

[A] 有界領域における反射 Brownian motion の構成

東京教育大理 福島 正俊

§ 1. 序

D を N 次元ユークリッド空間 R^N の有界領域とする. D の境界 ∂D の滑らかさは仮定しない.

$\lambda > 0$, $x, y \in D$, $x \neq y$ に対して定義された関数 $G_\lambda(x, y)$ が次の条件を満たすとき, G_λ は D 上の resolvent density とする.

$$G. 1) \quad G_\lambda(x, y) \geq 0, \quad \lambda > 0, \quad x, y \in D, \quad x \neq y.$$

$$G. 2) \quad \lambda \int_D G_\lambda(x, y) dy \leq 1, \quad \lambda > 0, \quad x \in D.$$

$$G. 3) \quad G_\alpha(x, y) - G_\beta(x, y) + (\alpha - \beta) \int_D G_\alpha(x, z) G_\beta(z, y) = 0, \quad \alpha, \beta > 0, \quad x, y \in D, \quad x \neq y.$$

$$G. 4) \quad \lambda > 0 \text{ を固定すると } G_\lambda(x, y) \text{ は } (x, y) \text{ に関して連続}$$

$G. 2)$ における等号が任意の λ, x について成立するときは $G_\lambda(x, y)$ は conservative とする.

目的は

i) D 上に G_λ が conservative resolvent density

を構成する。 $\square$

2) したがって D を含むある compact 集合 D^* 上の 拡散過程 (強マルコフ過程 ω path が連続なものを決定する) ω を決定する。 $\square$

1) 2) の拡散過程は ∂D が充分滑らかな場合には $D \cup \partial D$ 上の所謂 反射壁 Brown 運動 と同等である。 $\square$

これを三つの定理として述べよう。

$p(t, x, y)$, $t > 0$, $x, y \in D$, が D 上の transition density であるとは、したがって次の条件を満たすこと。

T. 1) $p(t, x, y) \geq 0$.

T. 2) $\int_D p(t, x, y) dy \leq 1$, $t > 0$, $x \in D$.

T. 3) $\int_D p(t, x, z) p(s, z, y) dz = p(t+s, x, y)$,
 $t, s > 0$, $x, y \in D$.

T. 4) $p(t, x, y)$ は三変数の連続関数

T. 2) に於て等号が全ての t, x について成立すること $\int p(t, x, y) dy = \text{conservative}$ である。

また $p'(t, x, y) \in D$ 上の 吸収壁 Brown 運動 の transition density である (15) ([8] 参照).

1. 1) $G_\alpha^\circ(x, y) = \int_0^{+\infty} e^{-\alpha t} p'(t, x, y) dt$, $\alpha > 0$,
 $x, y \in D$, $\alpha' < \alpha$ $G_\alpha^\circ(x, y)$ は D 上の α の

resolvent density 2) 有.

これは $\mathbb{R}$ の様に $t \geq 0$ である.

$$1.2) \quad G_\alpha^0(x, y) = \pi_\alpha(x, y) - E_x^0(e^{-\alpha \tau} \pi_\alpha(x_\tau, y)),$$

但し

$$\pi_\alpha(x, y) = \int_0^{+\infty} e^{-\alpha t} \frac{1}{(2\pi t)^{1/2}} e^{-\frac{|x-y|^2}{2t}} dt,$$

$x, y \in D$, E_x^0 は N -次元 Brown 運動の measure による平均. τ は D から exit time.

D 上の関数 u が α -harmonic である.

$(\alpha - \frac{1}{2}\Delta)u(x) = 0$ かつ $\forall \varepsilon > 0$ $x \in D$ について $x \in D$ である. 但し Δ は Laplace 微分作用素.

D 上の関数 u, v に対して

$$1.3) \quad (u, v) = \int_D u(x) v(x) dx,$$

$$D(u, v) = \int_D (\text{grad } u, \text{grad } v)(x) dx$$

である.

$$1.4) \quad H_\alpha = \{u : u \text{ は } D \text{ 上の } \alpha\text{-harmonic かつ } (u, u) < +\infty, D(u, u) < +\infty\}$$

である.

定理 1

(1) 各 $\alpha > 0$, $x \in D$ について $\mathbb{R}$ の性質を満たす H_α の元 $R_\alpha^x(y) = R_\alpha(x, y)$ が一意に存在する.

$$1.5) \quad D(R_\alpha^x, v) + \alpha(R_\alpha^x, v) = v(x), \quad \forall v \in H_\alpha.$$

(2) $G_\alpha(x, y) = G_\alpha^0(x, y) + R_\alpha(x, y)$ は D 上の conservative resolvent density 2) 有.

$G_\alpha(x, y)$ は $x, y \in D$ に対して定義される.

(3) $B(D) \subset C(D)$ 上の有界 (有界連続) 関数の全体である.

1.6) $G_\alpha f(x) = \int_D G_\alpha(x, y) f(y) dy$ として定義される.

G_α は $B(D) \subset C(D)$ に作用する. また,

$\lim_{\alpha \rightarrow +\infty} \alpha G_\alpha f(x) = f(x), \quad \forall f \in C(D), \quad \forall x \in D.$

(4) $K_1 \subset D_1 \subset K_2 \subset D$ である. 但し, K_1, K_2 は compact, D_1 は open. すると,

$$\sup_{x \in K_1, y \in D - K_2} G_\alpha(x, y) < +\infty.$$

(5) $\mathbb{R}$ の各 t に対して 1.7) を満たす D 上の transition density $p(t, x, y)$ が一意に存在する.

$$1.7) \quad G_\alpha(x, y) = \int_0^{+\infty} e^{-\alpha t} p(t, x, y) dt, \quad \alpha > 0, \quad x, y \in D, \quad x \neq y.$$

ここで $p(t, x, y)$ は conservative, $x, y \in D$ に対しては $\int_D p(t, x, y) f(y) dy$ は (t, x) に関する連続 (但し $f \in B(D)$) である.

