

ON N-SEMIGROUPS

By

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A semigroup S is called an N -semigroup, due to Petrich[8], if S is commutative, cancellative, archimedean, nonpotent(without idempotent). The concept of N -semigroups is very important in the theory of commutative semigroups([2], [4], [16]). N -semigroups were employed firstly in 1956, by Hewitt and Zuckerman[4] in a paper on the semilattice decomposition of commutative semigroups. Since then many papers on N -semigroups have appeared. In this paper we shall discuss synthetically major results of these.

1. TAMURA'S REPRESENTATION

In this section we shall discuss a faithful representation for N -semigroups which is fundamental in the study of N -semigroups. The following 1.1, 1.2, 1.3 are due to Tamura[17] and 1.4 is due to Sasaki[14].

Let S be an N -semigroup and let a be a fixed element of S . Then any element x of S is uniquely expressed as $x = a^n p_a$, where $p_a \in S \setminus aS$ and n is a non-negative integer, since $S = \bigcup_{i=0}^{\infty} P_i$, $P_0 = S \setminus aS$, $P_i =$

$a^i S \setminus a^{i+1} S$ and $P_i \neq \emptyset$, $P_i \cap P_j = \emptyset$ for $i \neq j$.

1.1. Group decomposition. We define two relations τ_a and ρ_a on S as follows:

$x \tau_a y$ if and only if $x = a^n y$ for some integer $n \geq 0$,

$x \rho_a y$ if and only if $x \tau_a y$ or $y \tau_a x$.

Then τ_a is a compatible partial ordering and ρ_a is a congruence generated by τ_a , hence we get the following:

Theorem 1. The factor semigroup $S_a^* = S/\rho_a$ is a commutative group, and is a homomorphic image of S . And each ρ_a -class S_λ of S is an infinite chain with respect to $\tau_a|_{S_\lambda}$ and it contains exactly one $\tau_a|_{S_\lambda}$ -maximal element, which is called the prime respecting a .

S_a^* is called the structure group of S with respect to a , and a is called the standard element for S_a^* . We note that the set of all primes respecting a coincides with $P_0 = S \setminus aS$. Let p_λ denote the prime respecting a contained in S_λ . If ε denotes the identity of S_a^* , $p_\varepsilon = a$. For $p_\alpha, p_\beta \in P_0$, there exists unique non-negative integer n such that $p_\alpha p_\beta = a^n p_{\alpha\beta}$. We define here $n = I(\alpha, \beta)$, then $I(\alpha, \beta)$ becomes an index function on S_a^* , which shall be called the index function corresponding to S_a^* . By an index function I on a commutative group G we mean a mapping of $G \times G$ into the additive non-negative integers satisfying the following conditions:

$$(1) I(\alpha, \beta) = I(\beta, \alpha),$$

$$(2) I(\alpha, \beta) + I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma) + I(\beta, \gamma),$$

(3) $I(\varepsilon, \varepsilon) = 1$, ε is the identity of G ,

(4) for any $\alpha \in G$, there exists a positive integer n such that $I(\alpha, \alpha^n) > 0$.

1.2. Construction. The following theorem is important.

Theorem 2. Let G be a commutative group and let I be an index function on G . Let S be the product set of the additive non-negative integers and G , and define a binary operation on S by $(m, \alpha)(n, \beta) = (m + n + I(\alpha, \beta), \alpha\beta)$. Then S is an N -semigroup under this operation and $S_{(0, \varepsilon)}^*$, ε is the identity of G , is isomorphic upon G . Also every N -semigroup may be obtained in this manner.

By $S(G; I)$ we denote the N -semigroup constructed from a commutative group G with an index function I in the above mentioned manner.

1.3. Faithful representation. From the foregoing description we see that a mapping $x = a^n p_\alpha \rightarrow (n, \bar{p}_\alpha)$, $\bar{p}_\alpha$ is the p_α -class of S containing p_α , gives a faithful representation of S by $S(S_\alpha^*; I)$, which shall play an important role in the study of N -semigroups.

In this paper, a representation of an N -semigroup shall mean always one as given in the above.

