Proposition 3.7.11, we shall follow some of the arguments from [6]. First of all, by using the formula from the Appendix of [2], and as in [6], one has

$$\begin{aligned} (Q_c(f, g), h) &= \int b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) [\hat{\Phi}_c(\xi_* - \xi^-) - \hat{\Phi}_c(\xi_*)] \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma. \\ &= \int_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} \cdots d\xi d\xi_* d\sigma + \int_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \cdots d\xi d\xi_* d\sigma \\ &= A_1(f, g, h) + A_2(f, g, h), \end{aligned}$$

where $\hat{f}(\xi)$ is the Fourier transform of f with respect to $v \in \mathbb{R}^3$ and $\xi^- = \frac{1}{2}(\xi - |\xi|\sigma)$.

Then, we write $A_2(f, g, h)$ as

$$\begin{aligned} A_2 &= \int b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \hat{\Phi}_c(\xi_* - \xi^-) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma \\ &\quad - \int b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \hat{\Phi}_c(\xi_*) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma \\ &= A_{2,1}(f, g, h) - A_{2,2}(f, g, h). \end{aligned}$$

While for A_1 , we use the Taylor expansion of $\hat{\Phi}_c$ at order 2 to have

$$A_1 = A_{1,1}(f, g, h) + A_{1,2}(f, g, h)$$

where

$$A_{1,1} = \int b \xi^- \cdot (\nabla \hat{\Phi}_c)(\xi_*) \mathbf{1}_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma,$$

and $A_{1,2}(f, g, h)$ is the remaining term corresponding to the second order term in the Taylor expansion of $\hat{\Phi}_c$. The $A_{i,j}$ with $i, j = 1, 2$ are estimated by the following lemmas.

Lemma 3.7.13. *For any $q \in [1, 2]$, let $\gamma > \max\{-3, -\nu - 3 + 3/q\}$. Furthermore, assume that $m \leq 3/2 - \nu/2$ if $q < 2$. Then we have*

$$|A_{1,1}| + |A_{1,2}| \lesssim \|f\|_{L^q} \|f\|_{H^{\nu/2+m}} \|h\|_{H^{\nu/2-m}}.$$

Proof. Considering firstly $A_{1,1}$, by writing

$$\xi^- = \frac{|\xi|}{2} \left(\left(\frac{\xi}{|\xi|} \cdot \sigma \right) \frac{\xi}{|\xi|} - \sigma \right) + \left(1 - \left(\frac{\xi}{|\xi|} \cdot \sigma \right) \right) \frac{\xi}{2},$$

we see that the integral corresponding to the first term on the right hand side vanishes because of the symmetry on $\mathbb{S}^2$. Hence, we have

$$A_{1,1} = \int_{\mathbb{R}^6} K(\xi, \xi_*) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_*,$$

where

$$K(\xi, \xi_*) = \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left(1 - \left(\frac{\xi}{|\xi|} \cdot \sigma \right) \right) \frac{\xi}{2} \cdot (\nabla \hat{\Phi}_c)(\xi_*) \mathbf{1}_{|\xi^-| \leq \frac{1}{2} \langle \xi_* \rangle} d\sigma.$$

Note that $|\nabla \hat{\Phi}_c(\xi_*)| \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma+1}}$, from the Appendix of [7]. If $\sqrt{2}|\xi| \leq \langle \xi_* \rangle$, then $|\xi^-| \leq \langle \xi_* \rangle/2$ and this implies the fact that $0 \leq \theta \leq \pi/2$, and we have

$$|K(\xi, \xi_*)| \lesssim \int_0^{\pi/2} \theta^{1-\nu} d\theta \frac{\langle \xi \rangle}{\langle \xi_* \rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right).$$

On the other hand, if $\sqrt{2}|\xi| \geq \langle \xi_* \rangle$, then

$$|K(\xi, \xi_*)| \lesssim \int_0^{\pi \langle \xi_* \rangle / (2|\xi|)} \theta^{1-\nu} d\theta \frac{\langle \xi \rangle}{\langle \xi_* \rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{\nu-1}.$$

Hence we obtain

$$\begin{aligned} |K(\xi, \xi_*)| &\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left\{ \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \mathbf{1}_{\sqrt{2}|\xi| \leq \langle \xi_* \rangle} \right. \\ &\quad \left. + \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle} + \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^\nu \mathbf{1}_{\langle \xi_* \rangle \leq |\xi|/2} \right\}. \end{aligned} \quad (3.7.5)$$

Notice that

$$\begin{cases} \langle \xi \rangle \lesssim \langle \xi_* \rangle \sim \langle \xi - \xi_* \rangle & \text{on supp } \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} \\ \langle \xi \rangle \sim \langle \xi - \xi_* \rangle & \text{on supp } \mathbf{1}_{\langle \xi_* \rangle \leq |\xi|/2} \\ \langle \xi \rangle \sim \langle \xi_* \rangle \gtrsim \langle \xi - \xi_* \rangle & \text{on supp } \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2} \end{cases} \quad (3.7.6)$$

Take an $\varepsilon > 0$ such that $3 + \gamma + \nu > 3/q + \varepsilon$. Then we have

$$\frac{1}{\langle \xi_* \rangle^{3+\gamma}} \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \mathbf{1}_{\sqrt{2}|\xi| \leq \langle \xi_* \rangle} \lesssim \frac{\langle \xi - \xi_* \rangle^{\nu/2+m} \langle \xi \rangle^{\nu/2-m} \langle \xi \rangle^{-3/2-\varepsilon}}{\langle \xi_* \rangle^{3/2+\gamma+\nu-\varepsilon}}$$

in view of $\nu/2 - m - 3/2 - \varepsilon < 1$. Replacing the factor $(\langle \xi \rangle / \langle \xi_* \rangle)^\nu \mathbf{1}_{\langle \xi_* \rangle \leq |\xi|/2}$ on the right hand side of (3.7.5) by

$$\frac{\langle \xi \rangle^{\nu/2-m} \langle \xi - \xi_* \rangle^{\nu/2+m}}{\langle \xi_* \rangle^\nu},$$

we obtain

$$\begin{aligned} |K(\xi, \xi_*)| &\lesssim \frac{\langle \xi - \xi_* \rangle^{\nu/2+m} \langle \xi \rangle^{\nu/2-m} \langle \xi \rangle^{-3/2-\varepsilon}}{\langle \xi_* \rangle^{3/2+\gamma+\nu-\varepsilon}} + \frac{\langle \xi \rangle^{\nu/2-m} \langle \xi - \xi_* \rangle^{\nu/2+m}}{\langle \xi_* \rangle^{3+\gamma+\nu}} \\ &\quad + \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* \rangle}}{\langle \xi_* \rangle^{3+\gamma+\nu/2-m}} \frac{\langle \xi \rangle^{\nu/2-m} \langle \xi - \xi_* \rangle^{\nu/2+m}}{\langle \xi - \xi_* \rangle^{\nu/2+m}}. \end{aligned} \quad (3.7.7)$$

Putting $\tilde{g}(\xi) = \langle \xi \rangle^{\nu/2+m} \hat{g}(\xi)$, $\tilde{h}(\xi) = \langle \xi \rangle^{\nu/2-m} \hat{h}(\xi)$, we have

$$\begin{aligned} |A_{1,1}|^2 &\lesssim \left(\int_{\mathbb{R}^3} \frac{|\tilde{h}(\xi)|}{\langle \xi \rangle^{3/2+\varepsilon}} \left(\int_{\mathbb{R}^3} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3/2+\gamma+\nu-\varepsilon}} |\tilde{g}(\xi - \xi_*)| d\xi_* \right) d\xi \right)^2 \\ &\quad + \int_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma+\nu}} |\tilde{g}(\xi - \xi_*)|^2 d\xi d\xi_* \int_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma+\nu}} |\tilde{h}(\xi)|^2 d\xi d\xi_* \\ &\quad + \int_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|^2}{\langle \xi_* \rangle^{6+2\gamma+\nu-2m}} \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* \rangle}}{\langle \xi - \xi_* \rangle^{\nu+2m}} d\xi d\xi_* \int_{\mathbb{R}^6} |\tilde{g}(\xi - \xi_*)|^2 |\tilde{h}(\xi)|^2 d\xi d\xi_* \\ &= \mathcal{K}^2 + \mathcal{AB} + \mathcal{DE}, \end{aligned}$$

by the Cauchy-Schwarz inequality. It follows from Hölder inequality that

$$\begin{aligned} \mathcal{K} &\lesssim \|g\|_{H^{\nu/2+m}} \|h\|_{H^{\nu/2-m}} \left(\int_{\mathbb{R}^3} \frac{|\hat{f}(\xi_*)|^2}{\langle \xi_* \rangle^{3+2(\gamma+\nu-\varepsilon)}} d\xi_* \right)^{1/2} \\ &\lesssim \|f\|_{L^q} \|g\|_{H^{\nu/2+m}} \|h\|_{H^{\nu/2-m}}, \end{aligned}$$

where we have used the fact that

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|\hat{f}(\xi_*)|^2}{\langle \xi_* \rangle^\ell} d\xi_* &\lesssim \|f\|_{L^q}^2 \\ \text{if } \ell &> -3 + 6/q \text{ for } q \in [1, 2) \text{ and } \ell \geq 0 \text{ for } q = 2 \end{aligned} \quad (3.7.8)$$

by means of $\|\hat{f}\|_{L^{q'}} \leq \|f\|_{L^q}$ with $1/q + 1/q' = 1$ for $q \in [1, 2]$.

Since it follows from $3 + \gamma + \nu > 3/q$ that $\langle \xi_* \rangle^{-(3+\gamma+\nu)} \in L^q$, the Hölder inequality again shows

$$\mathcal{A} \lesssim \int_{\mathbb{R}^3} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma+\nu}} d\xi_* \|g\|_{H^{\nu/2+m}}^2 \lesssim \|f\|_{L^q} \|g\|_{H^{\nu/2+m}}^2, \quad \mathcal{B} \lesssim \|f\|_{L^q} \|h\|_{H^{\nu/2-m}}^2.$$

Note that

$$\int \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* \rangle}}{\langle \xi - \xi_* \rangle^{\nu+2m}} d\xi \lesssim \begin{cases} \frac{1}{\langle \xi_* \rangle^{-3+\nu+2m}} & \text{if } \nu/2 + m < 3/2 \\ \log \langle \xi_* \rangle & \text{if } \nu/2 + m \geq 3/2. \end{cases}$$

Since $3 + 2(\gamma + \nu) > 0$ when $q = 2$, together with $6 + 2\gamma + \nu - 2m > 0$, we get $\mathcal{D} \leq \|f\|_{L^2}^2$, which concludes the desired bound for $A_{1,1}$ when $q = 2$. In the case where $q \in [1, 2)$, it follows from (3.7.8) that $\mathcal{D} \lesssim \|f\|_{L^q}^2$ because of $\nu/2 + m \leq 3/2$.