定義 Compact set D^* が D の compact set $\iff$ (i) D は D^* に含まれ; $\exists \varepsilon > 0$ open, dense.
(ii) D^* の D の topology は Euclidean topology と同等.

定理 2

(1) $\exists D^*$; $D \subset \text{compact } \mathbb{R}$.

$P(t, x, y)$ (in 定理 1) is $x \in D^*$ 及 拡張した 拡張した関数 $\tilde{P}(t, x, y)$

$\forall t \in \mathbb{R}^+ \quad T.1), T.2), T.3)$ for $x \in D^*, y \in D$

$\rho = \gamma \in \mathbb{R}^+$ 成立する. $\int_D P(t, x, y) f(y) dy$ is

$f \in B(D)$, $x \in D^*$ かつ $t \geq 0$ 連続

(2) $\exists X = \{X_t, P_x, x \in D^*\}$ Markov process on D^* . X is 条件を満たす.

(a) $\forall A$: Borel set of D^* ,

$$P_x(X_t \in A) = \int_{D \cap A} P(t, x, y) dy, \quad t \geq 0, x \in D^*.$$

(b) X is 連続 : $P_x(X_t \text{ is continuous in } \forall t \geq 0) = 1, \quad x \in D^*.$

(c) X is 確率過程.

(d) $\exists D_1^*$ Borel set in D^* . $D \subset D_1^*$.

$$P_x(X_0 = x) = 1, \quad x \in D_1^*.$$

$$P_x(X_t \in D_1^*, \forall t \geq 0) = 1, \quad x \in D_1^*.$$

$$(e) \quad P_x(X_t \in A, t < \tau) = \int_A P^0(t, x, y) dy,$$

$t \geq 0, x \in D, A \subset D \subset \text{Borel set},$

$$\tau = \inf \{t : X_t \in D^* - D\}.$$

数理論シニホジニエリニフ 定理 1 の証明を中
心に報告した。それは上の論文の §2 の部分
であり、ニニエリはそれを英文のフフのせり。
定理 2, 3 の証明は概略を報告した。

定理 2 についで

$G_\alpha(x, y)$ を 定理 1 のそれとする。 定理 2, (1)
の D^* とし、 D の $G_1(x, y)$ による Martin-倉持型
の completion をとる。 二の D^* は $\mathbb{R}^n$ の性
質による特徴づけられるものとする。

(D^* , 1) D^* は D の compact 区。

(D^* , 2) $\{G_\alpha f, f \in C_0(D)\}$ は D^* に連
続に拡張され、拡張された二の関数族は
 D^* の二点を分離する。

但し、 $C_0(D)$ は 有限 compact な D 上の連続
関数の全体を表わす。

勿論 $G_1(x, y)$ は 各 $y \in D$ についで x の関数

とし、 D^* 上に連続に拡張され、

$\forall \alpha > 0, \quad \forall x \in D^* - D,$

$$1.8) \quad G_\alpha(x, y) = G_1(x, y) - (\alpha - 1) \int_D G_\alpha(x, z) G_\alpha(z, y) dz$$

とあり、 $G_\alpha(x, y), x \in D^*, y \in D$ は

$G_1(1), G_1(2), G_1(3)$ を満たす二と知られる。

特 $G_2(x, y)$, $\lambda > 0$, $x, y \in D$ の conservativity
 $\in F$

1.9) $\lambda \int_D G_2(x, y) dy = \int_D G_1(x, y) dy \leq 1$, $\forall \lambda > 0$,
 $\forall x \in D^*$, (加 従 1.7) 次 2. 定理 2, (1)

を 満 ち $p(t, x, y)$, $t > 0$, $x \in D^*$, $y \in D$ は

$$1.10) \quad G_2(x, y) = \int_0^{+\infty} e^{-\lambda t} p(t, x, y) dt,$$

$\lambda > 0$, $x \in D^*$, $y \in D$,

存 在 関 係 を 満 ち 1.2 見 っ け 出 来 る.

2. 定理 2, (2), (A) の 条 件 を 満
 ち Markov process $X = (X_t, P_x, x \in D^*)$ を
 考 へ る と 其 の 適 当 if version が 得 ら れ る 過 程
 と な る こ と が 示 せ ら れ る. 二 の 証 明 は 先 ね 1.2
 の 性 質 ($G_1 f$, $f \in \mathcal{B}_0(D)$ の 連 続 性) に 注 意
 し て 行 っ て 来 る. X_t は 1.2 の 存 在 性 ($t \geq 0$),
 且 っ 左 極 限 を 有 る ($t > 0$) と 考 へ る こ と が 示 せ ら れ る.

Ray [20], 国 田・渡 辺 (未 公 表) [10] [11] に 採 取
 せ ら れ る 場 合 と 同 じ. D^* は branching point を
 含 有 せ ぬ (可 分). $P_x(X_0 = x) < 1$ なる x を
 branching point と 呼 ぶ 其 の 1/2 本 を Δ_0 と する.

$$\Delta_0 \subset D^* - D \text{ である. } D^* - \Delta_0 = D_1^* \text{ と 表 示 せ ら れ る.}$$

次 の こ と が 示 せ ら れ る.

$$P_x(X_t \notin \Delta_0, X_{t-} \notin \Delta_0, \forall t \geq 0) = 1, \quad \forall x \in D^*.$$

更に X に関する定理 2, (2), (c) の性質の証明に
 なる結果

$P_x(X_t \text{ is continuous at such } t \text{ that } X_t \text{ or } X_{t-} \in D) = 1, \quad x \in D^*, \quad \text{かつ} \quad (13),$

定理 2 の証明の第 2 本質的ステップは
 path の境界の jump を扱うことである。証明は
 二つの異なる定性的な方法を示す。

定義 $u \geq 0$ on D^* が 1-excessive とは
 $e^{-t} \int_D p(t, x, y) u(y) dy \uparrow u(x), \quad (t \downarrow 0), \quad x \in D^*$
 が成立すること。