1.4. Generalized index functions. Consider a mapping I of the product set $G \times G$ of a commutative group G and itself into the additive non-negative integers satisfying the conditions (1), (2)

and (3'), (4') below:

$$(3') \quad I(\varepsilon, \varepsilon) > 0,$$

$$(4') \quad \text{for any } \alpha \text{ there exists a positive integer } m \text{ such that } I(\alpha, \alpha) + \dots + I(\alpha, \alpha^m) \geq I(\varepsilon, \varepsilon).$$

Let G be a commutative group with the above generalized index function I , and let S be the product set of the additive non-negative integers and G . Define a binary operation on S by $(m, \alpha)(n, \beta) = (m + n + I(\alpha, \beta), \alpha\beta)$, then S becomes an N -semigroup. Such S shall be denoted by $S(G: I)$. Especially, in the case $I(\varepsilon, \varepsilon) = 1$, $S(G: I)$ shall be understood as $S(G: I)$. Let $G_{(0, \varepsilon)}$, $I_{(0, \varepsilon)}$ be the structure group of $S(G: I)$ with respect to $(0, \varepsilon)$, ε is the identity of G , and its corresponding index function. Then we can show that $S(G: I)$ is isomorphic upon $S(G_{(0, \varepsilon)}: I_{(0, \varepsilon)})$. Thus we have

Theorem 3. For given any $S(G: I)$ there exist a group-extension G' of a finite cyclic group by G and its corresponding index function I' such that $S(G: I) \cong S(G': I')$.

And we have

Theorem 4. For given any $S(G': I')$, whose G' has a cyclic subgroup of order $c > 1$, there exists $S(G: I)$ such that $S(G': I') \cong S(G: I)$, $I(\varepsilon, \varepsilon) = c$, $G' = G_{(0, \varepsilon)}$, $I' = I_{(0, \varepsilon)}$, where ε is the identity of G , if and only if it holds that $cI'(\overline{(m, \alpha)}, \overline{(n, \beta)}) + J(\overline{(m, \alpha)}, \overline{(n, \beta)}) - m - n = cI'(\overline{(0, \alpha)}, \overline{(0, \beta)}) + J(\overline{(0, \alpha)}, \overline{(0, \beta)})$ for all

$0 \leq m, n < c$, $\alpha, \beta \in G$, where $\overline{(m, \alpha)}$ is the equivalence class modulo p , $S(G: I)/p = G'$, containing $\overline{(m, \alpha)}$ and $J(\overline{(m, \alpha)}, \overline{(n, \beta)})$ is a non-negative integer valued function on $G' \times G'$ such that $\overline{(m, \alpha)} \overline{(n, \beta)} = \overline{(J(\overline{(m, \alpha)}, \overline{(n, \beta)}), \alpha\beta)}$, $0 \leq J(\overline{(m, \alpha)}, \overline{(n, \beta)}) < c$.

2. INDEX FUNCTIONS

It was shown in Theorem 2 that if an index function is defined on a commutative group G , then we can construct an N -semigroup such that its structure group is isomorphic upon G . Given a commutative group G , there always exists an index function, for example $I(\alpha, \beta) = 1$ for all $\alpha, \beta \in G$, but it is not easy to determine all the index functions on G . In this section we will discuss how to determine all the index functions on given a finitely generated commutative group and also discuss the semigroup of all generalized index functions on given a commutative group. The following 2.1 is due to Pigge, Tamura and Sasaki[1] and 2.2 is due to Sasaki[14].

2.1. Determination of all index functions. The results are as follows:

Theorem 5. If G is a cyclic group of order n generated by α , the index function values $I(\alpha, \alpha^k)$, $k = 1, \dots, n-1$, are independent up to relative size considerations and every other function value is determined from these $n-1$ values by $I(\alpha^i, \alpha^j) = I(\alpha, \alpha^{i+j-1}) + [j, i-1]_{i-1}$, $i \geq 2$, where $[m, n]_i$ denotes $\sum_{p=0}^{i-1} (I(\alpha, \alpha^{m+p}) - I(\alpha, \alpha^{n-p}))$ if