Now we consider $A_{1,2}(f, g, h)$, which comes from the second order term of the Taylor expansion. Note that

$$A_{1,2} = \int b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \int_0^1 d\tau (\nabla^2 \hat{\Phi}_c)(\xi_* - \tau\xi^-)(\xi^-)^2 \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \bar{h}(\xi) d\sigma d\xi d\xi_*.$$

Again from the Appendix of [7], we have

$$|(\nabla^2 \hat{\Phi}_c)(\xi_* - \tau\xi^-)| \lesssim \frac{1}{\langle \xi_* - \tau\xi^- \rangle^{3+\gamma+2}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma+2}},$$

because $|\xi^-| \leq \langle \xi_* \rangle/2$. Similar to $A_{1,1}$, we can obtain

$$|A_{1,2}| \lesssim \int_{\mathbb{R}^6} \tilde{K}(\xi, \xi_*) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \bar{h}(\xi) d\xi d\xi_*,$$

where $\tilde{K}(\xi, \xi_*)$ has the following upper bound

$$\begin{aligned} \tilde{K}(\xi, \xi_*) &\lesssim \int_0^{\min(\pi/2, \pi\langle \xi_* \rangle/(2|\xi|))} \theta^{1-\nu} d\theta \frac{\langle \xi \rangle^2}{\langle \xi_* \rangle^{3+\gamma+2}} \\ &\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left\{ \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^2 \mathbf{1}_{\sqrt{2}|\xi| \leq \langle \xi_* \rangle} + \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2} \right. \\ &\quad \left. + \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^\nu \mathbf{1}_{\langle \xi_* \rangle \leq |\xi|/2} \right\}, \end{aligned} \quad (3.7.9)$$

from which we obtain the same inequality as (3.7.7) for $\tilde{K}(\xi, \xi_*)$. Hence we obtain the desired bound for $A_{1,2}$. And this completes the proof of the lemma. $\square$

Lemma 3.7.14. *For any $q \in [1, 2]$, let $\gamma > \max\{-3, -\nu - 3 + 3/q\}$. Furthermore, assume that $m \leq 3/2 - \nu/2$ if $q < 2$. Then*

$$|A_{2,1}| + |A_{2,2}| \lesssim \|f\|_{L^q} \|f\|_{H^{\nu/2+m}} \|h\|_{H^{\nu/2-m}}.$$

Proof. In view of the definition of $A_{2,2}$, the fact that $|\xi| \sin(\theta/2) = |\xi^-| \geq \langle \xi_* \rangle/2$ and $\theta \in [0, \pi/2]$ imply $\sqrt{2}|\xi| \geq \langle \xi_* \rangle$. We can then directly compute the spherical

integral appearing inside $A_{2,2}$ together with $\hat{\Phi}_c$ as follows:

$$\begin{aligned} \left| \int b \left(\frac{\xi}{|\xi|} \cdot \sigma \right) \hat{\Phi}_c(\xi_*) \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} d\sigma \right| &\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \frac{\langle \xi \rangle^\nu}{\langle \xi_* \rangle^\nu} \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle} \\ &\lesssim \frac{\langle \xi \rangle^{\nu/2-m} \langle \xi - \xi_* \rangle^{\nu/2+m}}{\langle \xi_* \rangle^{3+\gamma+\nu}} \\ &\quad + \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* \rangle}}{\langle \xi_* \rangle^{3+\gamma+\nu/2-m}} \langle \xi \rangle^{\nu/2-m} \langle \xi - \xi_* \rangle^{\nu/2+m}, \end{aligned} \quad (3.7.10)$$

which yields the desired estimate for $A_{2,2}$.

We now turn to

$$A_{2,1} = \int b \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \hat{\Phi}_c(\xi_* - \xi^-) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\sigma d\xi d\xi_*.$$

Firstly, note that we can work on the set $|\xi_* \cdot \xi^-| \geq \frac{1}{2} |\xi^-|^2$. In fact, on the complementary of this set, we have $|\xi_* \cdot \xi^-| \leq \frac{1}{2} |\xi^-|^2$ so that $|\xi_* - \xi^-| \gtrsim |\xi_*|$, and in this case, we can proceed in the same way as for $A_{2,2}$. Therefore, it suffices to estimate

$$A_{2,1,p} = \int b \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \mathbf{1}_{|\xi_* \cdot \xi^-| \geq \frac{1}{2} |\xi^-|^2} \hat{\Phi}_c(\xi_* - \xi^-) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\sigma d\xi d\xi_*.$$

By

$$\mathbf{1} = \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi - \xi_* \rangle > \langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle < |\xi|/2}$$

we decompose

$$A_{2,1,p} = A_{2,1,p}^{(1)} + A_{2,1,p}^{(2)} + A_{2,1,p}^{(3)}.$$

On the sets for above integrals, we have $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi_* \rangle$, because $|\xi^-| \lesssim |\xi_*|$ that follows from $|\xi^-|^2 \leq 2|\xi_* \cdot \xi^-| \lesssim |\xi^-| |\xi_*|$. Furthermore, on the sets for $A_{2,1,p}^{(1)}$ and $A_{2,1,p}^{(2)}$ we have $\langle \xi \rangle \sim \langle \xi_* \rangle$, so that $\sup \left(b \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \right) \lesssim \mathbf{1}_{|\xi^-| \leq |\xi|/\sqrt{2}}$ and $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi \rangle$. Hence we have, in view of $\nu/2 - m \geq 0$,

$$\begin{aligned} |A_{2,1,p}^{(1)}|^2 &\lesssim \int \frac{|\hat{\Phi}_c(\xi_* - \xi^-)|^2 |\hat{f}(\xi_*)|^2 \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* - \xi^- \rangle}}{\langle \xi_* - \xi^- \rangle^{\nu-2m} \langle \xi - \xi_* \rangle^{\nu+2m}} d\xi d\xi_* d\sigma \\ &\quad \times \int |\langle \xi - \xi_* \rangle^{\nu/2+m} \hat{g}(\xi - \xi_*)|^2 |\langle \xi \rangle^{\nu/2-m} \hat{h}(\xi)|^2 d\sigma d\xi d\xi_*. \end{aligned}$$

Note that $3 + 2(\gamma + \nu) > 0$ when $q = 2$, together with $6 + 2\gamma + \nu - 2m > 0$. Then, with $u = \xi_* - \xi^-$ we have

$$\begin{aligned} |A_{2,1,p}^{(1)}|^2 &\lesssim \int |\hat{f}(\xi_*)|^2 \left\{ \sup_u \langle u \rangle^{-(6+2\gamma+\nu-2m)} \int \frac{\mathbf{1}_{\langle \xi^+ - u \rangle \leq \langle u \rangle}}{\langle \xi^+ - u \rangle^{\nu+2m}} d\xi^+ \right\} d\xi_* \\ &\quad \times \|g\|_{H^{\nu/2+m}}^2 \|h\|_{H^{\nu/2-m}}^2 \\ &\lesssim \|f\|_{L^2}^2 \|g\|_{H^{\nu/2+m}}^2 \|h\|_{H^{\nu/2-m}}^2, \end{aligned}$$

because $d\xi \sim d\xi^+$ on the support of $\mathbf{1}_{|\xi^-| \leq |\xi|/\sqrt{2}}$. In the case where $q < 2$, we use the condition $\nu/2 + m \leq 3/2$. If $q = 1$ then $\gamma + \nu > 0$, and by the change of variables $\xi_* - \xi^- \rightarrow u$ we have

$$\begin{aligned} |A_{2,1,p}^{(1)}|^2 &\lesssim \|\hat{f}\|_{L^\infty}^2 \|g\|_{H^{\nu/2+m}}^2 \|h\|_{H^{\nu/2-m}}^2 \int \langle u \rangle^{-(6+2\gamma+\nu-2m)} \int \frac{\mathbf{1}_{\langle w \rangle \leq \langle u \rangle}}{\langle w \rangle^{\nu+2m}} dw du \\ &\lesssim \|f\|_{L^1}^2 \|g\|_{H^{\nu/2+m}}^2 \|h\|_{H^{\nu/2-m}}^2. \end{aligned}$$

If $1 < q < 2$, then $3 + \gamma + \nu > 3/q$ and by the Hölder inequality and the change of variables $u = \xi_* - \xi^-$ we have

$$\begin{aligned} |A_{2,1,p}^{(1)}|^2 &\lesssim \|g\|_{H^{\nu/2+m}}^2 \|h\|_{H^{\nu/2-m}}^2 \left(\int |\hat{f}(\xi_*)|^{q/(q-1)} d\xi_* \right)^{2(q-1)/q} \\ &\quad \times \left(\int \left(\langle u \rangle^{-(6+2\gamma+\nu-2m)} \int \frac{\mathbf{1}_{\langle \xi^+ - u \rangle \leq \langle u \rangle}}{\langle \xi^+ - u \rangle^{\nu+2m}} d\xi^+ \right)^{q/(2-q)} du \right)^{2/q-1} \\ &\lesssim \|f\|_{L^q}^2 \|g\|_{H^{\nu/2+m}}^2 \|h\|_{H^{\nu/2-m}}^2. \end{aligned}$$

As for $A_{2,1,p}^{(2)}$ we have by the Cauchy-Schwarz inequality

$$\begin{aligned} |A_{2,1,p}^{(2)}|^2 &\lesssim \int \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi_* - \xi^- \rangle^\nu} |\langle \xi - \xi_* \rangle^{\nu/2+m} \hat{g}(\xi - \xi_*)|^2 d\sigma d\xi d\xi_* \\ &\quad \times \int \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi_* - \xi^- \rangle^\nu} |\langle \xi \rangle^{\nu/2-m} \hat{h}(\xi)|^2 d\sigma d\xi d\xi_*. \end{aligned}$$

Since it follows from $3 + \gamma + \nu < 3/q$ and Hölder inequality that

$$\int \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi_* - \xi^- \rangle^\nu} d\xi_* d\sigma \lesssim \|f\|_{L^q},$$

we have the desired estimates for $A_{2,1,p}^{(2)}$.

On the set $A_{2,1,p}^{(3)}$ we have $\langle \xi \rangle \sim \langle \xi - \xi_* \rangle$. Hence

$$\begin{aligned} |A_{2,1,p}^{(3)}|^2 &\lesssim \int b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi \rangle^\nu} |\langle \xi - \xi_* \rangle^{\nu/2+m} \hat{g}(\xi - \xi_*)|^2 d\sigma d\xi d\xi_* \\ &\quad \times \int b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi \rangle^\nu} |\langle \xi \rangle^{\nu/2-m} \hat{h}(\xi)|^2 d\sigma d\xi d\xi_*. \end{aligned}$$

We use the change of variables in ξ_* , $u = \xi_* - \xi^-$. Note that $|\xi^-| \geq \frac{1}{2}\langle u + \xi^- \rangle$ implies $|\xi^-| \geq \langle u \rangle/\sqrt{10}$. If $q = 1$ then $\gamma + \nu > 0$ and we have

$$\begin{aligned} &\int b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi \rangle^\nu} d\sigma d\xi_* \\ &\lesssim \|\hat{f}\|_{L^\infty} \int \left(\frac{|\xi|}{\langle u \rangle} \right)^\nu \langle u \rangle^{-(3+\gamma)} \langle \xi \rangle^{-\nu} du \lesssim \|f\|_{L^1}. \end{aligned}$$

On the other hand, if $q > 1$ then this integral is upper bounded by

$$\begin{aligned}
& \int b \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)|}{\langle \xi \rangle^{\nu/q} \langle \xi_* - \xi^- \rangle^{\nu/q'}} \frac{\langle \xi_* \rangle^{\nu/q'} |\hat{f}(\xi_*)|}{\langle \xi \rangle^{\nu/q'}} d\sigma d\xi_* \\
& \leq \left(\int b \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)|^q}{\langle \xi \rangle^{\nu} \langle \xi_* - \xi^- \rangle^{\nu q/q'}} d\sigma d\xi_* \right)^{1/q} \\
& \quad \times \left(\int b \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \frac{\langle \xi_* \rangle^{\nu} |\hat{f}(\xi_*)|^{q'}}{\langle \xi \rangle^{\nu}} d\sigma d\xi_* \right)^{1/q'} \\
& \leq \left(\int b \mathbf{1}_{|\xi^-| \gtrsim \langle u \rangle} \frac{|\hat{\Phi}_c(u)|^q}{\langle \xi \rangle^{\nu} \langle u \rangle^{\nu q/q'}} d\sigma du \right)^{1/q} \|\hat{f}\|_{L^{q'}} \\
& \lesssim \int \frac{du}{\langle u \rangle^{q(3+\gamma+\nu)}} \|f\|_{L^q},
\end{aligned}$$

where $1/q + 1/q' = 1$. Hence we also obtain the desired estimates for $A_{2,1,p}^{(3)}$. The proof of the lemma is complete. $\square$

Proposition 3.7.11 is then a direct consequence of Lemmas 3.7.13 and 3.7.14.

