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# Geometry of 7 lines on the real projective plane and the root system of type $E_7$

姫路工業大学理学部　関口次郎 (Jiro Sekiguchi)

**Abstract.** The configuration space of 7 points in the real projective plane is studied. In particular, the relationship between the space and the root system of type $E_7$ is established.

## 1. Introduction and the main results

The main purpose of this article is to clarify the relationship between the geometry of 7 lines on the real projective plane $\mathbf{P}^2(\mathbf{R})$ and the root system of type $E_7$.

We first introduce marked 7 lines $(l_1, l_2, \cdots, l_7)$ on $\mathbf{P}^2(\mathbf{R})$. We give conditions on these 7 lines:

1. The 7 lines $l_1, l_2, \cdots, l_7$ are mutually different.
2. No three of $l_1, l_2, \cdots, l_7$ intersect at a point.
3. There is no conic tangent to six of $l_1, l_2, \cdots, l_7$.

The totality of marked 7 lines on $\mathbf{P}^2(\mathbf{R})$ with conditions 1, 2 forms the configuration space $P(3,7)$: the space $P(3,7)$ is defined by

$$P(3,7) = GL(3,\mathbf{R})\backslash M'(3,7)/(\mathbf{R}^\times)^7,$$

where $M'(3,7)$ is the set of $3\times 7$ real matrices of which no 3-minor vanishes. On the other hand, the totality of marked 7 lines on $\mathbf{P}^2(\mathbf{R})$ with conditions 1, 2, 3 forms a subset of $P(3,7)$ which we denote by $P_0(3,7)$. Both $P(3,7)$, $P_0(3,7)$ are affine open subsets of $\mathbf{R}^6$. Permutations on the 7 lines $l_1, l_2, \cdots, l_7$ induce a biregular $S_7$-action on $P(3,7)$ (and also that on $P_0(3,7)$). It is stressed here that the $S_7$-action on $P_0(3,7)$ is naturally extended to a biregular $W(E_7)$-action (cf. [3], [4]). Let $\mathcal{P}_7$ be the set of connected components of $P_0(3,7)$. The $W(E_7)$-action on $P_0(3,7)$ naturally induces that on $\mathcal{P}_7$. Then we are in a position to state one of the main results of this article.

**Theorem 1** *The $W(E_7)$-action on $\mathcal{P}_7$ is transitive.*

Before entering into the main text, we are going to explain the motivations of this study briefly. It is an interesting problem to construct a tame compactification of the configurations space $P_{\mathbf{C}}(n,k)$ of marked $k$ hyperplanes on the complex projective space $\mathbf{P}^{n-1}(\mathbf{C})$.

In the case of $n = 2$, there is a nice compactification of $P_{\mathbf{C}}(2,k)$, the so called Terada model. We note here that $P_{\mathbf{C}}(2,k)$ admits $S_k$-action. What happens if we treat $P(2,k)$ instead of $P_{\mathbf{C}}(2,k)$, where $P(2,k)$ is the configuration space of $k$-points of $\mathbf{P}^1(\mathbf{R})$. In [8], it is shown that the $S_k$-action on the totality of connected components of $P(2,k)$ is transitive and that each connected component in question is described in terms of "*juzu*" introduced there. On the other hand, in the case $(n,k) = (3,6)$, there is a tame compactification of the configuration space of marked six lines of $\mathbf{P}^2(\mathbf{C})$ with conditions 1, 2, 3 above constructed in [2] which is called Naruki's cross ratio variety and denoted by $\mathcal{C}$. As an application of the results of [2], [3], already shown is the result which we are going to explain (cf. [7]). Let $P_0(3,6)$ be the space defined similarly to $P_0(3,7)$, changing the 7 lines with 6 lines. We note that $P_0(3,6)$ admits a $W(E_6)$-action (cf. [2], [3]). Then the result for the case $P_0(3,6)$ given in [8] similar to the case of Terada models is that there are 432 connected components in $P_0(3,6)$ and that the $W(E_6)$-action on the set is transitive. Theorem 1 is an analogue of these results to the case of 7 lines on the real projective plane.