定理 4

u が 1-excessive 且 $\int_D u(x) dx < +\infty$ ならば
 存在 $\nu : D^* \rightarrow \mathbb{R}_+$ 測度

$$u(x) = \int_{D^*} G_1(x, y) \nu(dy), \quad \forall x \in D^*.$$

且 $G_1(x, y), x \in D^*, y \in D$ から D^* へ
 1-excessive な核を導く。

この $\nu \in u$ に対する canonical measure とし

$$\mathcal{R} = GY_2(B(D^*)) \quad \text{と} \quad \mathcal{R}' \subset \mathcal{R}.$$

$u \in \mathcal{R}, \quad u = GY_2 f \quad \text{と} \quad \mathcal{R}'$

$$A_t^u = e^{-t/2} u(X_t) - u(X_0) + \int_0^t e^{-s/2} f(X_s) ds, \quad t \geq 0,$$

$$v_u(x) = E_x((A_{+\infty}^u)^2), \quad x \in D^* \text{ 且 } x' < x$$

v_u is 1-excessive 且 $\int_D v_u(x) < +\infty$.

定義

$u \in R$ is 正 v_u is 正 μ 正 canonical measure μ $v_u \in \mathcal{F}$.

$\|u\|_X = \sqrt{v_u(D_1^*)}$ is u on X is 正 Dirichlet norm $\sim$).

定理 5

$u \in R$ and $u \in \mathcal{F}$

$$\|u\|_X = \int_D (\text{grad } u, \text{grad } u)(x) dx \text{ and } \mu \text{ is } \mathcal{F}.$$

且 $v_u(D_1^* - D) = 0$ and $\mathcal{F}$.

定理 5 is 正 正 Brown 運動 正 正 正.

本正. 渡田 (18) [18] 正 正 正.

定理 5 is 正 正 正 正 正 正 正 正 正 正.

$E_x(A_t) = 0$, $E_x(A_t^2) < +\infty$ and X is 正 正 正.

additive functional A is 正 正 正 正 正 正 正.

$\int \chi_{D_1^* - D} \cdot dA$ is 正 正 正 正 正 正 正.

正 正 正 $\chi_{D_1^* - D}$ is D_1^* 正 indicator function.

正 正 正 path on $D_1^* - D$ is jump 正 正 正 正 正 正 正.

§2. Construction of a resolvent density (Proof of Theorem 1).

From now on, we fix an arbitrary bounded domain D of $\mathbb{R}^N$. The following criterion for a function on D to be α -harmonic is easily verified and will be frequently used in this paper.

Lemma 1. ²⁾ Let α be positive number. A function u on D is α -harmonic, if and only if, for each ball B with closure contained in D , it holds that

$$u(x) = \int_{\partial B} h_{\alpha}^B(x, y) u(y) \sigma(dy), \quad x \in B,$$

where $\sigma(dy)$ is the surface Lebesgue measure of ∂B and

$$h_{\alpha}^B(x, y) = \frac{1}{2} \frac{\partial}{\partial n_y} G_{\alpha}^B(x, y), \quad x \in B, y \in \partial B,$$

$G_{\alpha}^B(x, y)$ being the resolvent density defined by 1.1) for the Ball B .

For functions u and v on D , let $D(u, v)$ and (u, v) be quantities defined by 1.3) if they have the meaning anyway.. For each $\alpha > 0$, consider the function space H_α defined by 1.4). H_α contains the function which is identically zero on D . Later, we shall show that it contains also non-trivial elements. Let us put for $u, v \in H_\alpha$

$$2.1) \quad D_\alpha(u, v) = D(u, v) + 2\alpha(u, v).$$

Lemma 2.2. For each $\alpha > 0$, H_α forms a real Hilbert space with the inner product $D_\alpha(u, v)$. Moreover, any Cauchy sequence of functions in H_α with respect to the norm $\sqrt{D_\alpha(u, u)}$ converges on D uniformly on any compact subset of D .

Proof. Suppose that $u_n \in H_\alpha$, $n = 1, 2, \dots$, and

$$D_\alpha(u_n - u_m, u_n - u_m) \xrightarrow{n, m \rightarrow +\infty} 0.$$

Let K be any compact subset of D . We can choose $\varepsilon > 0$ such that the closed ball $K_\varepsilon(x)$ with radius ε centered at any point $x \in K$ is contained in D . Applying Lemma 2.1 to the α -harmonic function $u_n - u_m$, we have

$$2.2) \quad u_n(x) - u_m(x) = \frac{1}{V_\varepsilon} \int_{K_\varepsilon(x)} \eta_\alpha(|y-x|)(u_n(y) - u_m(y)) dy, \quad x \in K,$$

where V_ε is the volume of $K_\varepsilon(x)$, $|y-x|$ is the distance between x and y , and $\eta_\alpha(r)$ is a function of real $r > 0$ which depends only on $\alpha > 0$ and satisfies $0 < \eta_\alpha(r) < 1$. The Schwarz inequality applied to 2.2) leads to

$$(u_n(x) - u_m(x))^2 \leq \frac{1}{V_\varepsilon} (u_n - u_m, u_n - u_m)$$

$$\frac{1}{2\alpha V_\varepsilon} D_\alpha(u_n - u_m, u_n - u_m), \quad x \in K.$$

Thus, u_n converges to a function u on D uniformly on any compact subset of D . By virtue of Lemma ^{2.1} 1, u is also α -harmonic on D and the derivatives of u_n converge to those of u uniformly on any compact subset of D . On the other hand, since u_n , $n = 1, 2, \dots$, form a Cauchy sequence with respect to $D_\alpha(\cdot, \cdot)$, one can find, for any $\varepsilon > 0$, a compact subset $K \subset D$ such that

$$\int_{D-K} |\text{grad } u_n|^2(x) dx + 2 \int_{D-K} u_n(x)^2 dx < \varepsilon$$

uniformly in n . Hence, $u \in H_\alpha$ and $D_\alpha(u_n - u, u_n - u) \xrightarrow{n \rightarrow +\infty} 0$.

Lemma ^{2.3} 3. Let $\alpha > 0$ be fixed.