$l > 0$, and does 0 if $l = 0$. The relative sizes of the independent $I(\alpha, \alpha^k)$, $k = 1, \dots, n-1$, are determined as follows:

the case $n = 2$, $I(\alpha, \alpha) \geq 0$,

the case $n = 3$, $I(\alpha, \alpha) \geq 0$, $I(\alpha, \alpha^2) \geq \max\{0, I(\alpha, \alpha) - 1\}$,

the case $n = 4$, $I(\alpha, \alpha) \geq 0$, $I(\alpha, \alpha^2) \geq 0$, $I(\alpha, \alpha^3) \geq \max\{0, I(\alpha, \alpha) - I(\alpha, \alpha^2), I(\alpha, \alpha) - 1, I(\alpha, \alpha^2) - 1\}$,

the case $n \geq 5$, $I(\alpha, \alpha^k) \geq \bar{m}(k)$, $k = 1, \dots, n-2$, $I(\alpha, \alpha^{n-1}) \geq \max\{\bar{m}'(0), \bar{m}'(1), \dots, \bar{m}'(n-5), \bar{m}(n-1), \max_{1 \leq i \leq n-2} \{I(\alpha, \alpha^i) - 1\}\}$,

where $\bar{m}(k) = \max_{0 \leq i \leq \lfloor \frac{k-1}{2} \rfloor} \{[1, k-1]_i\}$ and $\bar{m}'(k) = \max_{1 \leq i \leq \lfloor \frac{n-k-3}{2} \rfloor} \{[k+1, n-2]_i + I(\alpha, \alpha^{i+k+1}) - 1\}$.

Theorem 6. If G is an infinite cyclic group generated by α , the function values $I(\alpha, \alpha^k)$, $k = \pm 1, \dots$, are independent up to relative size considerations and every other function value is determined from these by $I(\alpha^i, \alpha^j) = I(\alpha, \alpha^{i+j-1}) + [j, i-1]_{i-1}$ if $i \geq 2$ and $I(\alpha^i, \alpha^j) = I(\alpha, \alpha^i) + [i+1, j-1]_{-1}$ if $i \leq -1$. And the relative sizes of the $I(\alpha, \alpha^k)$, $k = \pm 1, \dots$, are given as follows: $I(\alpha, \alpha^k) \geq \bar{m}(k)$ for $k \geq 1$, $I(\alpha, \alpha^{-1}) \geq \bar{n}(-1)$, $\bar{n}'(-k) \geq I(\alpha, \alpha^{-k}) \geq \bar{n}(-k)$ for $k \geq 1$, and for any non-zero integer s there exists a positive integer t_s such that $[st_s, s-1]_s \geq 0$ if $s \geq 1$, $[s, st_s-1]_{-s} \geq 0$ if $s \leq -1$, where $\bar{n}(-k) = \max_{0 \leq i \leq \lfloor \frac{k-2}{2} \rfloor} \{[1, i-k]_i\}$, $\bar{n}'(-k) = \min_{0 \leq i \leq \lfloor \frac{k-2}{2} \rfloor} \{I(\alpha, \alpha^{-1}) + 1 + [i-1, i-k]_i\}$.

Theorem 7. Suppose that the direct product $G = A \times B$ of two commutative groups A, B and that index function values I_A for A and I_B for B are already given. Then the set $I_{A,B}$ of function values $I((\alpha, \varepsilon'), (\xi, \beta)), \alpha \in A \setminus \{\varepsilon\}, \beta \in B \setminus \{\varepsilon'\}$, where $\varepsilon, \varepsilon'$ are the identities of A and B , are independent up to relative size considerations and every other value $I((\alpha_1, \beta_1), (\alpha_2, \beta_2))$ is determined from I_A, I_B and $I_{A,B}$ by $I((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = I((\alpha_1, \varepsilon'), (\alpha_2, \varepsilon')) + I((\varepsilon, \beta_1), (\varepsilon, \beta_2)) + I((\alpha_1 \alpha_2, \varepsilon'), (\varepsilon, \beta_1 \beta_2)) - I((\alpha_1, \varepsilon'), (\varepsilon, \beta_1)) - I((\alpha_2, \varepsilon'), (\varepsilon, \beta_2))$.

The complete solution for the relative sizes of the independent elements in the above Theorem 7 is not yet obtained.