The second main result is the classification of the $S_7$-orbital structure of $\mathcal{P}_7$. This will be done in terms of tetradiagrams for the root system of type $E_7$. In particular, we conclude that $\mathcal{P}_7$ is decomposed into 14 $S_7$-orbits (cf. Theorem 4).

## 2. Review on the root system $\Delta$ of type $E_7$

We first recall the definition of the root system of type $E_7$. Let $\tilde{E}$ be an 8-dimensional Euclidean space with a standard basis $\{\varepsilon_j; 1 \le j \le 8\}$. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\tilde{E}$ defined by

$$\langle \varepsilon_j, \varepsilon_k \rangle = \delta_{jk}$$

and let $E$ be its linear subspace orthogonal to $\varepsilon_7 + \varepsilon_8$. We define the following sixty-three vectors of $E$:

$$\begin{aligned}
\gamma_1 &= \varepsilon_8 - \varepsilon_7, & \\
\gamma_j &= -\varepsilon_{j-1} + \gamma_0, & 1 < j < 8 \\
\gamma_{1j} &= -\varepsilon_{j-1} + \gamma_0, & 1 < j < 8 \\
\gamma_{jk} &= \varepsilon_{j-1} - \varepsilon_{k-1}, & 1 < j < k < 8 \\
\gamma_{1jk} &= -\varepsilon_{j-1} - \varepsilon_{k-1}, & 1 < j < k < 8 \\
\gamma_{ijk} &= -\varepsilon_{i-1} - \varepsilon_{j-1} - \varepsilon_{k-1} + \gamma_0, & 1 < i < j < k < 8
\end{aligned}$$

where

$$\gamma_0 = \frac{1}{2}\sum_{j=1}^{8} \varepsilon_j - \varepsilon_7.$$

Note that

$$\gamma_i \perp \gamma_{jk}, \quad \gamma_i \perp \gamma_{ijk}, \quad \gamma_{ij} \perp \gamma_{kl}, \quad \gamma_{ij} \perp \gamma_{ijk}, \quad \gamma_{ij} \perp \gamma_{klm}, \quad \gamma_{ijk} \perp \gamma_{ilm}.$$

The totality $\Delta$ of $\pm\gamma_j, \pm\gamma_{jk}, \pm\gamma_{ijk}$ forms a root system of type $E_7$. It is clear that

$$\gamma_{12}, \quad \gamma_{123}, \quad \gamma_{23}, \quad \gamma_{34}, \quad \gamma_{45}, \quad \gamma_{56}, \quad \gamma_{67} \tag{1}$$

can serve as a system of positive simple roots; its extended Dynkin diagram is given as

$$\begin{array}{ccccccccccccc} \gamma_1 & -- & \gamma_{12} & -- & \gamma_{23} & -- & \gamma_{34} & -- & \gamma_{45} & -- & \gamma_{56} & -- & \gamma_{67} \\ & & & & & & | & & & & & & \\ & & & & & & \gamma_{123} & & & & & & \end{array} \tag{2}$$

The set $\{\gamma_i, \gamma_{jk}, \gamma_{ijk}\}$ is the totality of positive roots of $\Delta$.

Let $s_i, s_{ij}, s_{ijk}$ be the reflections on $E$ with respect to $\gamma_i, \gamma_{ij}, \gamma_{ijk}$. These reflections act on $\Delta$ as

$$s_i: \quad \gamma_j \leftrightarrow \gamma_{ij}, \quad \gamma_{jk} \leftrightarrow \gamma_{jk}, \quad \gamma_{jkl} \leftrightarrow \gamma_{mnp}, \quad \{i,j,k,l,m,n,p\} = \{1,2,\ldots,7\},$$

$$s_{ij}: \quad \text{permutation of the indices } i \text{ and } j,$$

$$s_{123}: \ \gamma_1 \leftrightarrow \gamma_1, \quad \gamma_4 \leftrightarrow \gamma_{567}, \quad \gamma_{12} \leftrightarrow \gamma_{12}, \quad \gamma_{14} \leftrightarrow \gamma_{234}, \quad \gamma_{45} \leftrightarrow \gamma_{45}, \quad \gamma_{145} \leftrightarrow \gamma_{145}$$