1) For each $x \in D$, there exists a function $u^{(x)} \in H_\alpha$ uniquely such that

$$2.3) \quad D_\alpha(u^{(x)}, v) = 2v(x), \text{ for any } v \in H_\alpha.$$

2) $u^{(x)}$ in 1) is a unique element of H_α minimizing the value of the functional $\bar{\Psi}(u) = D_\alpha(u, u) - 4u(x)$ on H_α .

Proof. 1). For a fixed $x \in D$, define the linear mapping $\bar{\Phi}$ from H_α to $\mathbb{R}^1$ by $\bar{\Phi}(v) = 2v(x)$, $v \in H_\alpha$. $\bar{\Phi}$ is bounded, because, if $v_n \xrightarrow{n \rightarrow +\infty} 0$ in H_α , then, by Lemma ^{2.2} 2, $2v_n(x) \xrightarrow{n \rightarrow +\infty} 0$.

The Riesz representation theorem implies 1).

2). We have only to notice the equality $\Psi(u) = \Psi(u^{(x)}) + D_\alpha(u - u^{(x)}, u - u^{(x)})$, $u \in H_\alpha$.

Definition 1. For $\alpha > 0$ and $x \in D$, denote by $R_\alpha^x(y) = R_\alpha(x, y)$, $y \in D$, the function $u^{(x)}(y)$, $y \in D$, determined by Lemma 3.

Definition 2. Let $G_\alpha^0(x, y)$ be the resolvent density defined by 1.1). Define the function $G_\alpha(x, y)$, $\alpha > 0$, $x, y \in D$, by $G_\alpha(x, y) = G_\alpha^0(x, y) + R_\alpha(x, y)$.

Before examining properties of $G_\alpha(x, y)$ stated in Theorem 1, we prepare three lemmas.

An exhaustion of D is a sequence of domains D_n , $n = 1, 2, \dots$, such that the closure of D_n is contained in D_{n+1} and D_n converges monotonically to D . An exhaustion $\{D_n\}$ of D is called regular if ∂D_n are of class C^3 .

Lemma 4. Let $\alpha > 0$ be fixed.

- 1) Any non-negative α -harmonic function on D is either identically zero on D or strictly positive on D .
 - 2) The function $w = 1 - \alpha G_\alpha^0(1)$ is strictly positive on D . Moreover w is the unique element in H_α satisfying
- $$2.4) \quad D_\alpha(w, v) = 2\alpha(1, v) \quad \text{for all } v \in H_\alpha.$$

Proof.

- 1) Since Lemma 1 implies that the value of an α -harmonic function at any point of D is a weighted volume mean on the ball centered at the point, 1) is verified in the same manner as in the case of harmonic functions.

2) It is evident, by the expression 1.2) of G_α^0 , that w is α -harmonic and strictly positive on D . In order to show 2.4), consider a regular exhaustion $\{D_n\}$ of D .

Put $w_n = \chi_{D_n} - \alpha {}^n G_\alpha^0 \chi_{D_n}$, where χ_{D_n} is the indicator function of D_n , ${}^n G_\alpha^0 \chi_{D_n}(x) = \int_{D_n} {}^n G_\alpha^0(x, y) dy$ and ${}^n G_\alpha^0(x, y)$ is the resolvent density defined by 1.1) for D_n . w_n is α -harmonic in D_n , converges to w monotonically and (consequently) uniformly on any compact subset of D . On account of Lemma 2.1, the derivatives of w_n converge to those of w on D . Denote by $D_\alpha^n(\cdot, \cdot)$ the integral 2.1) on D_n . Since w_n belongs to $C^1(D_n \cup \partial D_n)$, we can apply Green's formula to w_n and $v \in H_\alpha$, obtaining $D_\alpha^n(w_n, v) = 2\alpha(\chi_{D_n}, v)$. This equality implies the inequality

$$D_\alpha^n(w_n, w_n) - 4\alpha(\chi_{D_n}, w_n) \leq D_\alpha^n(v, v) - 4\alpha(\chi_{D_n}, v)$$

for all $v \in H_\alpha$. Letting n tend to infinity and using Fatou's Lemma, we obtain

$$D_\alpha(w, w) - 4\alpha(1, w) \leq D_\alpha(v, v) - 4\alpha(1, v).$$

Thus, $w \in H_\alpha$, and if we put, instead of v , $w + \varepsilon v$ in the inequality above, we arrive at 2.4). Proof of the uniqueness is straightforward.

Lemma 2.5. Take an exhaustion $\{D_n\}$ of D arbitrarily.

Let ${}^n R_\alpha^x(y)$ and ${}^n G_\alpha(x, y)$, $\alpha > 0$, $x, y \in D_n$ be the functions defined by Definition 1 and Definition 2 for the

domain D_n . Then, $\lim_{n \rightarrow +\infty} {}^n G_\alpha(x, y) = G_\alpha(x, y)$, $\alpha > 0$,

$x, y \in D$, $x \neq y$. Moreover, for each $x \in D$, the equality

$$2.5) \quad \lim_{n \rightarrow +\infty} {}^n R_\alpha^x(y) = R_\alpha^x(y), \quad y \in D,$$

holds and the convergence is uniform in y on any compact subset of D .

Proof. Let ${}^n G_\alpha^0(x, y)$ be the resolvent density defined by 1.1) for the domain D_n . Since ${}^n G_\alpha^0(x, y)$ increases to $G_\alpha^0(x, y)$, we have only to show 2.5) together with the uniformity of the convergence.

Let us fix $x \in D$. We can assume that x is in D_1 . Let us denote by superscript n that we are concerned with D_n instead of D ; for instance, H_α^n and D_α^n . It is clear that, if $m < n$, the restriction of the function of H_α^n to D_m is an element of H_α^m .