The following lemma is a variant of [6, Lemma 4.5], where the roles of g and h are exchanged.

Lemma 3.7.15. *Let $0 < \nu < 2$ and $\gamma > \max\{-3, -\nu - 3/2\}$. Then for any $\alpha \geq 0$ and any $\beta, \beta' \in \mathbb{R}$ we have*

$$\begin{aligned}
& \left| \left(\Gamma(f, g), h \right)_{L^2} - \left(Q(\mu^{1/2} f, g), h \right)_{L^2} \right| \\
& \lesssim \|\mu^{1/10} f\|_{L^2}^{1/2} \|g\|_{L^2_{(\nu+\gamma)/2-\beta}} \left(\mathcal{D}(\mu^{1/4} |f|, \langle v \rangle^\beta h) \right)^{1/2} \\
& \quad + \|f\|_{L^2_{-\alpha}} \|g\|_{L^2_{(\nu+\gamma)/2+\alpha-\beta'}} \|h\|_{L^2_{(\nu+\gamma)/2+\beta'}} \\
& \quad + \|\mu^{1/10} f\|_{L^2} \|\mu^{1/10} g\|_{L^2} \|\mu^{1/10} h\|_{H^{\nu/2}}, \tag{3.7.11}
\end{aligned}$$

where

$$\mathcal{D}(f, g) = \iiint_{\mathbb{R}_v^3 \times \mathbb{R}_{v_*}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) f_*(g - g')^2 dv dv_* d\sigma.$$

Remark 3.7.16. *If $\gamma > -5/2$ then the last term of the right hand side of (3.7.11) disappears.*

Since it follows from Lemma 2.12 of [6] and its proof that

$$\mathcal{D}(\mu^{1/4} |f|, \langle v \rangle^\beta h) \lesssim \|\mu^{1/10} f\|_{L^2} \|\langle v \rangle^\beta h\|^2 \lesssim \|\mu^{1/10} f\|_{L^2} \|h\|_{H^{\nu/2}_{(\nu+\gamma)/2+\beta}}^2,$$

we have the following;

Corollary 3.7.17. *Let $0 < \nu < 2$ and $\gamma > \max\{-3, -\nu - 3/2\}$. Then for any $\alpha \geq 0$ and any $\beta, \beta' \in \mathbb{R}$ we have*

$$\begin{aligned} & \left| \left(\Gamma(f, g), h \right)_{L^2} - \left(Q(\mu^{1/2} f, g), h \right)_{L^2} \right| \\ & \lesssim \|\mu^{1/10} f\|_{L^2} \|g\|_{L^2_{(\gamma+\nu)/2-\beta}} \|h\|_{H^{\nu/2}_{(\gamma+\nu)/2+\beta}} \\ & \quad + \|f\|_{L^2_{-\alpha}} \|g\|_{L^2_{(\nu+\gamma)/2+\alpha-\beta'}} \|h\|_{L^2_{(\nu+\gamma)/2+\beta'}}. \end{aligned} \quad (3.7.12)$$

Proof of Lemma 3.7.15. We write

$$\begin{aligned} & \left(\Gamma(f, g), h \right)_{L^2} - \left(Q(\mu^{1/2} f, g), h \right)_{L^2} = \int B \left(\mu_*'^{1/2} - \mu_*^{1/2} \right) (f_*) gh' d\sigma dv_* dv \\ & = 2 \int B \left((\mu'_*)^{1/4} - \mu_*^{1/4} \right) (\mu_*^{1/4} f_*) gh d\sigma dv_* dv \\ & \quad + \int B \left((\mu'_*)^{1/4} - \mu_*^{1/4} \right)^2 f_* gh' d\sigma dv_* dv \\ & \quad + 2 \int B \left((\mu'_*)^{1/4} - \mu_*^{1/4} \right) (\mu_*^{1/4} f_*) g(h' - h) d\sigma dv_* dv \\ & = D_1 + D_2 + D_3. \end{aligned}$$

Note that

$$\begin{aligned} & \left((\mu'_*)^{1/4} - \mu_*^{1/4} \right)^2 \leq 2 \left((\mu'_*)^{1/8} - \mu_*^{1/8} \right)^2 \left((\mu'_*)^{1/4} + \mu_*^{1/4} \right) \\ & \lesssim \min(|v - v_*|, 1) \min(|v' - v'_*|, 1) (\mu'_*)^{1/4} + \left(\min(|v' - v'_*|, 1) \right)^2 \mu_*^{1/4}. \end{aligned}$$

By this decomposition we estimate

$$|D_2| \lesssim D_2^{(1)} + D_2^{(2)}$$

Since $|v - v_*| \sim |v - v'_*|$ on $\text{supp } b$, we have

$$\begin{aligned} \langle v_* \rangle & \lesssim \langle v - v_* \rangle + \langle v \rangle \lesssim \langle v - v'_* \rangle \left(1 + \frac{\langle v \rangle}{\langle v - v'_* \rangle} \right) \\ & \lesssim \langle v - v'_* \rangle \langle v'_* \rangle \lesssim \langle v - v_* \rangle \langle v'_* \rangle \text{ on } \text{supp } b, \end{aligned}$$

and hence

$$(\mu'_*)^{1/4} |f_*| \lesssim \langle v'_* \rangle^\alpha (\mu'_*)^{1/4} |\langle v_* \rangle^{-\alpha} f_*| \langle v - v_* \rangle^\alpha \text{ for any } \alpha \geq 0.$$

Noting $\langle v \rangle^{\beta'} \lesssim \langle v' - v'_* \rangle^{\beta'} \langle v'_* \rangle^{|\beta'|}$ on $\text{supp } b$ for any $\beta' \in \mathbb{R}$, by the Cauchy-

Schwarz inequality we have

$$\begin{aligned}
(D_2^{(1)})^2 &\lesssim \int B|v - v_*|^{2\alpha} \mathbf{1}_{|v-v_*| \geq 1} \min(|v - v_*|^2 \theta^2, 1) \left(\frac{f_* g}{\langle v_* \rangle^\alpha \langle v \rangle^{\beta'}} \right)^2 d\sigma dv_* dv \\
&\quad \times \int B|v - v_*|^{2\beta'} \mathbf{1}_{|v-v_*| \geq 1} \min(|v - v_*|^2 \theta^2, 1) (\mu_*^{1/8} h)^2 d\sigma dv_* dv \\
&\quad + \int B|v - v_*|^2 \mathbf{1}_{|v-v_*| < 1} \theta^2 (\mu_*^{1/100} \mu^{1/100} h)^2 d\sigma dv_* dv \\
&\quad \times \int B|v - v_*|^2 \mathbf{1}_{|v-v_*| < 1} \theta^2 (\mu_*^{1/100} f_* \mu^{1/100} g)^2 d\sigma dv_* dv \\
&\lesssim \|f\|_{L^2_{-\alpha}}^2 \|g\|_{L^2_{(\nu+\gamma)/2+\alpha-\beta'}}^2 \|h\|_{L^2_{(\nu+\gamma)/2+\beta'}}^2,
\end{aligned}$$

where we used the fact that $\langle v'_* \rangle \sim \langle v' \rangle \sim \langle v \rangle \sim \langle v_* \rangle$ on $\text{supp } b \cap \mathbf{1}_{|v-v_*| < 1}$, and $2\gamma + 4 > -3$. As for $D_2^{(2)}$, we have

$$(D_2^{(2)})^2 \lesssim \|\mu^{1/10} f\|_{L^2}^2 \|g\|_{L^2_{(\nu+\gamma)/2-\beta'}}^2 \|h\|_{L^2_{(\nu+\gamma)/2+\beta'}}^2,$$

thanks to the factor $\mu_*^{1/4}$ instead of $\mu_*'^{1/4}$.

By the Cauchy-Schwarz inequality we have for any $\beta \in \mathbb{R}$

$$\begin{aligned}
|D_3| &\lesssim \left(\int B \left((\mu'_*)^{1/4} - \mu_*^{1/4} \right)^2 |\mu_*^{1/4} f_*| (\langle v \rangle^{-\beta} g)^2 d\sigma dv_* dv \right)^{1/2} \\
&\quad \times \left(\int B \mu_*^{1/4} |f_*| \langle v \rangle^{2\beta} (h' - h)^2 d\sigma dv_* dv \right)^{1/2} \\
&= \left(\tilde{D}_3(f, \langle v \rangle^\beta g) \right)^{1/2} \left(\mathcal{D}_\beta(\mu^{1/4} f, h) \right)^{1/2}.
\end{aligned}$$

We have

$$\begin{aligned}
\mathcal{D}_\beta(\mu^{1/4} f, h) &\leq 2 \left(\mathcal{D}(|\mu^{1/4} f|, \langle v \rangle^\beta h) + \int B |\mu_*^{1/4} f_*| \left(\langle v \rangle^\beta - \langle v' \rangle^\beta \right)^2 h^2 dv dv_* d\sigma \right) \\
&\lesssim \mathcal{D}(|\mu^{1/4} f|, \langle v \rangle^\beta h) + \|\mu^{1/8} f\|_{L^2} \|h\|_{L^2_{\beta+\gamma/2}}^2,
\end{aligned}$$

because it follows from the same arguments in the proof of Lemma 2.12 in [6] that

$$\left| \langle v \rangle^\beta - \langle v' \rangle^\beta \right| \lesssim \sin \frac{\theta}{2} \left(\langle v \rangle^\beta \langle v_* \rangle^{2|\beta|+1} \mathbf{1}_{|v-v_*| > 1} + \langle v \rangle^{\beta-1} |v - v_*| \mathbf{1}_{|v-v_*| \leq 1} \right). \quad (3.7.13)$$

The similar method as for $D_2^{(2)}$ shows

$$\tilde{D}_3 \lesssim \|\mu^{1/10} f\|_{L^2} \|g\|_{L^2_{(\nu+\gamma)/2-\beta}}^2.$$

To estimate D_1 we use the Taylor formula

$$\begin{aligned}
(\mu'_*)^{1/4} - \mu_*^{1/4} &= (\nabla \mu^{1/4})(v_*) \cdot (v'_* - v_*) \\
&\quad + \int_0^1 (1 - \tau) (\nabla^2 \mu^{1/4})(v_* + \tau(v'_* - v_*)) (v'_* - v_*)^2 d\tau.
\end{aligned}$$