Theorem 8. Let $G = A_1 \times \dots \times A_n$ be the direct product of n commutative groups $A_1, \dots, A_n$. Suppose that I -values I_i for $A_i, i = 1, \dots, n$, are already given, and consider the sets I'_j of function values: $I'_j((\alpha_1, \dots, \alpha_{j-1}, \varepsilon_j, \dots, \varepsilon_n), (\varepsilon_1, \dots, \varepsilon_{j-1}, \alpha_j, \varepsilon_{j+1}, \dots, \varepsilon_n)), j = 2, \dots, n$, where $(\alpha_1, \dots, \alpha_{j-1}, \varepsilon_j, \dots, \varepsilon_n) \neq (\varepsilon_1, \dots, \varepsilon_n), (\varepsilon_1, \dots, \varepsilon_{j-1}, \alpha_j, \varepsilon_{j+1}, \dots, \varepsilon_n) \neq (\varepsilon_1, \dots, \varepsilon_n)$ and ε_k is the identity of A_k . Then the union of $I'_j, j = 2, \dots, n$, is a set of I -values independent up to relative size considerations and every other value is determined from $I_1, \dots, I_n, I'_2, \dots, I'_n$.

Since every finitely generated commutative group is the direct product of a finite number of cyclic groups, the results obtained above can be applied to any finitely generated commutative group.

All the index functions on groups of order ≤ 4 are given as the following tables:

Group of order 1.

	ε
ε	1

Group of order 2.

	ε	α
ε	1	1
α	1	1

$i \geq 0.$

Group of order 3.

	ε	α	α^2
ε	1	1	1
α	1	1	j
α^2	1	j	j+1-i

$i \geq 0,$
 $j \geq \max\{0, i-1\}.$

Group of order 4.

cyclic group.

	ε	α	α^2	α^3
ε	1	1	1	1
α	1	1	j	k
α^2	1	j	j+k-i	1+k-i
α^3	1	k	1+k-i	1+k-j

$i \geq 0,$
 $j \geq 0,$
 $k \geq \max\{0, i-j, i-1, j-1\}.$

Klein's group.

	ε	α	β	$\alpha\beta$
ε	1	1	1	1
α	1	1	j	i+1-j
β	1	j	k	k+1-j
$\alpha\beta$	1	i+1-j	k+1-j	k+i+1-2j

$j \geq 0,$
 $i \geq \max\{0, j-1\},$
 $k \geq \max\{0, j-1, 2j-i-1\}.$

2.2. Semigroups of generalized index functions. Let $\mathfrak{J}(G)$ be

the set of all generalized index functions satisfying (1), (2), (3'),

(4') in 1.4 on given a commutative group G . $\mathfrak{J}(G)$ is not empty.

Define $I_1 + I_2$ and aI_1 for $I_1, I_2 \in \mathfrak{J}(G)$ and non-negative integer a .

as follows:

$$(I_1 + I_2)(\alpha, \beta) = I_1(\alpha, \beta) + I_2(\alpha, \beta), \quad (aI_1)(\alpha, \beta) = a(I_1(\alpha, \beta)).$$

Then $I_1 + I_2 \in \mathfrak{J}(G)$, $aI_1 \in \mathfrak{J}(G)$ and

Theorem 9. $\mathfrak{J}(G)$ forms a commutative, cancellative, nonpotent semigroup under $(+)$ and satisfies $a(I_1 + I_2) = aI_1 + aI_2$, $(a + b)I_1 = aI_1 + bI_1$, $(ab)I_1 = a(bI_1)$.

Let I_1, I_2 be any elements of $\mathfrak{J}(G)$. Consider the direct product of $S(G: I_1)$ and $S(G: I_2)$ and define a relation σ on it as follows:

$((m, \alpha), (n, \beta)) \sigma ((m', \alpha'), (n', \beta'))$ if and only if $m + n = m' + n'$, $\alpha = \alpha'$, $\beta = \beta'$.

Then we have that $S(G: I_1 + I_2) \cong (S(G: I_1) \times S(G: I_2)) / \sigma$. And it is easily seen that $S(G: I_1)$ is isomorphic upon a subsemigroup of $S(G: aI_1)$ for any positive integer a . Thus we have

Theorem 10. $S(G: I_1 + I_2)$, $I_1, I_2 \in \mathfrak{J}(G)$, can be embedded isomorphically onto a homomorphic image of the direct product $S(G: I_1) \times S(G: I_2)$, and $S(G: I_1)$ be embedded into $S(G: aI_1)$ for any positive integer a .

3. STRUCTURE GROUPS

In this section we shall consider the relations between the structure groups of an N-semigroup S with respect to a and b of S and their corresponding index functions. All results in this section are due to Sasaki[12].