If $m < n$, we have

$$\begin{aligned} & D_\alpha^m({}^n R_\alpha^x - {}^m R_\alpha^x, {}^n R_\alpha^x - {}^m R_\alpha^x) \\ &= D_\alpha^m({}^n R_\alpha^x, {}^n R_\alpha^x) - 2 D_\alpha^m({}^m R_\alpha^x, {}^n R_\alpha^x) + D_\alpha^m({}^m R_\alpha^x, {}^m R_\alpha^x). \end{aligned}$$

We can apply Lemma 2.3) to each term of the last expression.

The first term is not greater than $D_\alpha^n({}^n R_\alpha^x, {}^n R_\alpha^x) = 2 {}^n R_\alpha^x(x)$.

The second and the third terms are equal to $-4 {}^n R_\alpha^x(x)$ and $2 {}^m R_\alpha^x(x)$, respectively. Therefore, for each N , it holds that

$$2.6) \quad 0 \leq D_\alpha^N({}^n R_\alpha^x - {}^m R_\alpha^x, {}^n R_\alpha^x - {}^m R_\alpha^x) \leq 2({}^m R_\alpha^x(x) - {}^n R_\alpha^x(x)),$$

for any m and n such that $N \leq m < n$. 2.6) implies that

${}^n R_\alpha^x(x)$ is non-increasing in n and since ${}^n R_\alpha^x(x) = \frac{1}{2} D_\alpha^n({}^n R_\alpha^x, {}^n R_\alpha^x)$ is non-negative, ${}^n R_\alpha^x(x)$ converges. Thus, 2.6) and Lemma 1 show that ${}^n R_\alpha^x(y)$ converges to an α -harmonic function $\tilde{R}_\alpha^x(y)$ on D uniformly on any compact subset of D , and for each N , the restriction of ${}^n R_\alpha^x$ to D_N converges to that of $\tilde{R}_\alpha^x$ in the norm D_α^N .

Let us prove that $\tilde{R}_\alpha^x(y) = R_\alpha^x(y)$, $y \in D$. Since R_α^x belongs to H_α^n , Lemma 3, 2) implies

$$D_\alpha^n({}^n R_\alpha^x, {}^n R_\alpha^x) - 4{}^n R_\alpha^x(x) \leq D_\alpha^n(R_\alpha^x, R_\alpha^x) - 4R_\alpha^x(x).$$

Letting n tend to infinity, we have, for each N ,

$$D_\alpha^N(\tilde{R}_\alpha^x, \tilde{R}_\alpha^x) - 4\tilde{R}_\alpha^x(x) \leq D_\alpha^N(R_\alpha^x, R_\alpha^x) - 4R_\alpha^x(x).$$

Let N tend to infinity, then

$$D_\alpha(\tilde{R}_\alpha^x, \tilde{R}_\alpha^x) - 4\tilde{R}_\alpha^x(x) \leq D_\alpha(R_\alpha^x, R_\alpha^x) - 4R_\alpha^x(x).$$

Thus, we see that $\tilde{R}_\alpha^x \in H_\alpha$ and that, by Lemma 3, 2), the inequality above is just the equality and $\tilde{R}_\alpha^x(x) = R_\alpha^x(x)$, $x \in D$. The proof of Lemma 5 is complete.

We have seen in §1 (in the paragraph below the description of Theorem 1) that, if ∂D_n is of class C^3 , ${}^n G_\alpha(x, y)$ is nothing but the Laplace transform of the fundamental solution of the heat equation with the reflecting barrier boundary condition for the domain D_n and the latter is a transition density on D_n . Hence, we have

Lemma 6. Let $\{D_n\}$, $\{{}^n R_\alpha(x, y)\}$ and $\{{}^n G_\alpha(x, y)\}$

be those in Lemma 5. If $\{D_n\}$ is regular, then for each n , we have,

$$2.7) \quad {}^nG_\alpha(x, y) \geq 0, \quad \alpha > 0, x, y \in D_n, x \neq y.$$

$$2.8) \quad {}^nR_\alpha(x, y) \geq 0, \quad \alpha > 0, x, y \in D_n.$$

$$2.9) \quad \alpha \int_{D_n} {}^nG_\alpha(x, y) dy \leq 1, \quad \alpha > 0, x \in D_n.$$

$$2.10) \quad {}^nG_\alpha(x, y) - {}^nG_\beta(x, y) + (\alpha - \beta) \int_{D_n} {}^nG_\alpha(x, z) {}^nG_\beta(z, y) dz = 0, \quad \alpha, \beta > 0, x, y \in D_n, x \neq y.$$

We note that 2.8) follows from 2.7).

Now, let us complete the proof of Theorem 1 by the following series of lemmas.

Lemma^{2.7}. $R_\alpha(x, y)$ is non-negative for $\alpha > 0$, $x, y \in D$ and $\alpha \int_D G_\alpha(x, y) dy \leq 1$, for $\alpha > 0, x \in D$.

$G_\alpha(x, y)$ is symmetric in $x, y \in D$ and continuous in (x, y) on $D \times D$ off the diagonal.

Proof. The first part of Lemma^{2.7} is an immediate consequence of Lemma^{2.5} and Lemma^{2.6}. It is well known that $G_\alpha^0(x, y)$ is symmetric in $x, y \in D$ and continuous in $(x, y) \in D \times D$ off the diagonal set. $R_\alpha(x, y)$ is symmetric because $D_\alpha(R_\alpha^x, R_\alpha^y) = 2R_\alpha^x(y) = 2R_\alpha^y(x)$, $x, y \in D$. We shall show that $R_\alpha(x, y)$ is continuous in $(x, y) \in D \times D$. Since $R_\alpha(x, y)$ is α -harmonic in x and in y , applying Lemma^{2.1} for any $x, y \in D$ and for sufficiently small balls B_1 and B_2 containing x and y , respectively, we have

$$R_\alpha(x, y) = \int_{\partial B_1} \int_{\partial B_2} h_\alpha^{B_1}(x, z) R_\alpha(z, z') h_\alpha^{B_2}(z, z') \sigma_1(dz) \sigma_2(dz'), \text{ where } \sigma_1(dz) \text{ and } \sigma_2(dz') \text{ are the surface Lebesgue}$$

measures of ∂B_1 and ∂B_2 , respectively. While, $R_\alpha(z, z')$ being continuous in z' for each z , $\int_{\partial B_2} R_\alpha(z, z') \sigma_2(dz')$

is finite and α -harmonic in z . Thus,

$$\int_{\partial B_1} \int_{\partial B_2} R_\alpha(z, z') \sigma_1(dz) \sigma_2(dz') < +\infty.$$

Since R_α is non-negative, Lebesgue's convergence theorem implies continuity of $R_\alpha(x, y)$. The proof of the latter half of Lemma²⁴ 7 is complete.