By writing

$$v'_* - v_* = \frac{|v - v_*|}{2} \{(\sigma \cdot \mathbf{k})\mathbf{k} - \sigma\} + \frac{v - v_*}{2} (1 - \mathbf{k} \cdot \sigma), \quad \mathbf{k} = \frac{v - v_*}{|v - v_*|},$$

we see that the integral corresponding the first term on the integral of D_1 vanishes because of the symmetry on $\mathbb{S}^2$. Therefore, we have

$$\begin{aligned} |D_1| &\lesssim \int B \min(|v - v_*| \theta^2, 1) \mu_*^{1/4} f_* g h d\sigma dv_* dv \\ &\quad + \int B \min(|v - v_*|^2 \theta^2, 1) \mu_*^{1/4} f_* g h d\sigma dv_* dv \\ &\lesssim \int B \mathbf{1}_{|v - v_*| \geq 1} \min(|v - v_*|^2 \theta^2, 1) \mu_*^{1/4} f_* g h d\sigma dv_* dv \\ &\quad + \int \mathbf{1}_{|v - v_*| < 1} |v - v_*|^{\gamma+1+\nu/2} \mu_*^{1/10} f_*(\mu^{1/10} g) \frac{\mu^{1/10} h}{|v - v_*|^{\nu/2}} dv_* dv \\ &\lesssim \|\mu^{1/10} f\|_{L^2} \|g\|_{L^2_{(\nu+\gamma)/2-\beta'}} \|h\|_{L^2_{(\nu+\gamma)/2+\beta'}} \\ &\quad + \|\mu^{1/10} f\|_{L^2} \|\mu^{1/10} g\|_{L^2} \|\mu^{1/10} h\|_{H^{\nu/2}} \end{aligned}$$

in view of $2\gamma + \nu + 2 > -3$. $\square$

Since we may replace $B \mathbf{1}_{|v - v_*| \geq 1}$ by $B_{\bar{c}}$ in the proof of Lemma 3.7.15, under the condition $\gamma > -3$, it follows that for any $\beta, \beta' \in \mathbb{R}$ and for any $\alpha \geq 0$ we have

$$\begin{aligned} &\left| \left(\Gamma_{\bar{c}}(f, g), h \right)_{L^2} - \left(Q_{\bar{c}}(\mu^{1/2} f, g), h \right)_{L^2} \right| \\ &\lesssim \|\mu^{1/10} f\|_{L^2} \|\langle v \rangle^{-\beta} g\|_{L^2_{(\nu+\gamma)/2}} \|\langle v \rangle^{\beta} h\| \\ &\quad + \|f\|_{L^2_{-\alpha}} \|g\|_{L^2_{(\nu+\gamma)/2+\alpha-\beta'}} \|h\|_{L^2_{(\nu+\gamma)/2+\beta'}}. \end{aligned} \quad (3.7.14)$$

By means of Lemma 3.2 of [8], we get, for any $\alpha \geq 0$ and any $\beta' \in \mathbb{R}$,

$$\begin{aligned} &\left| \left(\Gamma_{\bar{c}}(f, g), h \right)_{L^2} \right| \lesssim \|\mu^{1/10} f\|_{L^2} \|g\| \|h\| \\ &\quad + \|f\|_{L^2_{-\alpha}} \|g\|_{L^2_{(\nu+\gamma)/2+\alpha-\beta'}} \|h\|_{L^2_{(\nu+\gamma)/2+\beta'}}. \end{aligned} \quad (3.7.15)$$

Now we write

$$\begin{aligned}
\left(\Gamma_c(f, g), h\right)_{L^2} &= \int B_c \mu_*^{1/2} (f'_* g' - f_* g) h dv dv_* d\sigma \\
&= \int B_c \left(\mu_*^{1/4} - \mu_*'^{1/4}\right) (\mu^{1/4} f)_* g h dv dv_* d\sigma \\
&\quad + \int B_c \left(\mu_*^{1/4} - \mu_*'^{1/4}\right)^2 f_* g h' dv dv_* d\sigma \\
&\quad + \int B_c \left(\mu_*^{1/4} - \mu_*'^{1/4}\right) (\mu^{1/4} f)_* g (h' - h) dv dv_* d\sigma \\
&\quad + \int B_c \mu_*^{1/4} \left((\mu^{1/4} f)'_* g' - (\mu^{1/4} f)_* g\right) h dv dv_* d\sigma \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

The estimation for I_4 is just the same as in the arguments starting from the line 14 at the page 1021 of [8], by replacing f by $\mu^{1/4} f$. Then we have $|I_4| \lesssim \|\mu^{1/10} f\|_{L^2} \|g\| \|h\|$. On the other hand, the estimations for I_1, I_2, I_3 are quite the same as those for D_1, D_2, D_3 , respectively, in the proof of Lemma 3.7.15. Therefore, $\left(\Gamma_c(f, g), h\right)_{L^2}$ has the same bound as the right hand side of (3.7.15). Thus the proof of Lemma 3.7.1 is complete.

It remains to prove Proposition 3.7.10. To this end we state a variant of [6, Proposition 2.5] where the roles of g, h are exchanged.

Proposition 3.7.18. *Let $0 < \nu < 2$ and $\gamma > \max\{-3, -\nu - 3/2\}$. For any $\ell, \beta, \delta \in \mathbb{R}$ and any small $\varepsilon > 0$*

$$\left| \left(\langle v \rangle^\ell Q_c(f, g) - Q_c(f, \langle v \rangle^\ell g), h \right) \right| \lesssim \|f\|_{L_{\ell-1-\beta-\delta}^2} \|g\|_{L_\beta^2} \|h\|_{H_\delta^{(\nu-1+\varepsilon)^+}}.$$

Proof. It suffices to write

$$\begin{aligned}
\left(\langle v \rangle^\ell Q_c(f, g) - Q_c(f, \langle v \rangle^\ell g), h \right) &= \int B_c \left(\langle v' \rangle^\ell - \langle v \rangle^\ell \right) f_* g h dv dv_* d\sigma \\
&\quad + \int B_c \left(\langle v' \rangle^\ell - \langle v \rangle^\ell \right) f_* g (h' - h) dv dv_* d\sigma \\
&= J_1 + J_2.
\end{aligned}$$

The estimation for J_2 is quite the same as in the proof of [6, Proposition 2.5] As for J_1 , we use the Taylor expansion, with $v'_\tau = v + \tau(v' - v)$,

$$\langle v' \rangle^\ell - \langle v \rangle^\ell = \nabla \left(\langle v \rangle^\ell \right) \cdot (v' - v) + \int_0^1 (1 - \tau) \nabla^2 \left(\langle v'_\tau \rangle^\ell \right) d\tau (v' - v)^2. \quad (3.7.16)$$

Since we have

$$v' - v = \frac{|v - v_*|}{2} (\sigma - (\mathbf{k} \cdot \sigma) \mathbf{k}) + \frac{v - v_*}{2} (1 - \mathbf{k} \cdot \sigma)$$

and the integral corresponding to the first term vanishes because of the symmetry on $\mathbb{S}^2$, it follows that

$$\begin{aligned}
|J_1| &\lesssim \left| \int b(\cos \theta) \Phi_c \sin^2(\theta/2) |v - v_*| \left(|\nabla \left(\langle v \rangle^\ell \right)| \right. \right. \\
&\quad \left. \left. + \int_0^1 |\nabla^2 \left(\langle v'_\tau \rangle^\ell \right)| d\tau \right) |f_* g h| dv dv_* d\sigma \right| \\
&\lesssim \int_{|v-v_*| \lesssim 1} |v - v_*|^{\gamma+1+(\nu-1+\varepsilon)^+} |\langle v_* \rangle^{\ell-1-\beta-\delta} f_*| \\
&\quad \times |(\langle v \rangle^\beta g)| \frac{|\langle v \rangle^\delta h|}{|v - v_*|^{(\nu-1+\varepsilon)^+}} dv dv_* \\
&\lesssim \|f\|_{L_{\ell-1-\beta-\delta}^2} \|g\|_{L_\beta^2}^2 \|h\|_{H_\delta^{(\nu-1+\varepsilon)^+}}^2,
\end{aligned}$$

which, together with the estimate for J_2 , gives the desired estimate. $\square$

Proof of Proposition 3.7.10. Here we only prove the case $m \in [-\nu/2, 0]$ because the other case was essentially given in [6, Lemma 4.7]. Note

$$\begin{aligned}
(Q_c(f, g), h) &= (Q_c(f, \langle v \rangle^\ell g), \langle v \rangle^{-\ell} h) \\
&\quad + (\langle v \rangle^\ell Q_c(f, \langle v \rangle^\ell g) - Q_c(f, \langle v \rangle^\ell g), \langle v \rangle^{-\ell} h).
\end{aligned}$$

It follows from Propositions 3.7.11 and 3.7.18 with $\beta = \ell$, $\delta = 0$ that

$$|(Q_c(f, g), h)| \lesssim \|f\|_{L^2} \|g\|_{H_\ell^{\nu/2+m}} \|h\|_{H_{-\ell}^{\nu/2-m}},$$

where we have used $m \leq 0$. By means of [3, Theorem 2.1] (see also (2.1) of [6]) we have for any $m, \ell \in \mathbb{R}$,

$$|(Q_{\tilde{c}}(f, g), h)| \lesssim \|f\|_{L_{\ell^++(\gamma+\nu)^+}^1} \|g\|_{H_{(\ell+\gamma+\nu)^+}^{\nu/2+m}} \|h\|_{H_{-\ell}^{\nu/2-m}}.$$

Above two estimates and Corollary 3.7.17 concludes (3.7.4) when $-\nu/2 \leq m \leq 0$. $\square$

3.7.3 Commutator estimates with moments

Let $W_\delta(v) = \langle \delta v \rangle^{-N}$ for $0 < \delta < 1$ and $N \geq 1$.

$$\begin{aligned}
&(\mathcal{L}_1(g), W_\delta^2 g)_{L^2(\mathbb{R}_v^3)} - (\mathcal{L}_1(W_\delta g), W_\delta g)_{L^2(\mathbb{R}_v^3)} \\
&= \int B \mu_*^{1/2} \mu_*'^{1/2} (W_\delta - W_\delta') g' W_\delta g dv dv_* d\sigma \\
&= \frac{1}{2} \int B \mu_*^{1/2} \mu_*'^{1/2} (W_\delta - W_\delta')^2 g' g dv dv_* d\sigma \\
&\leq \frac{1}{2} \int B \mu_* (W_\delta' - W_\delta)^2 g^2 dv dv_* d\sigma,
\end{aligned}$$

where we used the change of variables $(v', v'_*, \sigma) \rightarrow (v, v_*, \mathbf{k})$ and the Cauchy-Schwarz inequality. Note that we have

$$\begin{aligned}
|W_\delta(v') - W_\delta(v)| &\leq \int_0^1 |\nabla W_\delta(v'_\tau)| d\tau |v' - v|, \quad v'_\tau = v + \tau(v' - v) \\
&\lesssim \delta \int_0^1 W_\delta(v'_\tau) \langle \delta v'_\tau \rangle^{-1} d\tau |v - v_*| \sin \frac{\theta}{2} \\
&\lesssim \delta^{\nu/2} |v - v_*|^{\nu/2} \theta W_\delta(v) \langle v_* \rangle^{N+2-\nu/2} \\
&\lesssim \delta^{\nu/2} \langle v \rangle^{\nu/2} \theta W_\delta(v) \langle v_* \rangle^{N+2}, \tag{3.7.17}
\end{aligned}$$

because $\langle \delta v'_\tau \rangle^{-1} \lesssim \langle \delta v \rangle^{-1} \langle v_* \rangle$ and $\delta |v - v_*| \leq \langle \delta v \rangle \langle v_* \rangle$. Since it follows from Lemma 2.5 of [7] that

$$\int |v - v_*|^{\gamma+\nu} \mu_*^{1/2} dv_* \lesssim \langle v \rangle^{\gamma+\nu},$$

we have

$$\left| \left(\mathcal{L}_1(g), W_\delta^2 g \right)_{L^2(\mathbb{R}_v^3)} - \left(\mathcal{L}_1(W_\delta g), W_\delta g \right)_{L^2(\mathbb{R}_v^3)} \right| \lesssim \delta^\nu \|g\|_{L^2_{(\nu+\gamma)/2}}^2. \tag{3.7.18}$$

Proposition 3.7.19. *Let $\gamma > \max\{-3, -\nu - 3/2\}$. Then we have*

$$\begin{aligned}
&\left| \left(\Gamma(f, g), W_\delta h \right)_{L^2(\mathbb{R}^3)} - \left(\Gamma(f, W_\delta g), h \right)_{L^2(\mathbb{R}^3)} \right| \\
&\lesssim \delta^{\nu/2} \|f\|_{L^2} \|W_\delta g\|_{L^2_{(\nu+\gamma)/2}} \left(\|h\|_{L^2_{(\nu+\gamma)/2}} + \|h\|_{H_{-N_1}^{\nu'/2}} \right) \\
&\quad + \delta^{\nu/2} \|f\|_{L^2}^{1/2} \|W_\delta g\|_{L^2_{(\nu+\gamma)/2}} \mathcal{D}(\mu^{1/2} f, h)^{1/2},
\end{aligned}$$

for any $N_1 > 0$ and any $0 \leq \nu' \leq \nu$ satisfying $\gamma + \nu' > -3/2$.