modulo signs. We define two reflection groups

$$G_1 = \langle s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{67} \rangle \cong S_7,$$

$$G = \langle G_1, s_{123} \rangle \cong W(E_7),$$

where $S_7$ is the symmetric group on seven numerals $\{1,2,\ldots,7\}$, $W(E_7)$ the Weyl group of type $E_7$ and $\langle a, b, \ldots \rangle$ denotes the group generated by $a, b, \ldots$ Note that $G$ acts on $\Delta$ transitively on $\Delta$.

## 3. Connected components of $\mathcal{C}'_{\mathbf{R}}(\Delta, D_4)$

We start with introducing a $3 \times 7$ matrix

$$X = \begin{pmatrix} t & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & -2 & 5 & 6 \\ 1 & 0 & 1 & 1 & 4 & -1 & -6 \end{pmatrix}.$$

Then by taking the entries of the vector $(1 \quad x \quad y) \cdot X$, we obtain seven lines of the real projective plane regarding $x, y$ as inhomogeneous coordinates:

$$\begin{array}{lcl} L_1 & : & x + y + t = 0 \\ L_2 & : & x = 0 \\ L_3 & : & y = 0 \\ L_4 & : & x + y + 1 = 0 \\ L_5 & : & -2x + 4y + 1 = 0 \\ L_6 & : & 5x - y + 1 = 0 \\ L_7 & : & 6x - 6y + 1 = 0 \end{array} \tag{3}$$

We assume that $t$ is real and sufficiently large.

By the seven lines above, we obtain ten triangles $(T_j)$ $(j = 1, 2, \cdots, 10)$ surrounded by the three lines

| | | |
|---|---|---|
| $(T_1)$ | $L_1L_2L_6$ | $\gamma_{126}$ |
| $(T_2)$ | $L_3L_6L_7$ | $\gamma_{367}$ |
| $(T_3)$ | $L_4L_5L_6$ | $\gamma_{456}$ |
| $(T_4)$ | $L_2L_3L_7$ | $\gamma_{237}$ |
| $(T_5)$ | $L_2L_3L_5$ | $\gamma_{235}$ |
| $(T_6)$ | $L_1L_2L_4$ | $\gamma_{124}$ |
| $(T_7)$ | $L_1L_3L_4$ | $\gamma_{134}$ |
| $(T_8)$ | $L_1L_3L_5$ | $\gamma_{135}$ |
| $(T_9)$ | $L_4L_5L_7$ | $\gamma_{457}$ |
| $(T_{10})$ | $L_1L_5L_7$ | $\gamma_{157}$ |

(4)

Corresponding to these ten triangles, there is a 6-polytope $P$ of $\mathcal{C}'_{\mathbf{R}}(\Delta, D_4)$ which we are going to define.

**Theorem 2** *There is a simply connected and connected component $P$ of $\mathcal{C}'_{\mathbf{R}}(\Delta, D_4)$ surrounded by the following hypersurfaces:*

| | |
|---|---|
| $A_1$ | $Y_{126}, Y_{367}, Y_{456}, Y_{235}, Y_{134}, Y_{157},$ |
| $A_2$ | $Y_{237}, Y_{124}, Y_{135}, Y_{457},$ |
| $A_3$ | $Z_{26,145}, Z_{237,456}, Z_{45,267}, Z_{37,246}, Z_{126,457}, Z_{14,236},$ $Z_{57,146}, Z_{12,467}, Z_{67,123}, Z_{46,357}, Z_{23,567}, Z_{124,367},$ |
| $A_4$ | $W_{1,37,45}, W_{3,26,57}, W_{3,12,45}, W_{1,23,46}, W_{5,14,67}, W_{5,12,37},$ |
| $A_5$ | $W_{1,23,45}, W_{5,14,37}, W_{3,12,57},$ |
| $A_6$ | $X_{123}, X_{145}, X_{357}$ |

(5)

Let $\mathcal{H}(P)$ be the totality of these hypersurfaces. Then, for each $w \in W(E_7)$, $w \cdot P$ is surrounded by the hypersurfaces contained in $w \cdot \mathcal{H}(P)$.