We will show (4) of Theorem 1.

Lemma²⁴ 8. Let K_1 and K_2 be compact subsets of D such that K_1 and the closure of $D - K_2$ are disjoint.

Then, $\sup_{x \in K_1, y \in D - K_2} G_\alpha(x, y)$ is finite.

Proof. Without loss of generality, we can assume that $S = \partial(D - K_2) \cap D$ is sufficiently regular. Consider an regular exhaustion $\{D_n\}$ of D such that $D_1 \supset K_2$. Let x be fixed in K_1 . For a fixed n , set $D' = D_n - K_2$ and $u(y) = {}^n G_\alpha(x, y)$, $y \in D' \cup \partial D'$. Since $\frac{\partial}{\partial n_y} u(y) = 0$,

$y \in \partial D_n$, we see by Green's formula that $D'_\alpha(u, v - u) = 0$ holds if $v \in \mathcal{C}^1(D' \cup \partial D')$ and $v = u$ on S .¹⁾ Hence, the equality

$$2.11) \quad D'_\alpha(u, u) = D'_\alpha(v, v) - D'_\alpha(u - v, u - v)$$

is valid for each v belonging to $\tilde{\mathcal{D}}_u = \{v; v \text{ is square summable, } \underbrace{v}_{\text{on } D'} \text{ has square summable weak-derivatives, } \underbrace{v}_{\text{on } D'} \in \mathcal{C}(D' \cup S) \text{ and } v = u \text{ on } S\}$.²⁾ Set $\delta = \sup_{y \in S} u(y)$ and $u_1(y) =$

$\min(u(y), \delta)$, $y \in D' \cup S$. Obviously, $D'_\alpha(u, u) \geq D'_\alpha(u_1, u_1)$. But, since $u_1 \in \tilde{\mathcal{D}}_u$, 2.11) holds for $v = u_1$ and consequently $u_1(y) = u(y)$ on D' .

We have proved that, if $x \in K_1$ and $y \in D_n - K_2$, then $n G_\alpha(x, y) \leq \sup_{y \in S} n G_\alpha(x, y)$. Letting n tend to infinity, we see by virtue of Lemma 5, $G_\alpha(x, y) \leq \sup_{y \in S} G_\alpha(x, y)$, $x \in K_1$, $y \in D - K_2$. Thus,

$$\sup_{x \in K_1, y \in D - K_2} G_\alpha(x, y) \leq \sup_{x \in K_1, y \in S} G_\alpha(x, y).$$

The right hand side above is finite by Lemma 7.

Let us show (3) of Theorem 2.

Lemma 9. The operator G_α defined by 1.6) maps $B(D)$ into $C(D)$. Moreover, if $f \in C(D)$, $\lim_{\alpha \rightarrow +\infty} \alpha G_\alpha f(x) = f(x)$, $x \in D$.

Proof. We note that G_α^0 has the properties of Lemma 9. 3) For $f \in B(D)$, $R_\alpha f(x) = \int_D R_\alpha(x, y) f(y) dy$ is α -harmonic and bounded on account of Lemma 1 and Lemma 7. Moreover, we see by Lemma 1 that for any $x \in D$ and sufficiently small ball B containing x ,

$$\begin{aligned} |\alpha R_\alpha f(x)| &\leq \int_{\partial B} h_\alpha^B(x, y) |\alpha R_\alpha f(y)| \sigma(dy) \\ &\leq \sup_{x \in D} |f(x)| \int_{\partial B} h_\alpha^B(x, y) \sigma(dy) \xrightarrow{\alpha \rightarrow +\infty} 0. \end{aligned}$$

The proof of Lemma 9 is complete.

The following lemmas are (2) and (5) of Theorem 1.

Lemma ²10. $G_\alpha(x, y)$ is a conservative resolvent density on D . $R_\alpha(x, y)$ is strictly positive.

Proof. We must prove that $G_\alpha(x, y)$ satisfies the conditions G.1) ~ G.4) stated in the beginning of §1 and the conservativity condition. G.1), G.2) and G.3) are already proved in Lemma ²7.

Proof of resolvent equation G.4). Take a regular exhaustion $\{D_n\}$ of D . Let f and g be non-negative continuous functions on D with compact supports. Owing to 2.10) of Lemma ²6, we have for sufficiently large n ,

$$2.12) \quad (f, {}^n G_\alpha g)_n - (f, {}^n G_\beta g)_n + (\alpha - \beta)({}^n G_\alpha f, {}^n G_\beta g)_n = 0,$$

where $(u, v)_n$ denotes the integral of $u \cdot v$ on D_n .

Note that $0 \leq {}^n G_\alpha f(x) {}^n G_\beta g(x) \leq \frac{1}{\alpha\beta} \sup_{x \in D} f(x) \cdot \sup_{x \in D} g(x)$

and that ${}^n G_\alpha g$ converges to $G_\alpha g$ on D (since, ${}^n G_\alpha^0 g$ increases to $G_\alpha^0 g$ and ${}^n R_\alpha^x(y)$ converges uniformly on any compact subset). Hence, we can delete both superscript and subscript n in 2.12). Owing to Lemma ²8 and Lemma ²9, the left hand side of G.4) is, for each $x \in D$, continuous in $y \in D - \{x\}$, and we can see that G.4) is valid.