Proof. Note that

$$\begin{aligned}
&\left(\Gamma(f, g), W_\delta h \right)_{L^2(\mathbb{R}^3)} - \left(\Gamma(f, W_\delta g), h \right)_{L^2(\mathbb{R}^3)} \\
&= \int B \mu_*^{1/2} (W'_\delta - W_\delta) f_* g h' dv dv_* d\sigma \\
&= \int B \mu_*^{1/2} (W'_\delta - W_\delta) f_* g h dv dv_* d\sigma \\
&\quad + \int B \left(\mu_*^{1/2} - \mu_*^{1/2} \right) f_* (W'_\delta - W_\delta) g h' dv dv_* d\sigma \\
&\quad + \int B \mu_*^{1/2} f_* (W'_\delta - W_\delta) g (h' - h) dv dv_* d\sigma \\
&= A_1 + A_2 + A_3.
\end{aligned}$$

Note that

$$\begin{aligned} |v - v_*|^\gamma &\lesssim \langle v \rangle^\gamma \langle v_* \rangle^{|\gamma|} \text{ or } \langle v \rangle^\gamma \langle v'_* \rangle^{|\gamma|} \text{ if } |v - v_*| \geq 1, \\ \langle v \rangle &\sim \langle v_* \rangle \sim \langle v' \rangle \sim \langle v'_* \rangle \text{ if } |v - v_*| < 1. \end{aligned}$$

We divide

$$A_2 = \int_{|v-v_*|<1} \cdots dv dv_* d\sigma + \int_{|v-v_*|\geq 1} \cdots dv dv_* d\sigma = A_{2,1} + A_{2,2}.$$

Using that

$$|\mu_*'^{1/2} - \mu_*^{1/2}| \lesssim \min(|v - v_*|, 1) \mu_*^{1/4} + \min(|v - v'_*|, 1) \mu_*'^{1/4}$$

we estimate as follows: $|A_{2,1}| \lesssim |A_{2,1}^*| + |A_{2,1}^{*\prime}|$. Then for any $N_1 > 0$ we have

$$\begin{aligned} |A_{2,1}^{*\prime}| &\lesssim \delta^{\nu/2} \int_{|v-v_*|<1} |v - v_*|^{\gamma+1+\nu/2} b \theta^2 \left| f_* \frac{W_\delta g}{\langle v \rangle^{N_1}} \frac{h'}{\langle v' \rangle^{N_1}} \right| dv dv_* d\sigma \\ &\lesssim \delta^{\nu/2} \left(\int \left(\int_{|v-v_*|<1} |v - v_*|^{2\gamma+2+\nu} dv_* \right) \left| \frac{W_\delta g}{\langle v \rangle^{N_1}} \right|^2 dv \right)^{1/2} \\ &\quad \times \left(\int b \theta^2 |f_*|^2 \left| \frac{h'}{\langle v' \rangle^{N_1}} \right|^2 dv dv_* d\sigma \right)^{1/2} \\ &\lesssim \delta^{\nu/2} \|f\|_{L^2} \|W_\delta g\|_{L^2_{-N_1}} \|h\|_{L^2_{-N_1}}. \end{aligned}$$

Similarly we have the same upper bound for $A_{2,1}^*$. As for $A_{2,2}$, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} A_{2,2}^2 &\leq 2 \int_{|v-v_*|\geq 1} B f_*^2 (W_\delta - W'_\delta)^2 g^2 (\mu_*'^{1/2} + \mu_*^{1/2}) dv dv_* d\sigma \\ &\quad \times \int B h^2 (\mu_*'^{1/4} - \mu_*^{1/4})^2 dv dv_* d\sigma \\ &\lesssim \delta^\nu \int \langle v \rangle^{\gamma+\nu} b \theta^2 |f_* W_\delta g|^2 dv dv_* d\sigma \|h\|_{L^2_{(\nu+\gamma)/2}}^2 \\ &\lesssim \delta^\nu \|f\|_{L^2}^2 \|W_\delta g\|_{L^2_{(\nu+\gamma)/2}}^2 \|h\|_{L^2_{(\nu+\gamma)/2}}^2, \end{aligned}$$

where we have used the fact that, if $|v - v_*| \geq 1$,

$$|v - v_*|^\gamma (W_\delta - W'_\delta)^2 \lesssim \delta^\nu \langle v \rangle^{\gamma+\nu} \theta^2 W_\delta^2 \min\{\langle v_* \rangle^{|\gamma|+2N+4}, \langle v'_* \rangle^{|\gamma|+2N+4}\},$$

because of (3.7.17) and it with v_* replaced by v'_* . By means of the Cauchy-Schwarz inequality, we have

$$A_3^2 \leq \int B \mu_*^{1/2} f_* (W'_\delta - W_\delta)^2 g^2 dv dv_* d\sigma \times \mathcal{D}(\mu^{1/2}|f|, h).$$

The first factor on the right hand side is estimated above from $C\delta^\nu$ times

$$\begin{aligned} & \|\mu^{1/4}f\|_{L^1}\|W_\delta g\|_{L^2_{(\nu+\gamma)/2}}^2 \\ & + \int \left(\int_{|v-v_*|<1} |v-v_*|^{2(\gamma+\nu)} dv_* \right)^{1/2} \|\mu^{1/4}f\|_{L^2}\|W_\delta g\|_{L^2_{-N_1}}^2 \end{aligned}$$

because it follows again from (3.7.17) that

$$|v-v_*|^\gamma (W_\delta - W'_\delta)^2 \mu_*^{1/4} \lesssim \delta^\nu \theta^2 W_\delta^2 \left\{ \langle v \rangle^{\gamma+\nu} \mathbf{1}_{|v-v_*|\geq 1} + \frac{|v-v_*|^{\gamma+\nu}}{\langle v \rangle^{N_1}} \mathbf{1}_{|v-v_*|<1} \right\}.$$

Therefore we also have the desired bound for A_3 .

In order to estimate A_1 , we use the following Taylor expansion of the second order; with $v'_\tau = v + \tau(v' - v)$,

$$\begin{aligned} W_\delta(v') - W_\delta(v) &= \nabla W_\delta(v) \cdot (v' - v) \\ &+ \int_0^1 (1-\tau) \nabla^2 W_\delta(v'_\tau) d\tau (v' - v)^2. \end{aligned} \quad (3.7.19)$$

Similar as in (3.7.16), we can estimate the factor $W'_\delta - W_\delta$ by

$$\begin{aligned} & |\nabla W_\delta(v)| |v-v_*| \theta^2 + |\nabla^2 W_\delta(v'_\tau)| |v-v_*|^2 \theta^2 \\ & \lesssim \delta W_\delta(v) \langle \delta v \rangle^{-1} |v-v_*| \theta^2 + \delta^2 W_\delta(v'_\tau) \langle \delta v'_\tau \rangle^{-2} |v-v_*|^2 \theta^2 \\ & \lesssim \delta^{\nu/2} W_\delta(v) |v-v_*|^\nu \theta^2 \langle v_* \rangle^{1-\nu/2} + \delta^\nu W_\delta(v) |v-v_*|^\nu \theta^2 \langle v_* \rangle^{N+4-\nu}. \end{aligned} \quad (3.7.20)$$

Consequently, for any $0 \leq \nu' \leq \nu$ satisfying $\gamma + \nu' > -3/2$, we have

$$\begin{aligned} |A_1| & \lesssim \delta^{\nu/2} \iint \langle v \rangle^{\gamma+\nu} \mathbf{1}_{|v-v_*|\geq 1} \mu_*^{1/4} |f_* W_\delta g h| dv dv_* \\ & + \delta^{\nu/2} \iint |v-v_*|^{\gamma+(\nu+\nu')/2} \mathbf{1}_{|v-v_*|<1} \left| f_* \frac{W_\delta g}{\langle v \rangle^{N_1}} \frac{h}{|v-v_*|^{\nu'/2} \langle v \rangle^{N_1}} \right| dv dv_* \\ & \lesssim \delta^{\nu/2} \|\mu^{1/4}f\|_{L^1} \|W_\delta g\|_{L^2_{(\nu+\gamma)/2}} \|h\|_{L^2_{(\nu+\gamma)/2}} \\ & + \delta^{\nu/2} \left(\int |f_*|^2 \int \left| \frac{h}{|v-v_*|^{\nu'/2} \langle v \rangle^{N_1}} \right|^2 dv \right) dv_*^{1/2} \\ & \quad \times \left(\int \left| \frac{W_\delta g}{\langle v \rangle^{N_1}} \right|^2 \int_{|v-v_*|<1} |v-v_*|^{2\gamma+\nu+\nu'} dv_* \right) dv^{1/2} \\ & \lesssim \delta^{\nu/2} \|f\|_{L^2} \left(\|W_\delta g\|_{L^2_{(\nu+\gamma)/2}} \|h\|_{L^2_{(\nu+\gamma)/2}} + \|W_\delta g\|_{L^2_{-N_1}} \|h\|_{H^{\nu'/2}_{-N_1}} \right). \end{aligned}$$

□

The following corollary is a variant of Lemma 4.9 in [6], which is used to prove the non-negativity of solutions.

Corollary 3.7.20. *Let $\gamma > \max\{-3, -\nu - 3/2\}$ and let $\varphi(v, x) = (1 + |v|^2 + |x|^2)^{\alpha/2}$ for $\alpha > 3/2$. If we put $W_\varphi = \varphi(v, x)^{-1}$, then we have*

$$\begin{aligned} & \left| \left(\Gamma(f, g), W_\varphi h \right)_{L^2(\mathbb{R}^3)} - \left(\Gamma(f, W_\varphi g), h \right)_{L^2(\mathbb{R}^3)} \right| \\ & \lesssim \|f\|_{L^2} \|W_\varphi g\|_{L^2_{\gamma/2}} \left(\|h\|_{L^2_{(\nu+\gamma)/2}} + \|h\|_{H_{-N_1}^{\nu'/2}} \right) \\ & \quad + \|f\|_{L^2}^{1/2} \|W_\varphi g\|_{L^2_{\gamma/2}} \mathcal{D}(\mu^{1/2} f, h)^{1/2}, \end{aligned}$$

for any $N_1 > 0$ and any $0 \leq \nu' \leq \nu$ satisfying $\gamma + \nu' > -3/2$.