Let $S = S(G: I)$ be an N -semigroup. Let $G_{(m, \alpha)}$ and $G_{(n, \beta)}$ be the structure groups of S with respect to (m, α) and $(n, \beta) \in S$ respectively, and let $\rho_{(m, \alpha)}, \rho_{(n, \beta)}$ be congruences on S such that $G_{(m, \alpha)} = S/\rho_{(m, \alpha)}, G_{(n, \beta)} = S/\rho_{(n, \beta)}$ respectively. For simplicity we shall denote by $\overline{(l, \xi)}$ and $\overline{(l, \xi)}$ the equivalence classes of S modulo $\rho_{(m, \alpha)}$ and $\rho_{(n, \beta)}$ containing $(l, \xi) \in S$ respectively. Then the following holds:

Theorem 11. $G_{(m, \alpha)}/[\overline{(n, \beta)}]$ is isomorphic upon $G_{(n, \beta)}/[\overline{(m, \alpha)}]$, where $[\eta]$ means the cyclic group generated by η .

From the above theorem we easily see that $G_{(m, \alpha)}$ is the group-extension of $[\overline{(0, \xi)}]$ by G .

For $(m, \alpha) \in S(G: I)$, $\xi \in G$ and for a non-negative integer p , we put $\prod_p((m, \alpha), \xi) = mp + \rho_1^{p-1}(\alpha) + I(\alpha^p, \alpha^{-p}\xi)$, where $\rho_r^s(\alpha)$ means

$$\sum_{i=r}^s I(\alpha, \alpha^i) \text{ if } s-r \geq 0, \text{ does } 0 \text{ if } s-r = -1, \text{ and } -\sum_{i=s+1}^{r-1} I(\alpha, \alpha^i) \text{ if } s-r \leq -2.$$

Then the following three conditions are equivalent:

- a) $(0, \alpha^{-p}\xi) \in S \setminus (m, \alpha)S$,
- b) $m + I(\alpha, \alpha^{-p-1}\xi) > 0$,
- c) $\prod_{p+1}((m, \alpha), \xi) - \prod_p((m, \alpha), \xi) > 0$.

Therefore we can state how to construct $G_{(n, \beta)}$ from given $G_{(m, \alpha)}$:

- (i) Select all primes of type $(0, \xi)$ respecting (m, α) ,
(ii) for each $(0, \xi)$ in (i), take $(0, \xi), (0, \alpha\xi), \dots, (0, \alpha^p \xi)$,
where p is a positive integer such that $0 = mp + \rho_1^{p-1}(\alpha) + I(\alpha^p, \xi)$
 $< m(p+1) + \rho_1^p(\alpha) + I(\alpha^{p+1}, \xi)$,
(iii) and put $(0, \alpha^i \xi), (1, \alpha^i \xi), \dots, (n-1 + I(\beta, \beta^{-1} \alpha^i \xi), \alpha^i \xi)$
for all $i (0 \leq i \leq p)$ such that $n + I(\beta, \beta^{-1} \alpha^i \xi) > 0$.

Then all members in (iii) form just the structure group $G_{(n, \beta)}$.

Let $(r, \xi), (s, \eta)$ be any primes of $S(G: I)$ respecting (m, α) .

Then the index function $I_{(m, \alpha)}$ corresponding to $G_{(m, \alpha)}$ is given by

$I_{(m, \alpha)}(\overline{(r, \xi)}, \overline{(s, \eta)}) = p$, where p is a non-negative integer such
that $\prod_p((m, \alpha), \xi \eta) \leq r + s + I(\xi, \eta) < \prod_{p+1}((m, \alpha), \xi \eta)$. Let $I_{(m, \alpha)}$

and $I_{(n, \beta)}$ be the index functions corresponding to $G_{(m, \alpha)}$ and $G_{(n, \beta)}$

respectively. Then we get the following:

Theorem 12. Let (r, ξ) and (s, η) be primes of S respecting
 (m, α) . If $\prod_p((m, \alpha), \xi \eta) \leq r + s + I(\xi, \eta) < \prod_{p+1}((m, \alpha), \xi \eta)$

and $\prod_q((n, \beta), \xi \eta) \leq r + s + I(\xi, \eta) < \prod_{q+1}((n, \beta), \xi \eta)$, then

$I_{(m, \alpha)}(\overline{(r, \xi)}, \overline{(s, \eta)}) = p$ and $I_{(n, \beta)}(\overline{(r, \xi)}, \overline{(s, \eta)}) = q - k - h$,

where k and h are non-negative integers such that $\prod_k((n, \beta), \xi) \leq r <$

$\prod_{k+1}((n, \beta), \xi), \prod_h((n, \beta), \eta) \leq s < \prod_{h+1}((n, \beta), \eta)$.