Proof of conservativity. If we show that $R_\alpha 1 \in \mathbb{H}_\alpha$

and that

$$2.13) \quad \mathbb{D}_\alpha(\alpha R_\alpha 1, v) = 2\alpha(1, v),$$

holds for all $v \in \mathbb{H}_\alpha$, then, we have, by (2) of Lemma ²4,

$$1 - \alpha G_\alpha^0 1 = \alpha R_\alpha 1 \quad \text{and} \quad \alpha G_\alpha 1 = 1.$$

Let $\{D_n\}$ be an exhaustion of D . Integrating

$\mathbb{D}_\alpha(R_\alpha^x, R_\alpha^y) = 2R_\alpha(x, y)$ by $dx dy$ on $D_m \times D_n$, we obtain

$$2.14) \quad \mathbb{D}_\alpha(R_\alpha \chi_{D_m}, R_\alpha \chi_{D_n}) = 2 \int_{D_m} \int_{D_n} R_\alpha(x, y) dy.$$

Here, we have used the Fubini theorem, which is possible, because, if $m \leq n$,

$$\begin{aligned} & \int_{D_m} \int_{D_n} dx dy \int_D |(\text{grad}_z R_\alpha^x(z), \text{grad}_z R_\alpha^y(z))| dz \\ & \leq \int_{D_n} \int_{D_n} \sqrt{\mathbb{D}_\alpha(R_\alpha^x, R_\alpha^x)} \sqrt{\mathbb{D}_\alpha(R_\alpha^y, R_\alpha^y)} dx dy \\ & = \left(\int_{D_n} \sqrt{2R_\alpha(x, x)} dx \right)^2 \leq 2 \int_{D_n} R_\alpha(x, x) dx \times \text{Lebesgue} \end{aligned}$$

measure of D_n , the integral in the last expression being finite by Lemma ^{2.4}17. In view of Lemma ^{2.4}17, $R_\alpha(x, y) \geq 0$ and

$$\int_D \int_D R_\alpha(x, y) dx dy \leq \frac{1}{\alpha} \times \text{Lebesgue measure of } D.$$

Therefore, $R_\alpha \chi_{D_n}$ forms a Cauchy sequence in $\mathbb{H}_\alpha$ and,

by Lemma ^{2.4}2, converges to $R_\alpha 1$ in $\mathbb{H}_\alpha$. We have

$\mathbb{D}_\alpha(R_\alpha 1, R_\alpha 1) = 2(1, R_\alpha 1)$. In the same way, 2.13) is obtained. Strict positivity of $R_\alpha(x, y)$ follows from Lemma 2.4.

Lemma ^{2.4}11. There is a transition density $P(t, x, y)$

on D uniquely which satisfies the following conditions.

$$(1) \quad G_\alpha(x, y) = \int_0^{+\infty} e^{-\alpha t} P(t, x, y) dt, \quad \alpha > 0.$$

(2) For each $t > 0$, $f \in \mathbb{B}(D)$,

$\int_D P(t, x, y) f(y) dy$ is continuous in $(t, x) \in (0, +\infty) \times D$.

(3) $P(t, x, y)$ is symmetric in $x, y \in D$ and it is conservative.

(4) Set $\gamma(t, x, y) = P(t, x, y) - P^0(t, x, y)$, then

$$\frac{1}{t} \int_D \gamma(t, x, y) dy \xrightarrow[t \rightarrow 0]{} 0 \quad \text{uniformly in } x \text{ on}$$

any compact subset of D .

Proof. First of all, we will show the existence of a non-negative function $\gamma(t, x, y)$ continuous in $t > 0$, satisfying

$$2.15) \quad R_\alpha(x, y) = \int_0^{+\infty} e^{-\alpha t} \gamma(t, x, y) dt, \quad \alpha > 0, x, y \in D.$$

If $x \neq y$, $R_\alpha(x, y)$ is completely monotonic in $\alpha \in (0, +\infty)$. In fact, by the resolvent equations G. 4) for G_α and G_α^0 , we have, if $x \neq y$,

$$2.16) \quad (-1)^n \frac{d^n}{d\alpha^n} R_\alpha(x, y) = n! [G_\alpha^{[n+1]}(x, y) - (G_\alpha^0)^{[n+1]}(x, y)],$$

$n = 0, 1, 2, \dots$. Here $G_\alpha^{[1]}(x, y) = G_\alpha(x, y)$ and $G_\alpha^{[n+1]}(x, y) = \int_D G_\alpha^{[n+1]}(x, z) G_\alpha(z, y) dz$, $n = 1, 2, \dots$.

$(G_\alpha^0)^{[n]}$ is similarly defined. Evidently, the right hand side of 2.16) is non-negative and, by Lemma 2.8, finite.

Hence, $R_\alpha(x, y)$ is expressed by a measure on $[0, +\infty)$ as

$$2.17) \quad R_\alpha(x, y) = \int_0^{+\infty} e^{-\alpha s} \gamma(ds, x, y), \quad x \neq y, \quad \alpha > 0.$$

Take a ball B with closure contained in D . Since $R_\alpha(x, y)$ is α -harmonic in x , we see, by Lemma 2.1, for any $x \in B$ and any $y \in D$,

$$2.18) \quad R_\alpha(x, y) = \int_{\partial B} h_\alpha^B(x, z) R_\alpha(z, y) \sigma(dz).$$

Note that $h_\alpha^B(x, z)$ is written in the form

$$2.19) \quad h_\alpha^B(x, z) = \int_0^{+\infty} e^{-\alpha t} h^B(t, x, z) dt, \quad x \in B, z \in \partial B,$$

where $h^B(t, x, z) = \frac{1}{2} \frac{\partial}{\partial n_z} P_B^0(t, x, z)$, P_B^0 being the

transition density P^0 for B . Let us put, for $t > 0$,

$$2.20) \quad \gamma(t, x, y) = \int_{\partial B} \int_0^t h^B(t-s, x, z) \gamma(ds, z, y) dt \sigma(dz).$$

Owing to 2.17), 2.18) and 2.19), $\gamma(t, x, y)$ of 2.20) satisfies the desired equation 2.15). On the other hand, for any ball B' such as $B' \cup \partial B' \subset B$, the obvious identity

$$h^B(t, x, z) = \int_{\partial B'} \int_0^t h^{B'}(t-s, x, z') h^B(s, z', z) ds \sigma'(dz'),$$

$x \in B', z \in \partial B$, leads us to

$$2.21) \quad \gamma(t, x, y) = \int_{\partial B'} \int_0^t h^{B'}(t-s, x, z') \gamma(s, z', y) ds \sigma'(dz'),$$

$t > 0, x \in B', y \in D$, which implies the continuity of $\gamma(t, x, y)$ in $(t, x) \in (0, +\infty) \times B'$.