Proof. Instead of (3.7.17) and (3.7.20), respectively, it suffices to use

$$\begin{aligned} |W_\varphi(v') - W_\varphi(v)| & \leq \int_0^1 |\nabla W_\varphi(v'_\tau)| d\tau |v' - v|, \quad v'_\tau = v + \tau(v' - v) \\ & \lesssim \int_0^1 W_\varphi(v'_\tau) \langle v'_\tau \rangle^{-1} d\tau |v - v_*| \sin \frac{\theta}{2} \\ & \lesssim \theta W_\varphi(v) \langle v_* \rangle^{\alpha+2} \left(\mathbf{1}_{|v-v_*| \geq 1} + |v - v_*| \mathbf{1}_{|v-v_*| < 1} \right) \end{aligned}$$

and

$$\begin{aligned} & |\nabla W_\varphi(v)| |v - v_*| \theta^2 + |\nabla^2 W_\varphi(v'_\tau)| |v - v_*|^2 \theta^2 \\ & \lesssim W_\varphi(v) \langle v \rangle^{-1} |v - v_*| \theta^2 + W_\varphi(v'_\tau) \langle v'_\tau \rangle^{-2} |v - v_*|^2 \theta^2 \\ & \lesssim \theta^2 W_\varphi(v) \langle v_* \rangle^{\alpha+4} \left(\mathbf{1}_{|v-v_*| \geq 1} + |v - v_*| \mathbf{1}_{|v-v_*| < 1} \right). \end{aligned}$$

□

3.7.4 Commutator with v derivative mollifier

Since the kinetic factor of the collision cross section is singular, we must confine ourselves to a special order of the mollifier. Let $1 \geq N_0 \geq \nu/2$ and let $M^\delta(\xi) = \frac{1}{(1 + \delta \langle \xi \rangle)^{N_0}}$ for $0 < \delta \leq 1$.

Proposition 3.7.21. *Assume that $0 < \nu < 2$ and $\gamma > \max\{-3, -\frac{3}{2} - \nu\}$. Then for any $\nu' > 0$ satisfying $\nu - 1 \leq \nu' < \nu$ and $\gamma + \nu' > -3/2$ we have*

$$\begin{aligned} & \left| \left(M^\delta(D_v) Q_c(f, g) - Q_c(f, M^\delta(D_v) g), h \right) \right| \\ & \lesssim \|f\|_{L^2} \|M^\delta(D_v) g\|_{H^{\nu'/2}} \|h\|_{H^{\nu'/2}}. \end{aligned}$$

Proof. We shall follow similar arguments of Proposition 3.4 from [9]. By using the formula from the Appendix of [2], we have

$$\begin{aligned} (Q_c(f, g), h) & = \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) [\hat{\Phi}_c(\xi_* - \xi^-) - \hat{\Phi}_c(\xi_*)] \\ & \quad \times \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma, \end{aligned}$$

where $\xi^- = \frac{1}{2}(\xi - |\xi|\sigma)$. Therefore

$$\begin{aligned}
& \left(M^\delta(D) Q_c(f, g) - Q_c(f, M^\delta(D) g), h \right) \\
&= \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) [\hat{\Phi}_c(\xi_* - \xi^-) - \hat{\Phi}_c(\xi_*)] \\
&\quad \times \left(M^\delta(\xi) - M^\delta(\xi - \xi_*) \right) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma \\
&= \iiint_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} \cdots d\xi d\xi_* d\sigma + \iiint_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \cdots d\xi d\xi_* d\sigma \\
&= \mathcal{A}_1(f, g, h) + \mathcal{A}_2(f, g, h).
\end{aligned}$$

Then, we write $\mathcal{A}_2(f, g, h)$ as

$$\begin{aligned}
\mathcal{A}_2 &= \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \hat{\Phi}_c(\xi_* - \xi^-) \cdots d\xi d\xi_* d\sigma \\
&\quad - \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \hat{\Phi}_c(\xi_*) \cdots d\xi d\xi_* d\sigma \\
&= \mathcal{A}_{2,1}(f, g, h) - \mathcal{A}_{2,2}(f, g, h).
\end{aligned}$$

On the other hand, for $\mathcal{A}_1$ we use the Taylor expansion of $\hat{\Phi}_c$ of order 2 to have

$$\mathcal{A}_1 = \mathcal{A}_{1,1}(f, g, h) + \mathcal{A}_{1,2}(f, g, h),$$

where

$$\begin{aligned}
\mathcal{A}_{1,1} &= \iiint b \xi^- \cdot (\nabla \hat{\Phi}_c)(\xi_*) \mathbf{1}_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} \left(M^\delta(\xi) - M^\delta(\xi - \xi_*) \right) \\
&\quad \times \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \bar{\hat{h}}(\xi) d\xi d\xi_* d\sigma,
\end{aligned}$$

and $\mathcal{A}_{1,2}(f, g, h)$ is the remaining term corresponding to the second order term in the Taylor expansion of $\hat{\Phi}_c$.

We first consider $\mathcal{A}_{1,1}$. By writing

$$\xi^- = \frac{|\xi|}{2} \left(\left(\frac{\xi}{|\xi|} \cdot \sigma \right) \frac{\xi}{|\xi|} - \sigma \right) + \left(1 - \left(\frac{\xi}{|\xi|} \cdot \sigma \right) \right) \frac{\xi}{2},$$

we see that the integral corresponding to the first term on the right hand side vanishes because of the symmetry on $\mathbb{S}^2$. Hence, we have

$$\mathcal{A}_{1,1} = \iint_{\mathbb{R}^6} K(\xi, \xi_*) \left(M^\delta(\xi) - M^\delta(\xi - \xi_*) \right) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \bar{\hat{h}}(\xi) d\xi d\xi_*,$$

where

$$K(\xi, \xi_*) = \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left(1 - \left(\frac{\xi}{|\xi|} \cdot \sigma \right) \right) \frac{\xi}{2} \cdot (\nabla \hat{\Phi}_c)(\xi_*) \mathbf{1}_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} d\sigma.$$

Note that $|\nabla\hat{\Phi}_c(\xi_*)| \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma+1}}$, from the Appendix of [7]. If $\sqrt{2}|\xi| \leq \langle \xi_* \rangle$, then $\sin(\theta/2) |\xi| = |\xi^-| \leq \langle \xi_* \rangle/2$ because $0 \leq \theta \leq \pi/2$, and we have

$$|K(\xi, \xi_*)| \lesssim \int_0^{\pi/2} \theta^{1-\nu} d\theta \frac{\langle \xi \rangle}{\langle \xi_* \rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right).$$

On the other hand, if $\sqrt{2}|\xi| \geq \langle \xi_* \rangle$, then

$$|K(\xi, \xi_*)| \lesssim \int_0^{\pi\langle \xi_* \rangle/(2|\xi|)} \theta^{1-\nu} d\theta \frac{\langle \xi \rangle}{\langle \xi_* \rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{\nu-1}.$$

Hence we obtain

$$\begin{aligned} |K(\xi, \xi_*)| &\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left\{ \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right) \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} \right. \\ &\quad \left. + \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2} + \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{\nu-1} \mathbf{1}_{|\xi|/2 \geq \langle \xi_* \rangle} \right\}. \end{aligned} \quad (3.7.21)$$

Similarly to $\mathcal{A}_{1,1}$, we can also write

$$\mathcal{A}_{1,2} = \int_{\mathbb{R}^6} \tilde{K}(\xi, \xi_*) \left(M^\delta(\xi) - M^\delta(\xi - \xi_*) \right) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \bar{h}(\xi) d\xi d\xi_*,$$

where

$$\tilde{K}(\xi, \xi_*) = \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \int_0^1 (1-\tau) (\nabla^2 \hat{\Phi}_c)(\xi_* - \tau\xi^-) \cdot \xi^- \cdot \xi^- \mathbf{1}_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} d\tau d\sigma.$$

Again from the Appendix of [7], we have

$$|(\nabla^2 \hat{\Phi}_c)(\xi_* - \tau\xi^-)| \lesssim \frac{1}{\langle \xi_* - \tau\xi^- \rangle^{3+\gamma+2}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma+2}},$$

because $|\xi^-| \leq \langle \xi_* \rangle/2$, which leads to

$$\begin{aligned} |\tilde{K}(\xi, \xi_*)| &\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left\{ \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^2 \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} \right. \\ &\quad \left. + \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2} + \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^\nu \mathbf{1}_{|\xi|/2 \geq \langle \xi_* \rangle} \right\}. \end{aligned} \quad (3.7.22)$$

We employ (3.4) of Lemma 3.1 in [9] with $p = 1$ and $\lambda = 0$, that is,

$$\begin{aligned} |M^\delta(\xi) - M^\delta(\xi - \xi_*)| &\leq CM^\delta(\xi - \xi_*) \left\{ \left(\frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right) \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} \right. \\ &\quad \left. + \left(M^\delta(\xi_*) (1 + \delta\langle \xi - \xi_* \rangle)^{N_0} + 1 \right) \mathbf{1}_{\sqrt{2}|\xi| > \langle \xi_* \rangle \geq |\xi|/2} + \frac{\langle \xi_* \rangle}{\langle \xi \rangle} \mathbf{1}_{|\xi|/2 > \langle \xi_* \rangle} \right\}. \end{aligned} \quad (3.7.23)$$

It follows from (3.7.21) and (3.7.22) that we have

$$|\mathcal{A}_1| \lesssim |\mathcal{A}_{1,1}| + |\mathcal{A}_{1,2}| \lesssim A_1 + A_2 + A_3,$$

with

$$A_1 = \iint_{\mathbb{R}^6} \left| \frac{\hat{f}(\xi_*)}{\langle \xi_* \rangle^{3+\gamma}} \right| |M^\delta(\xi - \xi_*) \hat{g}(\xi - \xi_*)| |\hat{h}(\xi)| \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} d\xi_* d\xi, \quad (3.7.24)$$

and

$$\begin{aligned} A_2 &= \iint_{\mathbb{R}^6} \left| \frac{\hat{f}(\xi_*)}{\langle \xi_* \rangle^{3+\gamma}} \right| |M^\delta(\xi - \xi_*) \hat{g}(\xi - \xi_*)| |\hat{h}(\xi)| \\ &\quad \times (M^\delta(\xi_*) (1 + (\delta \langle \xi - \xi_* \rangle)^{N_0}) + 1) \mathbf{1}_{\sqrt{2}|\xi| > \langle \xi_* \rangle \geq |\xi|/2} d\xi_* d\xi; \\ A_3 &= \iint_{\mathbb{R}^6} \left| \frac{\hat{f}(\xi_*)}{\langle \xi_* \rangle^{3+\gamma}} \right| |M^\delta(\xi - \xi_*) \hat{g}(\xi - \xi_*)| |\hat{h}(\xi)| \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{\nu-1} \mathbf{1}_{|\xi|/2 > \langle \xi_* \rangle} d\xi_* d\xi. \end{aligned}$$

