

On estimations for parametrized operator means 作用素平均族の評価式について

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For a nonnegative operator monotone function f on $[0, \infty)$, Kubo and Ando [2] introduced an *operator mean* for positive operators m_f :

$$A m_f B = A^{\frac{1}{2}} f \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}$$

where the semi-continuity $\lim_{\varepsilon \downarrow 0} (A + \varepsilon I) m_f (B + \varepsilon I) \downarrow A m_f B$ assures that we may assume operators are invertible. Recently Kittaneh-Manasrah [1] gave a refined Young inequality, which is immediately extended to an inequality among operator means in the sense of by Furuichi-Lin [3]: Let t be a weight $t \in [0, 1]$, then

$$A \nabla_t B - A \#_t B \geq \min\{2t, 2(1-t)\} (A \nabla B - A \# B)$$

where $A \nabla_t B = (1-t)A + tB$; the arithmetic mean and $A \#_t B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}}$; the geometric one (for convenience' sake, we omit t if $t = \frac{1}{2}$).

In this talk, we generalize it for parametrized operator means: For $-1 \leq r \leq 1$, it is known that the functions $f_{r,t}(x) = (1-t+tx^r)^{\frac{1}{r}}$ are operator monotone, so they define operator means

$$A m_{r,t} B = A^{\frac{1}{2}} \left((1-t)I + t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r \right)^{\frac{1}{r}} A^{\frac{1}{2}}.$$

If $r = 0$, $A \#_t B$ is the limit $\lim_{r \rightarrow 0} A m_{r,t} B$. Then we have

Theorem. For $0 \leq s \leq r \leq 1$,

$$A m_{r,t} B - A m_{s,t} B \geq \min\{(2t)^{\frac{1}{r}}, (2(1-t))^{\frac{1}{r}}\} (A m_r B - A m_s B).$$

We have only to show the numerical inequality:

$$(0) \quad f_{r,t}(x) - f_{s,t}(x) \geq \min\{(2t)^{\frac{1}{r}}, (2(1-t))^{\frac{1}{r}}\} \left(f_{r,\frac{1}{2}}(x) - f_{s,\frac{1}{2}}(x) \right).$$

It is easy to show the above inequality if $r = 1/n$, but I have not yet a simple proof for the general case. I have shown it by the following properties:

Lemma. The following properties hold:

- (0) f is operator monotone.
- (1) $F_{t,x}(r) = (1 - t + tx^r)^{\frac{1}{r}}$ is monotone-increasing for r .
- (2) $\ell_x(r) = \frac{x^r - 1}{r}$ is monotone-increasing for r .
- (3) If $0 < x \leq 1$, then $H(r) = g(r)^{\frac{1}{r}-1} = (1 - t + tx^r)^{\frac{1}{r}-1}$ is monotone-increasing for r .
- (4) Let $J_{r,x}(t) = (1 - t + tx^r)^{\frac{1}{r}}$. Then, $rtJ'_{r,x}(t) = J_{r,x}(t) - J_{r,x}(t)^{1-r}$.
Moreover, if $0 < x \leq 1$, then $J'_{r,x}(t) \geq J'_{s,x}(t)$.

Proof. (0) Considering the analytic continuation of f to the upper half plane $\text{Im } z > 0$, we have

$$0 < \text{Arg}(1 - t + tz^r) < \text{Arg } z^r,$$

so that

$$0 < \text{Arg } f_{r,t}(z) = \text{Arg}(1 - t + tz^r)^{1/r} < \text{Arg } z.$$

It follows that $\text{Im } f_{r,t}(z) > 0$, which shows f is operator monotone.

(1) Let $g(r) = 1 - t + tx^r$, $y = F_{t,x}(r) = g(r)^{1/r}$. Then $\log y = \log g(r)/r$. By the convexity of $\eta(x) = x \log x$, we have

$$\begin{aligned} F'_{t,x}(r) = y' &= \frac{yg'(r)}{rg(r)} - \frac{y \log g(r)}{r^2} = \frac{y^{1-r}rg'(r)}{r^2} - \frac{y \log g(r)}{r^2} \\ &= \frac{y^{1-r}}{r^2} (tx^r \log x^r - g(r) \log g(r)) = \frac{y^{1-r}}{r^2} (t\eta(x^r) - \eta(g(r))) \\ &\geq \frac{y^{1-r}}{r^2} (t\eta(x^r) - (1-t)\eta(1) - t\eta(x^r)) = 0, \end{aligned}$$

which shows $F_{t,x}(r)$ is monotone increasing.

(2) By the Klein inequality $\log y \geq 1 - 1/y$,

$$\ell'_x(r) = \frac{rx^r \log x - (x^r - 1)}{r^2} = \frac{x^r \log x^r - (x^r - 1)}{r^2} \geq \frac{x^r - 1 - (x^r - 1)}{r^2} = 0.$$

(3) Suppose $0 < x \leq 1$ and $1 - t + tx^r \leq 1$. By $1 - r < 1 - s$ and (1), we have

$$H(r) = (1 - t + tx^r)^{(1-r)/r} \geq (1 - t + tx^s)^{(1-r)/s} \geq (1 - t + tx^s)^{(1-s)/s} = H(s).$$

(4) The former inequality follows from

$$\begin{aligned} rtJ'_{r,x}(t) &= t(1 - t + tx^r)^{\frac{1-r}{r}}(x^r - 1) = (1 - t + tx^r)^{\frac{1-r}{r}}(1 - t + tx^r - 1) \\ &= (1 - t + tx^r)^{\frac{1}{r}} - (1 - t + tx^r)^{\frac{1-r}{r}} = J_{r,x}(t) - J_{r,x}(t)^{1-r}. \end{aligned}$$