Here, we have used the following estimate which is a consequence of 2.17), 2.20) and Lemma 2.8.

$$2.22) \quad \sup_{0 < t \leq T, x \in B', y \in D} \gamma(t, x, y) \leq C \cdot e^T \cdot \sup_{z \in \partial B, y \in D} R_1(z, y)$$

$< +\infty$, where T is an arbitrary positive number and C is a constant determined by T, B and B' . Hence, we see that, for any x and y in D , $\gamma(t, x, y)$ defined by 2.20)

is independent of the ball B such that $x \in B$ and $B \cup \partial B \subset D$, because it satisfies 2. 15) and it is continuous in t . It is symmetric in x, y because of the symmetry of $R_\alpha(x, y)$ (Lemma ^{2.4}7). Henceforce, it is continuous in y and 2. 21) and 2. 22) imply its continuity in $(t, x, y) \in (0, +\infty) \times D \times D$. In view of 2. 22), we see that $\int_D \gamma(t, x, y) f(y) dy$ is continuous in $(t, x) \in (0, +\infty) \times D$ for each $f \in \mathcal{B}(D)$.

Now put, for $t > 0, x, y \in D$,

$$2. 23) \quad P(t, x, y) = P^0(t, x, y) + \gamma(t, x, y).$$

Then, $P(t, x, y)$ is continuous in $(t, x, y) \in (0, +\infty) \times D \times D$ and satisfies (1), (2) and ^{the first half of 2.} (3) of Lemma 11. Particularly,

$\int_D P(t, x, y) dy$ is continuous in t , and so, the conservativity of $P(t, x, y)$ follows from that of $G_\alpha(x, y)$. For each

$x, y \in D$, $P(t+s, x, y)$ and $\int_D P(t, x, z) P(s, z, y) dz$ are

continuous in $(t, s) \in (0, +\infty) \times (0, +\infty)$, and so, they

are identical by virtue of G. 4) for $G_\alpha(x, y)$. Thus,

$P(t, x, y)$ is a transition density. (4) of Lemma 11 follows

from 2. 21) and the inequality $\int_D \gamma(t, x, y) dy \leq 1, t > 0,$

$x \in D$.

References.

- [1] C. Constantinescu and A. Cornea, Ideale rander Riemannscher
Flachen, Springer, 1963.
- [2] J.L.Doob, Stochastic processes, Wiley, New York, 1953.
- [3] E.B.Dynkin, Foundations of the theory of Markov processes,
Moscow,1959. (Russian).
- [4] E. B. Dynkin, Markov processes, Moscow, 1963. (Russian).
- [5] M. Fukushima, Resolvent kernels on a Martin space, Proc. Japan.
Acad., 41 (1964), 167 -175.
- [6] N. Ikeda, K.Sato, H.Tanaka and T. Ueno, Boundary problems
in multi-dimensional diffusion processes, Seminar on Probability,
vol 5, 1960; vol 6, 1961. (Japanese).
- [7] S. Ito, Fundamental solutions of parabolic differential
equations and boundary value problems, Japan. J. Math., 27,
55 - 102 (1957).
- [8] K. Ito and H. P. McKean Jr, Diffusion processes and their
sample paths, Springer, 1965.
- [9] H. Kunita and H. Nomoto, On a method of the compactification
in the theory of the Markov process, Seminar on Probability,
vol 14, 1962. (Japanese).
- [10] H. Kunita and T. Watanabe, On certain reversed processes and
their application to potential theory and boundary theory,
J. Math. Mech., 15 (1966), 393 - 434.
- [11] H. Kunita and T. Watanabe, Some theorems concerning resovents
over locally compact spaces, to appear in Proc. of 5-th Berkeley
Symposium.
- [12] Z. Kuramochi, Mass ditributions on the ideal boundaries of
abstract Riemann surfaces. II, Osaka Math. J., 8 (1956),
145 - 186.

- [13] R. S Martin, Minimal positive harmonic functions, Tran. Am. Math. Soc., 49 (1941) 137 - 172.
- [14] P. A. Meyer, Fonctionnelles multiplicatives et additives de Markov, Ann. Inst. Fourier Grenoble, 12(1962), 123 - 230.
- [15] P. A. Meyer, Probabilites et potentiel, Hermann, Paris, 1966.
- [16] C. Miranda, Equazioni alle derivate parziali di tipo ellittico, Springer, 1955.
- [17] M. Motoo, The sweeping- out of additive functionals and processes on the boundary. Ann. Inst. Stat. Math., 16 (1964), 317 - 345.
- [18] M. Motoo and S. Watanabe, On a class of additive functionals of Markov processes, J. Math. Kyoto Univ. 4 (1965), 429 - 469.
- [19] M. Ohtsuka, An elementary introduction of Kuramochi boundary, J. Sci. Hiroshima Univ. Ser A - I 28(1964), 271 - 299.
- [20] D. Ray, Resolvents, transition functions and strongly Marlovian processes, Ann. Math, 70 (1959) , 43 - 78.
- [21] K. Sato and T. Ueno, Multi-dimensional diffusion and the Markov process on the boundary. J. Math. Kyoto Univ. 4(1965), 529 - 605.
- [22] S. Watanabe, On discontinuous additive functionals and Levy measures of a Markov process. Jap. J. Math., vol 34 (1964), 53 - 70.
- [23] D. V. Widder, The Laplace transform, Princeton, 1946.

Tokyo University of Education.