Setting $\hat{G}(\xi) = \langle \xi \rangle^{\nu'/2} M^\delta(\xi) \hat{g}(\xi)$ and $\hat{H}(\xi) = \langle \xi \rangle^{\nu'/2} \hat{h}(\xi)$, we get

$$\begin{aligned} A_1 &\leq \int_{\mathbb{R}^3} \frac{|\hat{H}(\xi)|}{\langle \xi \rangle^{3/2+\varepsilon}} \left(\int_{\mathbb{R}^3} \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{3/2+\varepsilon-\nu'/2} \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} \frac{|\hat{f}(\xi_*)| |\hat{G}(\xi - \xi_*)|}{\langle \xi_* \rangle^{3/2+\gamma+\nu'-\varepsilon}} d\xi_* \right) d\xi \\ &\lesssim \|h\|_{H^{\nu'/2}} \|f\|_{L^2} \|M^\delta g\|_{H^{\nu'/2}}, \end{aligned}$$

because $3/2 + \varepsilon - \nu'/2 \geq 0$ and $3/2 + \gamma + \nu' - \varepsilon \geq 0$ for a sufficiently small $\varepsilon > 0$. Here we have used the fact that $\langle \xi_* \rangle \sim \langle \xi - \xi_* \rangle$ if $\langle \xi_* \rangle \geq \sqrt{2}|\xi|$. Noticing the third formula of (3.7.6), we get

$$\begin{aligned} |A_2|^2 &\lesssim \left\{ \int_{\mathbb{R}^3} \frac{|\hat{f}(\xi_*)|^2 d\xi_*}{\langle \xi_* \rangle^{6+2\gamma+\nu'}} \int_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* \rangle} \left(\frac{\langle \xi_* \rangle^{-2N_0}}{\langle \xi - \xi_* \rangle^{\nu'-2N_0}} + \frac{1}{\langle \xi - \xi_* \rangle^{\nu'}} \right) d\xi \right\} \\ &\quad \times \left(\iint_{\mathbb{R}^6} |\hat{G}(\xi - \xi_*)|^2 |\hat{H}(\xi)|^2 d\xi d\xi_* \right) \\ &\lesssim \int_{\mathbb{R}^3} \frac{|\hat{f}(\xi_*)|^2}{\langle \xi_* \rangle^{3+2(\gamma+\nu')}} d\xi_* \|M_\lambda^\delta g\|_{H^{\nu'/2}}^2 \|h\|_{H^{\nu'/2}}^2 \lesssim \|f\|_{L^2}^2 \|M_\lambda^\delta g\|_{H^{\nu'/2}}^2 \|h\|_{H^{\nu'/2}}^2, \end{aligned}$$

because $3 + 2(\gamma + \nu') > 0$. Since $\nu' \geq \nu - 1$ and $6 + 2(\gamma + \nu') > 3$, we have

$$\begin{aligned} |A_3|^2 &\lesssim \left(\int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 d\xi_* \int_{\mathbb{R}^3} |\hat{H}(\xi)|^2 d\xi \right) \\ &\quad \times \left(\int_{\mathbb{R}^3} \frac{d\xi_*}{\langle \xi_* \rangle^{6+2(\gamma+\nu')}} \int_{\mathbb{R}^3} \left(\frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right)^{2\{2s'-(\nu-1)\}} \mathbf{1}_{|\xi|/2 \geq \langle \xi_* \rangle} |\hat{G}(\xi - \xi_*)|^2 d\xi \right) \\ &\lesssim \|f\|_{L^2}^2 \|M_\lambda^\delta g\|_{H^{\nu'/2}}^2 \|h\|_{H^{\nu'/2}}^2. \end{aligned}$$

The above three estimates yield the desired estimate for $\mathcal{A}_1(f, g, h)$.

Next consider $\mathcal{A}_2(f, g, h) = \mathcal{A}_{2,1}(f, g, h) - \mathcal{A}_{2,2}(f, g, h)$. Since $\theta \in [0, \pi/2]$ and $|\xi^-| = |\xi| \sin(\theta/2) \geq \langle \xi_* \rangle / 2$, we have $\sqrt{2}|\xi| \geq \langle \xi_* \rangle$. Write

$$\mathcal{A}_{2,j} = \iint_{\mathbb{R}^6} K_j(\xi, \xi_*) \left(M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*) \right) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \bar{\hat{h}}(\xi) d\xi d\xi_*.$$

Then we have

$$\begin{aligned} |K_2(\xi, \xi_*)| &= \left| \int b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \hat{\Phi}_c(\xi_*) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} d\sigma \right| \\ &\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \frac{\langle \xi \rangle^\nu}{\langle \xi_* \rangle^\nu} \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle} \\ &\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left\{ \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2} + \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^\nu \mathbf{1}_{|\xi|/2 \geq \langle \xi_* \rangle} \right\}, \end{aligned}$$

which shows the desired estimate for $\mathcal{A}_{2,2}$, by exactly the same way as the estimation on A_2 and A_3 .

As for $\mathcal{A}_{2,1}$, it suffices to work under the condition $|\xi_* \cdot \xi^-| \geq \frac{1}{2}|\xi^-|^2$. In fact, on the complement of this set, we have $|\xi_* - \xi^-| > |\xi_*|$, and $\hat{\Phi}_c(\xi_* - \xi^-)$ is the same as $\hat{\Phi}_c(\xi_*)$. Therefore, we consider $\mathcal{A}_{2,1,p}$ which is defined by replacing $K_1(\xi, \xi_*)$ by

$$K_{1,p}(\xi, \xi_*) = \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \hat{\Phi}_c(\xi_* - \xi^-) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \mathbf{1}_{|\xi_* \cdot \xi^-| \geq \frac{1}{2}|\xi^-|^2} d\sigma.$$

By writing

$$\mathbf{1} = \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi - \xi_* \rangle \leq 2\langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi - \xi_* \rangle > 2\langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle < |\xi|/2},$$

we decompose respectively

$$\mathcal{A}_{2,1,p} = B_1 + B_2 + B_3.$$

On the sets corresponding to the above integrals, we have $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi_* \rangle$, because $|\xi^-| \lesssim |\xi_*|$ that follows from $|\xi^-|^2 \leq 2|\xi_* \cdot \xi^-| \lesssim |\xi^-| |\xi_*|$. Furthermore, on the sets for B_1 and B_2 we have $\langle \xi \rangle \sim \langle \xi_* \rangle$, so that $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi \rangle$ and $b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2}$ is bounded. Putting again $\hat{G}(\xi) = \langle \xi \rangle^{\nu'/2} M^\delta(\xi) \hat{g}(\xi)$ and $\hat{H}(\xi) = \langle \xi \rangle^{\nu'/2} \hat{h}(\xi)$, by means of (3.7.23) we have

$$\begin{aligned} |B_1|^2 &\lesssim \left[\iiint \left| \frac{\hat{\Phi}_c(\xi_* - \xi^-)}{\langle \xi_* - \xi^- \rangle^{s'}} \right|^2 |\hat{f}(\xi_*)|^2 b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \right. \\ &\quad \times \left\{ M^\delta(\xi_*)^2 \left(\frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* - \xi^- \rangle}}{\langle \xi - \xi_* \rangle^{\nu'}} + \frac{\delta^{2N_0} \mathbf{1}_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* - \xi^- \rangle}}{\langle \xi - \xi_* \rangle^{\nu' - 2N_0}} \right) \right. \\ &\quad \left. \left. + \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* - \xi^- \rangle}}{\langle \xi - \xi_* \rangle^{\nu'}} \right\} d\xi d\xi_* d\sigma \right] \left(\iiint |\hat{G}(\xi - \xi_*)|^2 |\hat{H}(\xi)|^2 d\sigma d\xi d\xi_* \right). \end{aligned}$$

Putting $u = \xi_* - \xi^-$, we have $\langle u \rangle \lesssim \langle \xi_* \rangle$, and $M^\delta(\xi_*)^2 \lesssim (1 + \delta\langle u \rangle)^{-2N_0}$. Therefore,

$$\begin{aligned} |B_1|^2 &\lesssim \int |\hat{f}(\xi_*)|^2 \left\{ \sup_u \langle u \rangle^{-(6+2\gamma+\nu')} \int_{\langle \xi^+ - u \rangle \leq \langle u \rangle} \left(\frac{\delta^{2N_0} (1 + \delta\langle u \rangle)^{-2N_0}}{\langle \xi^+ - u \rangle^{\nu' - 2N_0}} + \right. \right. \\ &\quad \left. \left. + \frac{1}{\langle \xi^+ - u \rangle^{\nu'}} \right) d\xi^+ \right\} d\xi_* \|M^\delta(D)g\|_{H^{\nu'/2}}^2 \|h\|_{H^{\nu'/2}}^2 \\ &\lesssim \|f\|_{L^2}^2 \|M^\delta(D)g\|_{H^{\nu'/2}}^2 \|h\|_{H^{\nu'/2}}^2 \sup_u \frac{1}{\langle u \rangle^{3+2(\gamma+\nu')}}. \end{aligned}$$

Here we have used the change of variables $\xi \rightarrow \xi^+$ whose Jacobian is

$$\begin{aligned} \left| \frac{\partial \xi^+}{\partial \xi} \right| &= \frac{\left| I + \frac{\xi}{|\xi|} \otimes \sigma \right|}{8} \\ &= \frac{|1 + \frac{\xi}{|\xi|} \cdot \sigma|}{8} = \frac{\cos^2(\theta/2)}{4} \geq \frac{1}{8}, \quad \theta \in [0, \frac{\pi}{2}]. \end{aligned}$$

As for B_2 , we first note that, on the set of the integration, $\xi^+ = \xi - \xi_* + u$ implies

$$\frac{\langle \xi - \xi_* \rangle}{2} \leq \langle \xi - \xi_* \rangle - |u| \leq \langle \xi^+ \rangle \leq \langle \xi - \xi_* \rangle + |u| \lesssim \langle \xi - \xi_* \rangle,$$

so that

$$(M^\delta(\xi) \sim) M^\delta(\xi^+) \sim M^\delta(\xi - \xi_*),$$

and hence we have by the Cauchy-Schwarz inequality

$$\begin{aligned} |B_2|^2 &\lesssim \iiint |\hat{f}(\xi_*)|^2 |\hat{G}(\xi - \xi_*)|^2 d\sigma d\xi d\xi_* \\ &\quad \times \iiint \frac{|\hat{\Phi}_c(\xi_* - \xi^-)|^2}{\langle \xi_* - \xi^- \rangle^{2\nu'}} |\hat{H}(\xi)|^2 d\sigma d\xi d\xi_* \\ &\lesssim \|f\|_{L^2}^2 \|M^\delta(D)g\|_{H^{\nu'/2}}^2 \|h\|_{H^{\nu'/2}}^2, \end{aligned}$$

because $6 + 2(\gamma + \nu') > 3$. On the set of the integration for B_3 we recall $\langle \xi \rangle \sim \langle \xi - \xi_* \rangle$ and

$$|M^\delta(\xi) - M^\delta(\xi - \xi_*)| \lesssim \frac{\langle \xi_* \rangle}{\langle \xi \rangle} M^\delta(\xi - \xi_*),$$

so that

$$\begin{aligned} |B_3|^2 &\lesssim \iiint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \left(\frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right)^\nu |\hat{f}(\xi_*)|^2 |\hat{G}(\xi - \xi_*)|^2 d\sigma d\xi d\xi_* \\ &\quad \times \iiint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \left(\frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right)^{2-\nu} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)|^2}{\langle \xi \rangle^{2\nu'}} |\hat{H}(\xi)|^2 d\sigma d\xi d\xi_*. \end{aligned}$$