We consider automorphisms of $P$ in $W(E_7)$. Let $s_i, s_{ij}, s_{ijk}$ be the reflections with respect to the roots $\gamma_i, \gamma_{ij}, \gamma_{ijk}$, respectively. Using this notation, we put

$$\sigma_1 = s_{12}s_{34}s_{57}s_6,$$
$$\sigma_2 = s_{35}s_{24}s_{56}s_2.$$

Then it is easy to show that

$$\sigma_1 \cdot P = \sigma_2 \cdot P = P.$$

Let $G_P$ be the group generated by $\sigma_1, \sigma_2$. Then $G_P$ is isomorphic to $S_4$, the symmetric group on four letters. This is proved as follows. First we note that

$$\sigma_1^2 = 1, \quad \sigma_2^3 = 1, \quad (\sigma_1\sigma_2)^4 = 1.$$

Putting

$$\tau_1 = \sigma_2\sigma_1\sigma_2^{-1}, \quad \tau_2 = \sigma_1, \quad \tau_3 = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1,$$

we find that the correspondence

$$\tau_1 \to (12), \quad \tau_2 \to (23), \quad \tau_3 \to (34)$$

induces the isomorphism between $<\tau_1, \tau_2, \tau_3>$ and $S_4$. On the other hand, since $\tau_2\tau_1\tau_2\tau_3 = \sigma_2^{-1}$, it follows that $G_P =<\tau_1, \tau_2, \tau_3>$.

**Lemma 1** *By the $G_P$-action, the hypersurfaces in $A_j$ are transitive ($j = 1, 2, 3, 4, 5, 6$).*

On the other hand, the center of $W(E_7)$ which is isomorphic to $\mathbf{Z}_2$ acts on the 6-polytope $P$ as the identity transformation. Noting that $A_2$ consists of four hypersurfaces, we obtain the following.

**Proposition 1** *The isotropy subgroup of $P$ in $W(E_7)$ is isomorphic to the group $S_4 \times \mathbf{Z}_2$.*

## 4. 6-polytopes adjacent to $P$

To study the 6-polytopes adjacent to $P$, it is better to treat $P' = s_{34}s_{23}s_{236}s_{16}s_{57} \cdot P$ instead of $P$. Then $P'$ is surrounded by the hypersurfaces in TABLE I below:

TABLE I

| | |
|---|---|
| $A_1$ | $Y_{12}, Y_{127}, Y_7, Y_{56}, Y_{34}, Y_{567}$, |
| $A_2$ | $Y_{67}, Y_{23}, Y_{45}, Y_1$, |
| $A_3$ | $Z_{67,345}, Z_{67}, Z_{345}, Z_{45,123}, Z_{12}, Z_{234,567}, Z_{567}, Z_{456}, Z_{17}, Z_{123}, Z_{234}, Z_{23,456}$, |
| $A_4$ | $W_{12,345}, W_{34,567}, W_{127}, W_{567}, W_{56,234}, W_{123,456}$, |
| $A_5$ | $W_{167}, W_{1,23,67}, W_{123}$, |
| $A_6$ | $X_0, X_{167}, X_{123}$ |

(6)