4. THE ISOMORPHISM PROBLEM

The problem of distinct representations for isomorphic N-semigroups was proposed by Tamura[17], and was discussed by Sasaki[11], [12]. Lately Higgins[5] gave an isomorphism theorem, for finitely generated N-semigroups, which depends on the canonical representation. The following theorem is due to Sasaki [11], [12].

Theorem 13. Let $S(G: I)$, $S(G': I')$ be N-semigroups. $S(G: I)$ is isomorphic upon $S(G': I')$ if and only if there exist cyclic subgroups $[\omega]$ of G and $[\omega']$ of G' such that $G/[\omega] \cong G'/[\omega']$ (under

ψ) and there exist representative systems $\Gamma = \{\xi_\alpha\}$ of the cosets of $[\omega]$ in G and $\Gamma' = \{\xi'_\alpha\}$ of the cosets of $[\omega']$ in G' satisfying

(1) for any $\xi_\alpha, \xi_\beta \in \Gamma$, if $\xi_\alpha \xi_\beta = \omega^l \xi_\gamma$, $\xi_\gamma \in \Gamma$, then $\xi_\alpha \tau \cdot \xi_\beta \tau = \omega'^{l'} \cdot \xi_\gamma \tau$ and $I'(\xi_\alpha \tau, \xi_\beta \tau) = n'l' + l + \rho_{-l}^0(\omega') - I'(\omega'^{-l'}, \xi_\alpha \tau \cdot \xi_\beta \tau)$, $l' = -nl + I(\xi_\alpha, \xi_\beta) - \rho_{-l}^0(\omega) + I(\omega^{-l}, \xi_\alpha \xi_\beta)$,

(2) for any integer s and $\xi_\alpha \in \Gamma$, $\xi'_\beta \in \Gamma'$

(i) $s + n't(s) + \rho_1^{t(s)-1}(\omega') + I'(\omega'^{t(s)}, \xi_\alpha \tau) \geq 0$, $t(s) = -ns - \rho_1^{s-1}(\omega) - I(\omega^s, \xi_\alpha)$ and

(ii) $s + nt'(s) + \rho_1^{t'(s)-1}(\omega) + I(\omega^{t'(s)}, \xi'_\beta \tau^{-1}) \geq 0$, $t'(s) = -n's - \rho_1^{s-1}(\omega') - I'(\omega'^s, \xi'_\beta)$, where τ is a mapping of Γ onto

Γ' such that $\tau: \xi_\alpha \rightarrow \xi'_\alpha$ if $([\omega]\xi_\alpha)\psi = [\omega']\xi'_\alpha$ and n, n' are non-

negative integers such that $h = n'h' + \rho_1^{h'}(\omega')$ and $h' = nh + \rho_1^h(\omega)$

for the orders h and h' of ω and ω' , where $h = 0$ means the order of ω is infinite and h' does so. $\rho_r^S(\alpha')$ is the symbol obtained by replacing I, α with I', α' in $\rho_r^S(\alpha)$.

Suppose that S is a finitely generated N -semigroup. Then all the structure groups of S are finite (Chrislock[3]), so there is an element $a \in S$ such that the structure group G_a of S with respect to a is of minimal order. Such a is called a normal standard. Let G_a and I_a be the structure group of S with respect to a normal standard element a of S and its corresponding index function. Then the representation $S(G_a : I_a)$ for S is called a canonical, due to Higgins[5]. We should note that an N -semigroup S may have many normal standard elements, so the canonical representation for S is not unique in general. Higgins[5] gave the following theorem:

Theorem 14. Let S and S' be finitely generated N -semigroups and let $S(G : I)$ and $S(G' : I')$ be canonical representations for S and S' respectively. Then S is isomorphic upon S' if and only if G'' with I'' equivalent to G' with I' can be obtained from G with I by changing normal standard elements.