We use the change of variables $\xi_* \rightarrow u = \xi_* - \xi^-$. Note that $|\xi^-| \geq \frac{1}{2}\langle u + \xi^- \rangle$ implies $|\xi^-| \geq \langle u \rangle / \sqrt{10}$, and that

$$\langle \xi_* \rangle \lesssim \langle \xi_* - \xi^- \rangle + |\xi| \sin \theta / 2,$$

which yields

$$\left(\frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right)^{2-\nu} \lesssim \left(\frac{\langle u \rangle}{\langle \xi \rangle} \right)^{2-\nu} + \theta^{2-\nu}.$$

Then we have

$$\begin{aligned} \iint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \left(\frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right)^{2-\nu} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)|^2}{\langle \xi \rangle^{2\nu'}} d\sigma d\xi_* &\lesssim \int \frac{\mathbf{1}_{\langle u \rangle \lesssim |\xi|}}{\langle u \rangle^{6+2(\gamma+\nu')}} \left(\frac{\langle u \rangle}{\langle \xi \rangle} \right)^{2\nu'} \\ &\times \left(\int b \mathbf{1}_{|\xi^-| \gtrsim \langle u \rangle} \left(\frac{\langle u \rangle}{\langle \xi \rangle} \right)^{2-\nu} d\sigma + \int b \theta^{2-\nu} \mathbf{1}_{|\xi^-| \gtrsim \langle u \rangle} d\sigma \right) du \\ &\lesssim \int \frac{du}{\langle u \rangle^{6+2(\gamma+\nu')}} < \infty, \end{aligned}$$

because

$$\int b \theta^{2-\nu} \mathbf{1}_{|\xi^-| \gtrsim \langle u \rangle} d\sigma \begin{cases} \lesssim \left(\frac{\langle u \rangle}{\langle \xi \rangle} \right)^{2-2\nu} & \text{if } \nu > 1 \\ \lesssim \log \frac{\langle \xi \rangle}{\langle u \rangle} & \text{if } \nu = 1 \\ < \infty & \text{if } \nu < 1. \end{cases}$$

Thus we have the same bound for B_3 . The proof of the proposition is then completed. $\square$

Let us recall Proposition 2.9 from [3].

Proposition 3.7.22. *Let $M(\xi)$ be a positive symbol in $S_{1,0}^0$ in the form of $M(\xi) = \tilde{M}(|\xi|^2)$. Assume that there exist constants $c, C > 0$ such that for any $s, \tau > 0$*

$$c^{-1} \leq \frac{s}{\tau} \leq c \quad \text{implies} \quad C^{-1} \leq \frac{\tilde{M}(s)}{\tilde{M}(\tau)} \leq C,$$

and $M(\xi)$ satisfies

$$|M^{(\alpha)}(\xi)| = |\partial_\xi^\alpha M(\xi)| \leq C_\alpha M(\xi) \langle \xi \rangle^{-|\alpha|},$$

for any $\alpha \in \mathbb{N}^3$. Then, if $0 < \nu < 1$, for any $N > 0$ there exists a $C_N > 0$ such that

$$\begin{aligned} &|(M(D_v)Q_{\bar{c}}(f, g) - Q_{\bar{c}}(f, M(D_v)g), h)_{L^2}| \\ &\leq C_N \|f\|_{L_{\gamma+}^1} \left(\|M(D_v)g\|_{L_{\gamma+}^2} + \|g\|_{H_{\gamma+}^{-N}} \right) \|h\|_{L^2}. \end{aligned} \quad (3.7.25)$$

Furthermore, if $1 < \nu < 2$, for any $N > 0$ and any $\varepsilon > 0$, there exists a $C_{N,\varepsilon} > 0$ such that

$$\begin{aligned} & |(M(D_v)Q_{\bar{c}}(f, g) - Q_{\bar{c}}(f, M(D_v)g), h)_{L^2}| \\ & \leq C_{N,\varepsilon} \|f\|_{L^1_{(\nu+\gamma-1)+}} \left(\|M(D_v)g\|_{H^{\nu-1+\varepsilon}_{(\nu+\gamma-1)+}} + \|g\|_{H^{-N}_{\gamma+}} \right) \|h\|_{L^2}. \end{aligned} \quad (3.7.26)$$

When $\nu = 1$ we have the same estimate as (3.7.26) with $(\nu + \gamma - 1)$ replaced by $(\gamma + \kappa)$ for any small $\kappa > 0$.

3.7.5 Boundedness of $M^\delta(D_v)$ on the triple norm

Instead of $M^\delta(\xi)$, we consider a little more general symbol $M(\xi) \in S^0_{1,0}$ satisfying conditions in Proposition 3.7.22.

Lemma 3.7.23. *Let $\gamma > -3$ and $0 < \nu < 2$. Then we have*

$$\|M(D_v)g\|^2 \lesssim \|g\|^2.$$

Proof. The proof is based on arguments in the subsection 2.3 of [7]. Since $J_2^{\Phi_\gamma}(g) \sim \|g\|_{L^2_{(\nu+\gamma)/2}}^2$ it suffices to show

$$J_1^{\Phi_\gamma}(Mg) \lesssim J_1^{\Phi_\gamma}(g) + \|g\|_{L^2_{(\nu+\gamma)/2}}^2 + \|g\|_{H^s_{\gamma/2}}^2, \quad (3.7.27)$$

in view of Proposition 2.2 of [7], that is, there exist two generic constants $C_1, C_2 > 0$ such that

$$C_1 \left\{ \|g\|_{H^s_{\gamma/2}(\mathbb{R}_v^3)}^2 + \|g\|_{L^2_{(\nu+\gamma)/2}(\mathbb{R}_v^3)}^2 \right\} \leq \|g\|^2 \leq C_2 \|g\|_{H^s_{(\nu+\gamma)/2}(\mathbb{R}_v^3)}^2.$$

Since $J_1^{\Phi_\gamma}(g) \sim J_1^{\Phi_0}(\langle v \rangle^{\gamma/2} g)$ modulo $\|g\|_{L^2_{(\nu+\gamma)/2}}^2$, and we have

$$\begin{aligned} J_1^{\Phi_0}(\langle v \rangle^{\gamma/2} Mg) & \leq 2 \left(J_1^{\Phi_0}(M \langle v \rangle^{\gamma/2} g) + J_1^{\Phi_0}([\langle v \rangle^{\gamma/2}, M]g) \right) \\ & \lesssim J_1^{\Phi_0}(M \langle v \rangle^{\gamma/2} g) + \|([\langle v \rangle^{\gamma/2}, M]g)\|_{H^s}^2, \end{aligned}$$

it suffices to show (3.7.27) with $\gamma = 0$. We recall (2.21) of [7];

$$\begin{aligned} J_1^{\Phi_0}(g) & = \iiint b(\cos \theta) \mu_*(g' - g)^2 dv_* d\sigma dv \\ & = \frac{1}{(2\pi)^3} \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left(\widehat{\mu}(0) |\widehat{g}(\xi) - \widehat{g}(\xi^+)|^2 \right. \\ & \quad \left. + 2\operatorname{Re} \left(\widehat{\mu}(0) - \widehat{\mu}(\xi^-) \right) \widehat{g}(\xi^+) \widehat{\bar{g}}(\xi) \right) d\xi d\sigma. \end{aligned} \quad (3.7.28)$$

Since $\widehat{\mu}(\xi)$ is real-valued, it follows that

$$\begin{aligned} \operatorname{Re} \left(\widehat{\mu}(0) - \widehat{\mu}(\xi^-) \right) \widehat{g}(\xi^+) \widehat{\bar{g}}(\xi) & = \left(\int (1 - \cos(v \cdot \xi^-)) \mu(v) dv \right) \operatorname{Re} \widehat{g}(\xi^+) \widehat{\bar{g}}(\xi) \\ & \lesssim \min\{\langle \xi \rangle^2 \theta^2, 1\} |\widehat{g}(\xi^+) \widehat{\bar{g}}(\xi)|. \end{aligned}$$

Therefore the second term of the right hand side of (3.7.28) is estimated by $\|g\|_{H^{\nu/2}}^2$ with a constant factor. Similarly we have

$$\begin{aligned}
J_1^{\Phi_0}(Mg) &= \iiint b(\cos \theta) \mu_*((Mg)' - Mg)^2 dv_* d\sigma dv \\
&\lesssim \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \widehat{\mu}(0) |M(\xi) \widehat{g}(\xi) - M(\xi^+) \widehat{g}(\xi^+)|^2 d\xi d\sigma + \|g\|_{H^{\nu/2}}^2 \\
&\lesssim J_1^{\Phi_0}(g) + \iint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) |M(\xi) - M(\xi^+)|^2 |\widehat{g}(\xi)|^2 d\xi d\sigma + \|g\|_{H^{\nu/2}}^2 \\
&\lesssim J_1^{\Phi_0}(g) + \|g\|_{H^{\nu/2}}^2,
\end{aligned}$$

because $|M(\xi) - M(\xi^+)| \lesssim M(\xi) \theta^2$ (see (2.3.5) of [3], for example). $\square$

3.7.6 Mollifier with respect to x variable

Lemma 3.7.24. *Let $S \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq S \leq 1$ and*

$$S(\tau) = 1, \quad |\tau| \leq 1; \quad S(\tau) = 0, \quad |\tau| \geq 2.$$

Put $S_\delta(D_x) = S(\delta D_x)$ for $\delta > 0$. Then

$$\begin{aligned}
&\left| \int_0^T \left(S_\delta(D_x) \Gamma(f, g) - \Gamma(f, S_\delta(D_x)g), h \right)_{L^2(\mathbb{R}_{x,v}^6)} dt \right| \\
&\lesssim \|\nabla f\|_{L^\infty([0,T] \times \mathbb{R}_x^3; L_v^2)} \|\langle v \rangle^{|\gamma|/2+\nu} g\|_{L^2([0,T] \times \mathbb{R}_{x,v}^6)} \\
&\quad \times \|\delta h\|_{L^2([0,T] \times \mathbb{R}_x^3; H_{\gamma/2}^\nu(\mathbb{R}_v^3))}. \tag{3.7.29}
\end{aligned}$$

Proof. The proof is similar as the one of Lemma 3.4 in [3]. If

$$K_\delta(z) = -\frac{z}{\delta^4} \mathcal{F}^{-1}(S)\left(\frac{z}{\delta}\right)$$

then we have

$$\begin{aligned}
&\left| \int_0^T \left(S_\delta(D_x) \Gamma(f, g) - \Gamma(f, S_\delta(D_x)g), h \right)_{L^2(\mathbb{R}^6)} dt \right| \\
&= \left| \int_0^1 \left\{ \int_{\mathbb{R}_x^3 \times \mathbb{R}_y^3} K_\delta(x - y) \right. \right. \\
&\quad \times \left. \int_0^T \left(\Gamma(\nabla f(t, x + \tau(y - x)), \cdot), \delta g(t, y, \cdot) \right), h(t, x, \cdot) \right)_{L^2(\mathbb{R}_v^3)} dt dx dy \Big\} d\tau \right| \\
&\lesssim \delta \|\nabla f\|_{L^\infty([0,T] \times \mathbb{R}_x^3; L_v^2)} \int_0^T \int_{\mathbb{R}_x^3} \left(|K_\delta| * \|g(t, \cdot)\|_{L_{|\gamma|/2+\nu}^2} \right)(x) \|h(t, x)\|_{H_{\gamma/2}^\nu} dx dt
\end{aligned}$$

where we have used (3.7.4). We get (3.7.29) since $\|K_\delta\|_{L_x^1} = \|K_1\|_{L_x^1}$. $\square$

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