**Lemma 2** *The following hold.*

$$\begin{aligned}
\mathcal{H}(P') \cap \mathcal{H}(s_{12} \cdot P') &= \{H \in \mathcal{H}(P'); s_{12} \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{Y_{12}} \neq \emptyset\} \\
\mathcal{H}(P') \cap \mathcal{H}(s_{67} \cdot P') &= \{H \in \mathcal{H}(P'); s_{67} \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{Y_{67}} \neq \emptyset\} \\
\mathcal{H}(P') \cap \mathcal{H}(s_{13} \cdot P') &= \{H \in \mathcal{H}(P'); s_{13} \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{Z_{123}} \neq \emptyset\} \\
\mathcal{H}(P') \cap \mathcal{H}(s_{14}s_{23} \cdot P') &= \{H \in \mathcal{H}(P'); s_{14}s_{23} \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{W_{567}} \neq \emptyset\} \\
\mathcal{H}(P') \cap \mathcal{H}(s_{47}s_{56} \cdot P') &= \{H \in \mathcal{H}(P'); s_{47}s_{56} \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{W_{123}} \neq \emptyset\} \\
\mathcal{H}(P') \cap \mathcal{H}(us_1 \cdot P') &= \{H \in \mathcal{H}(P'); us_1 \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{X_0} \neq \emptyset\}
\end{aligned}$$

*where $u = (1765432) \in W(E_7)$.*

As a consequence of Lemma 2, we find the following.

**Theorem 3** *The following statements hold:*

*(i) The 6-polytopes $P'$ and $s_{12} \cdot P'$ are adjacent to each other and $\overline{P'} \cap s_{12} \cdot \overline{P'} \subset Y_{12}$.*
*(ii) The 6-polytopes $P'$ and $s_{67} \cdot P'$ are adjacent to each other and $\overline{P'} \cap s_{67} \cdot \overline{P'} \subset Y_{67}$.*
*(iii) The 6-polytopes $P'$ and $s_{13} \cdot P'$ are adjacent to each other and $\overline{P'} \cap s_{13} \cdot \overline{P'} \subset Z_{123}$.*
*(iv) The 6-polytopes $P'$ and $s_{14}s_{23} \cdot P'$ are adjacent to each other and $\overline{P'} \cap s_{14}s_{23} \cdot \overline{P'} \subset W_{567}$.*
*(v) The 6-polytopes $P'$ and $s_{47}s_{56} \cdot P'$ are adjacent to each other and $\overline{P'} \cap s_{47}s_{56} \cdot \overline{P'} \subset W_{123}$.*
*(vi) The 6-polytopes $P'$ and $us_1 \cdot P'$ are adjacent to each other and $\overline{P'} \cap us_1 \cdot \overline{P'} \subset X_0$.*

Theorem 1 is a consequence of the theorem above.

## 5. Diagrams corresponding to 6-polytopes

Let $P'$ be the 6-polytope introduced in the previous section. From TABLE I, we find that there are 10 hypersurfaces in $A_1$, $A_2$ corresponding to positive roots of type $E_7$ below:

TABLE II

| | |
|---|---|
| $A_1$ | $\gamma_{12}, \gamma_{127}, \gamma_7, \gamma_{56}, \gamma_{34}, \gamma_{567}$ |
| $A_2$ | $\gamma_{67}, \gamma_{23}, \gamma_{45}, \gamma_1$ |

(7)

From these 10 roots, it is possible to construct a diagram similar to Dynkin diagrams. Namely, the diagram in question consists of 10 circles attached with the 10 roots in TABLE II. Two circles are connected by a segment if and only if the roots attached to them are not orthogonal each other. Then we find that there are 12 segments in this diagram.

In the same way, to each 6-polytope in $\mathcal{P}_7$, there associates a diagram which is called a *tetradiagram* for the reason that in the case of $P'$, the 4 circles corresponding to the roots in $A_2$ of TABLE II are regarded as 4 vertices of a tetrahedron and the 6 circles corresponding to the roots in $A_1$ of TABLE II are regarded as 6 middle points of the sides of the referred tetrahedron.

To avoid the complexity, we neglect the $\pm$ signs of tetradiagrams. In this manner, we finally obtain a tetra-diagram corresponding to each $P'' \in \mathcal{P}_7$. We denote by $\mathcal{TP}_7$ the set of tetra-diagrams.

We here note that the classification of the arrangements of 7 lines $(l_1, l_2, \cdots, l_7)$ on the real projective plane with conditions 1, 2, 3 is accomplished if we determine the decomposition of $\mathcal{TP}_7$ into $S_7$-orbits. This will be done in the next section.