Since

$$J'_{r,x}(t) = (1 - t + tx^r)^{1/r-1} \frac{x^r - 1}{r},$$

It follows from (2) and (3) that $J'_{r,x}(t) \geq J'_{s,x}(t)$. □

Proof of theorem. Suppose $0 < x \leq 1$. For $t \leq 1/2$, put $K(t) = \frac{J_{r,x}(t) - J_{s,x}(t)}{(2t)^{1/r}}$. It follows from Lemma that

$$\begin{aligned} K'(t) &= \frac{J'_{r,x}(t) - J'_{s,x}(t)}{(2t)^{1/r}} - \frac{2}{r} \frac{J_{r,x}(t) - J_{s,x}(t)}{(2t)^{1/r+1}} \\ &= \frac{2}{r(2t)^{1/r+1}} (tr(J'_{r,x}(t) - J'_{s,x}(t)) - (J_{r,x}(t) - J_{s,x}(t))) \\ &= \frac{2}{r(2t)^{1/r+1}} (J_{s,x}(t)^{1-s} - J_{r,x}(t)^{1-r}) = \frac{2}{r(2t)^{1/r+1}} (H(s) - H(r)) \leq 0, \end{aligned}$$

which shows K is monotone decreasing and attains the minimum $J_{r,x}(1/2) - J_{s,x}(1/2)$ at $t = 1/2$.

Next suppose $t > 1/2$. Putting $L(t) = \frac{J_{r,x}(t) - J_{s,x}(t)}{(2(1-t))^{1/r}}$, we have by Lemma that

$$\begin{aligned} L'(t) &= \frac{2}{r(2(1-t))^{1/r+1}} ((1-t)r(J'_{r,x}(t) - J'_{s,x}(t)) + (J_{r,x}(t) - J_{s,x}(t))) \\ &= \frac{2}{r(2(1-t))^{1/r+1}} (r(J'_{r,x}(t) - J'_{s,x}(t)) + (J_{r,x}(t)^{1-r} - J_{s,x}(t)^{1-s})) \\ &= \frac{2}{r(2(1-t))^{1/r+1}} (r(J'_{r,x}(t) - J'_{s,x}(t)) + (H(r) - H(s))) \geq 0. \end{aligned}$$

Thus L is monotone decreasing and attains the maximum at $t = 1/2$. Therefore

$$J_{r,x}(t) - J_{s,x}(t) \geq (2 \min\{1-t, t\})^{\frac{1}{r}} (J_{r,x}(1/2) - J_{s,x}(1/2)),$$

that is,

$$(1 - t + tx^r)^{1/r} - (1 - t + tx^s)^{1/s} \geq (2 \min\{1-t, t\})^{\frac{1}{r}} \left(\left(\frac{1+x^r}{2} \right)^{1/r} - \left(\frac{1+x^s}{2} \right)^{1/s} \right)$$

holds for $0 < x \leq 1$. By the homogeneity of x , it also holds for $x > 1$. □

If the following conjecture holds, we immediately have a simple proof for (0):

Conjecture. For $0 < s \leq r \leq 1$,

$$\begin{cases} \left(\frac{(1-t+tx^r)^{\frac{1}{r}} - (t(1+x^r))^{\frac{1}{r}}}{1-(2t)^{\frac{1}{r}}} \right)^{\frac{1}{r}} \geq \left(\frac{(1-t+tx^s)^{\frac{1}{r}} - (t(1+x^s))^{\frac{1}{r}}}{1-(2t)^{\frac{1}{r}}} \right)^{\frac{1}{s}} & (t < \frac{1}{2}) \\ \left(\frac{(1-t+tx^r)^{\frac{1}{r}} - ((1-t)(1+x^r))^{\frac{1}{r}}}{1-(2(1-t))^{\frac{1}{r}}} \right)^{\frac{1}{r}} \geq \left(\frac{(1-t+tx^s)^{\frac{1}{r}} - ((1-t)(1+x^s))^{\frac{1}{r}}}{1-(2(1-t))^{\frac{1}{r}}} \right)^{\frac{1}{s}} & (t > \frac{1}{2}). \end{cases}$$

In fact, it is also shown easily for $r = \frac{1}{n} > s$: For the case $t < 1/2$, the Jensen inequality for the power function for $ns < 1$ implies

$$\begin{aligned}
\text{LHS} &= \left(\frac{(1 - t + tx^{\frac{1}{n}})^n - \left(2t \left(\frac{1+x^{\frac{1}{n}}}{2}\right)\right)^n}{1 - (2t)^n} \right)^n \\
&= \left(\frac{\sum_{k=0}^{n-1} (2t)^k (1 - t + tx^{\frac{1}{n}})^{n-k-1} \left(\frac{1+x^{\frac{1}{n}}}{2}\right)^k}{\sum_{k=0}^{n-1} (2t)^k} \right)^{ns \cdot \frac{1}{s}} \\
&\geq \left(\frac{\sum_{k=0}^{n-1} (2t)^k ((1 - t + tx^{\frac{1}{n}})^{ns})^{n-k-1} \left(\left(\frac{1+x^{\frac{1}{n}}}{2}\right)^{ns}\right)^k}{\sum_{k=0}^{n-1} (2t)^k} \right)^{\frac{1}{s}} \\
&= \left(\frac{\sum_{k=0}^{n-1} (2t)^k (1 - t + tx^s)^{n-k-1} \left(\frac{1+x^s}{2}\right)^k}{\sum_{k=0}^{n-1} (2t)^k} \right)^{\frac{1}{s}} \\
&= \left(\frac{(1 - t + tx^s)^n - (t(1 + x^s))^n}{1 - (2t)^n} \right)^{\frac{1}{s}} = \text{RHS}.
\end{aligned}$$

参考文献

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