We say that G'' with I'' is equivalent to G' with I' if G'' is isomorphic upon G' under some mapping φ such that $I''(\xi, \eta) = I'(\xi\varphi, \eta\varphi)$ for $\xi, \eta \in G''$.

The theorem corresponding to Theorem 14 in general case is given as follows (Sasaki[13]):

Theorem 15. Let S and S' be N -semigroups and let $S(G: I)$ and $S(G': I')$ be representations for S and S' respectively. Then S is isomorphic upon S' if and only if there is an element $(0, \alpha) \in S$ such that $G_{(0, \alpha)}$ with $I_{(0, \alpha)}$ is equivalent to G' with I' , where $G_{(0, \alpha)}$, $I_{(0, \alpha)}$ are the structure group of S with respect to $(0, \alpha)$ and its corresponding index function.

5. FINITELY GENERATED N -SEMIGROUPS

The following classification for N -semigroups is due to Petrich[8]:

N -semigroups	$\left\{ \begin{array}{l} \text{power-joined} \\ \text{not power-joined} \end{array} \right.$	$\left\{ \begin{array}{l} \text{finitely generated} \\ \text{not finitely generated} \end{array} \right.$	$\left\{ \begin{array}{l} 1 \text{ generator} \\ 2 \text{ generators} \\ \text{more than 2 generators} \end{array} \right.$
		$\left\{ \begin{array}{l} \text{with indecomposable element (indec. e.)} \\ \text{without indec. e.} \end{array} \right.$	
			$\left\{ \begin{array}{l} \text{with indec. e.} \\ \text{without indec. e.} \end{array} \right.$

An element a is said to be indecomposable if $a \neq bc$ for all b, c .

Chrislock[3] showed the following in a more general form:

An N -semigroup S is a finitely generated if and only if all the structure groups of S are finite, S is a power joined, i.e. for

any a, b there exist positive integers m, n such that $a^m = b^n$, if and only if all the structure groups of S are periodic.

Petrich[8] has given a representation for N -semigroups with two generators by the set of ordered pairs of non-negative integers and Higgins[5] has given a new and somewhat distinct representation for finitely generated N -semigroups. In this section we shall treat of Higgins' representation for finitely generated N -semigroups, that is, of the embedding of finitely generated N -semigroups in the direct product of a finite commutative group and the additive positive integers. Excepting theorems 16 and 17, all the results in this section are due to Higgins[5].

5.1. Homomorphisms. We shall consider the homomorphisms of finitely generated N -semigroups into the additive positive integers and onto groups. The following two theorems are required:

Theorem 16(Tamura[18]). Let Ξ be a set of implications. Let τ be the class of all semigroups satisfying all implications in Ξ . Then every semigroup has a greatest homomorphic image of type τ .

Theorem 17(Levin and Tamura[6]). Any power joined and power cancellative N -semigroup can be embedded in the additive positive rational numbers.

Let S be a finitely generated N -semigroup. Then, by Theorems 16 and 17, S has a greatest power joined, power cancellative

homomorphic image M which is isomorphic upon a subsemigroup of the additive positive integers. Since any homomorphism between subsemigroups of the additive positive integers is an isomorphism, any positive integer homomorphic image of S is isomorphic upon M . Thus we have

Theorem 18. Let S be a finitely generated N -semigroup. Then the homomorphic image of S into the additive positive integers is uniquely determined.

Let G be any group homomorphic image of a finitely generated N -semigroup S . Take some element a of the pre-image of the identity of G , then G is a homomorphic image of the structure group of S with respect to a . Therefore we get

Theorem 19. Let S be a finitely generated N -semigroup. Let G be any group homomorphic image of S . Then G is a homomorphic image of some structure group of S .

5.2. J-functions. Let G be a finite commutative group.

Consider a mapping J of G into the positive integers satisfying the following conditions:

- (1) $J(e) = |G|$, where e is the identity of G ,
- (2) for all $a, b \in G$, it holds $J(a) + J(b) - J(ab) = k|G|$, where k is a non-negative integer,
- (3) for every $a \in G$, there is a positive integer n such that

$J(a) + J(a^n) - J(a^{n+1}) = h|G|$, and h is a positive integer. Then we can show that there is one and only one J -function which can be defined on any structure group of a finitely generated N -semigroup S and that $J(a)$ is equal to the number of elements of S which are prime to a , hence the J -function may be considered as a function defined on S . And we can also show that the J -function on a finitely generated N -semigroup S is a homomorphism of S into the additive positive integers. Thus the foregoing homomorphic image M of S in 5.1 may be constructed from the J -function.