## 6. Tetradiagrams

We have already introduced tetradiagrams in the previous section. In this section, we treat them in the language of root systems systematic manner.

**Definition 1** *Let $a_i\,(i = 1,2,3,4)$, $b_{ij}\,(1 \le i < j \le 4)$ be roots of $\Delta$. (Assume that $b_{ij} = b_{ji}$ for all $i, j$.) Then the set*

$$\mathsf{A} = \{a_i; i = 1,2,3,4\} \cup \{b_{ij}; 1 \le i < j <\le 4\}$$

*is called a tetrahedral set if the following conditions hold:*
*(i) $\langle a_i, a_j\rangle = 0 \quad (i \neq j)$.*
*(ii) $\langle b_{ij}, b_{i'j'}\rangle = 0 \quad (\{i,j\} \neq \{i',j'\})$.*
*(iii) $|\langle a_i, b_{jk}\rangle| = 0$ if and only if $i \notin \{j,k\}$.*

**Example 1** *The set*

$$\mathsf{U} = \{\gamma_{345}, \gamma_{123}, \gamma_{136}, \gamma_{256}, \gamma_{135}, \gamma_{167}, \gamma_{347}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}$$

*is a tetrahedral set. In particular, the correspondence*

$$\begin{array}{lll}
\gamma_{345} \longrightarrow \mathrm{A}_1 & \gamma_{135} \longrightarrow \mathrm{B}_{12} & \gamma_{124} \longrightarrow \mathrm{B}_{23} \\
\gamma_{123} \longrightarrow \mathrm{A}_2 & \gamma_{167} \longrightarrow \mathrm{B}_{13} & \gamma_{236} \longrightarrow \mathrm{B}_{24} \\
\gamma_{146} \longrightarrow \mathrm{A}_3 & \gamma_{347} \longrightarrow \mathrm{B}_{14} & \gamma_{257} \longrightarrow \mathrm{B}_{34} \\
\gamma_{256} \longrightarrow \mathrm{A}_4 & &
\end{array}$$

*induces a tetradiagram for $\mathsf{U}$.*

For a tetrahedral set $\mathsf{A} = \{a_i\} \cup \{b_{ij}\}$, we put

$$\tilde{\mathsf{A}} = \{\pm a_i\} \cup \{\pm b_{ij}\}$$

and call it an extended tetrahedral set. Let $\mathsf{A}'$ be also a tetrahedral set. Then $\mathsf{A}$ and $\mathsf{A}'$ are *equivalent* if and only if $\tilde{\mathsf{A}} = \tilde{\mathsf{A}}'$. In this case, we confuse a tetradiagram for $\mathsf{A}$ and that for $\mathsf{A}'$, for simplicity.

The $G_1$-orbit structure will be also important. For this purpose, we will introduce fourteen extended tetrahedral sets in TABLE III. If $\mathsf{A}$ is a tetrahedral set whose extended tetrahedral set has the name X in TABLE III, we denote by $O_{\mathrm{X}}$ the $S_7$-orbit of $\tilde{\mathsf{A}}$. If $\mathsf{B}$ is a tetrahedral set such that $\tilde{\mathsf{B}} \in O_{\mathrm{X}}$, we call $\mathsf{B}$ a tetrahedral set of type X. Similarly, we call a tetradiagram for $\mathsf{B}$ that of type X.

We are going to give a classification of $S_7$-orbital structure of extended tetrahedral sets. For this purpose, we define

$$L = \{\mathrm{A}, \mathrm{B1}, ..., \mathrm{B5}, \mathrm{C1}, ..., \mathrm{C4}, \mathrm{D1}, ...\mathrm{D4}\}.$$

**Theorem 4** *The set $\mathcal{T}$ of extended tetrahedral sets is decomposed into fourteen $S_7$-orbits $O_{\mathrm{X}}$ $(X \in L)$.*

It is clear from the definition that $\mathcal{T}$ and $\mathcal{TP}_7$ are isomorphic. Therefore the classification of $S_7$-orbits of $\mathcal{T}$ is accomplished by Theorem 4.

TABLE III

| Name | Extended tetrahedral set | Isotropy |
|---|---|---|
| A | $\pm\{\gamma_{345}, \gamma_{123}, \gamma_{146}, \gamma_{256}, \gamma_{135}, \gamma_{167}, \gamma_{347}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}$ | $\mathbf{Z}_3$ |
| B1 | $\pm\{\gamma_{345}, \gamma_{123}, \gamma_{146}, \gamma_{256}, \gamma_{25}, \gamma_{167}, \gamma_{347}, \gamma_{34}, \gamma_{16}, \gamma_{257}\}$ | $\mathbf{Z}_3$ |
| B2 | $\pm\{\gamma_{345}, \gamma_{123}, \gamma_{146}, \gamma_{256}, \gamma_{14}, \gamma_{2}, \gamma_{57}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}$ | 1 |
| B3 | $\pm\{\gamma_{14}, \gamma_{25}, \gamma_{146}, \gamma_{256}, \gamma_{135}, \gamma_{167}, \gamma_{347}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}$ | 1 |
| B4 | $\pm\{\gamma_{345}, \gamma_{123}, \gamma_{146}, \gamma_{256}, \gamma_{6}, \gamma_{167}, \gamma_{23}, \gamma_{17}, \gamma_{236}, \gamma_{45}\}$ | 1 |
| B5 | $\pm\{\gamma_{2}, \gamma_{123}, \gamma_{47}, \gamma_{256}, \gamma_{135}, \gamma_{167}, \gamma_{347}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}$ | 1 |
| C1 | $\pm\{\gamma_{15}, \gamma_{24}, \gamma_{36}, \gamma_{7}, \gamma_{25}, \gamma_{167}, \gamma_{17}, \gamma_{34}, \gamma_{346}, \gamma_{6}\}$ | 1 |
| C2 | $\pm\{\gamma_{25}, \gamma_{14}, \gamma_{146}, \gamma_{256}, \gamma_{345}, \gamma_{5}, \gamma_{27}, \gamma_{34}, \gamma_{16}, \gamma_{257}\}$ | 1 |
| C3 | $\pm\{\gamma_{36}, \gamma_{7}, \gamma_{15}, \gamma_{24}, \gamma_{246}, \gamma_{167}, \gamma_{347}, \gamma_{356}, \gamma_{145}, \gamma_{257}\}$ | $\mathbf{Z}_3$ |
| C4 | $\pm\{\gamma_{6}, \gamma_{37}, \gamma_{146}, \gamma_{256}, \gamma_{157}, \gamma_{26}, \gamma_{347}, \gamma_{34}, \gamma_{267}, \gamma_{15}\}$ | 1 |
| D1 | $\pm\{\gamma_{56}, \gamma_{123}, \gamma_{13}, \gamma_{256}, \gamma_{136}, \gamma_{157}, \gamma_{57}, \gamma_{124}, \gamma_{24}, \gamma_{1}\}$ | 1 |
| D2 | $\pm\{\gamma_{47}, \gamma_{123}, \gamma_{2}, \gamma_{256}, \gamma_{14}, \gamma_{146}, \gamma_{57}, \gamma_{6}, \gamma_{236}, \gamma_{23}\}$ | 1 |
| D3 | $\pm\{\gamma_{46}, \gamma_{123}, \gamma_{146}, \gamma_{23}, \gamma_{14}, \gamma_{147}, \gamma_{367}, \gamma_{7}, \gamma_{25}, \gamma_{257}\}$ | 1 |
| D4 | $\pm\{\gamma_{37}, \gamma_{6}, \gamma_{146}, \gamma_{256}, \gamma_{157}, \gamma_{136}, \gamma_{57}, \gamma_{124}, \gamma_{1}, \gamma_{24}\}$ | 1 |

(Here $\pm\{\gamma_{345}, \gamma_{123}, \gamma_{146}, \ldots\}$ is an abbreviation of $\{\pm\gamma_{345}, \pm\gamma_{123}, \pm\gamma_{146}, \ldots\}$.)

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