5.3. Subdirect products. A semigroup S is a subdirect product of the direct product $R \times T$ of two semigroups R and T if S is a sub-semigroup of $R \times T$ and the projections of S into R and T exactly coincide with R and T respectively. It is easily seen that S is isomorphic upon a subdirect product of $R \times T$ if and only if there exist homomorphisms H and K of S onto R and T , and the intersection of the pre-images of $r \in R$ and $t \in T$ in S contains at most one element of S . Let S be a finitely generated N -semigroup, and let Q be a homomorphism of S onto a finite commutative group G , which may be chosen as the structure group of some representation for S by Theorem 19. As a homomorphism of S into the additive positive integers we may use the J -function. Let M be the homomorphic image of S under J . Then we can show that the intersection of the pre-images of $r \in G$

and $t \in M$ in S , as given by Q and J , contains at most one element of S . Therefore

Theorem 20. A finitely generated N-semigroup is isomorphic upon a subdirect product of the direct product of a finite commutative group and a subsemigroup of the additive positive integers. The converse also holds.

The following is also obtained.

Theorem 21. A finitely generated N-semigroup S is isomorphic upon the direct product of a structure group G and a subsemigroup of the additive positive integers if and only if, for some representation for S in terms of G and its corresponding index function I , every element of the form $(0, \alpha)$ is a normal standard element.

6. POWER JOINED N-SEMIGROUPS

In this section we shall show that in the case of power joined N-semigroups we can obtain the theorems corresponding to Higgins' theorems 18~21. The results in this section are due to Tamura and Sasaki[15].

6.1. Homomorphisms. Let R' be a subsemigroup of the additive positive rational numbers R . Then R' is the union of ascending chain of finitely generated subsemigroups of R :

$$R' = \bigcup_{n=1}^{\infty} R'_n, \quad R'_1 \subseteq R'_2 \subseteq \cdots, \quad R'_n = R' \cap [1/n!].$$

Using Theorems 16 and 17, therefore, we get

Theorem 22. Let S be a power joined N -semigroup. Then the homomorphic image of S into the additive positive rational numbers is uniquely determined.

And we have easily

Theorem 23. Let S be a power joined N -semigroup. Let G be any group homomorphic image of S . Then G is a homomorphic image of some structure group of S .

6.2. $\bar{J}$ -functions. Let G be a periodic commutative group. Define a mapping $\bar{J}$ of G into the additive positive rational numbers as follows:

- (1) $\bar{J}(e) = 1$, e is the identity of G ,
- (2) for all $a, b \in G$, $\bar{J}(a) + \bar{J}(b) - \bar{J}(ab)$ is a non-negative integer.

Then we can prove that there is one and only one $\bar{J}$ -function which can be defined on any structure group of a power joined N -semigroup S . Let G and its corresponding index function I be a representation for S : $S = S(G; I)$. Then we may say the $\bar{J}$ -function on G is given as $\bar{J}(\alpha) = \rho_1^S(\alpha)/s$, where s is the order of α . For $(m, \alpha) \in S$ we define $\bar{J}((m, \alpha)) = m + \bar{J}(\alpha)$. Then the function $\bar{J}$ on S becomes a homomorphism of S into the additive positive rational numbers, hence the foregoing homomorphic image of S in Theorem 22 may be constructed from the $\bar{J}$ -function

6.3. Subdirect products. Using the same argument with the case of finitely generated, we obtain the following theorems:

Theorem 24. A power joined N-semigroup is isomorphic upon a subdirect product of the direct product of a periodic commutative group and a subsemigroup of the additive positive rational numbers. The converse also holds.

Theorem 25. A power joined N-semigroup is isomorphic upon the direct product of a structure group G and a subsemigroup M of the additive positive rational numbers if and only if M is a subsemigroup of the additive positive integers and for some representation for S in terms of G and its corresponding index function I , $\bar{J}$ -values of all elements of the form $(0, a)$ are equal.

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