

A note on scrolls of smallest embedded codimension

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§0. Introduction

The situation considered in this note is as follows. Let M be an algebraic submanifold of $\mathbf{P}^N$ with $n = \dim M$. M is said to be a scroll over S if there is a surjective morphism $\pi : M \rightarrow S$ such that every fiber $F_x = \pi^{-1}(x)$ over $x \in S$ is a linear subspace in $\mathbf{P}^N$ of dimension $r - 1 = n - s$, where $s = \dim S$. This is equivalent to saying that $M \cong \mathbf{P}_S(\mathcal{E})$ for some vector bundle $\mathcal{E}$ of rank $r = n - s + 1$ and the tautological bundle $H(\mathcal{E})$ is the hyperplane section bundle of M .

We have $b_i(M) = b_i(\mathbf{P}^N)$ for $i \leq 2n - N$ by Barth-Lefschetz Theorem, hence $N \geq 2n - 1$ for scrolls, since otherwise $1 + b_2(S) = b_2(M) = b_2(\mathbf{P}^N) = 1$, contradiction. Thus we want to study scrolls such that $N = 2n - 1$.

The case $s = 1$ was studied by Lanteri-Turrini [LT], who showed that M is the Segre scroll over $\mathbf{P}^1$ in this case; namely $S \cong \mathbf{P}^1$, $\mathcal{E} \cong \mathcal{O}(1)^{\oplus n}$, $M \cong \mathbf{P}^1 \times \mathbf{P}^{n-1}$ and $M \subset \mathbf{P}^N$ is the Segre embedding. In this paper we are interested mainly in the case $s = 2$.

This problem was studied by Ottaviani [Ot] and Beltrametti-Schneider-Sommese [BSS], [BS] in case $n = 3$ and by Ionescu-Toma [IT] for general n . Their results are as follows.

Theorem. (cf. [Ot]). *Let $M \subset \mathbf{P}^5$ be a three-dimensional scroll over a surface S . Then one of the following conditions is satisfied.*

- (1) (Segre scroll) $S \cong \mathbf{P}^2$, $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 2}$, $d = \deg M = 3$.
- (2) (Bordiga scroll) $S \cong \mathbf{P}^2$, $c_1(\mathcal{E}) = \mathcal{O}(4)$, $c_2(\mathcal{E}) = 10$ and $d = 6$.
- (3) (Palatini scroll) S is isomorphic to a cubic surface, $c_1(\mathcal{E}) = \mathcal{O}_S(2)$, $c_2(\mathcal{E}) = 5$ and $d = 7$.
- (4) (K3 scroll) S is a K3-surface obtained as a linear section of the Grassman variety parametrizing lines in $\mathbf{P}^5$ embedded by Plücker, and $\mathcal{E}$ is the restriction of the tautological vector bundle. $c_2(\mathcal{E}) = 5$ and $d = 9$.

All these four cases actually occur.

Theorem. (cf. [IT]). *Let $M \subset \mathbf{P}^N$ be a scroll over a surface S with $N = 2n - 1$, $n > 3$ and let $\mathcal{E}$ and $d = \deg M$ be as above. Then, M is one of the following types:*

- (S_n) $S \cong \mathbf{P}^2$, $\mathcal{E} \cong \mathcal{O}(1)^{\oplus(n-1)}$, $M \cong \mathbf{P}^2 \times \mathbf{P}^{n-2}$, $d = n(n-1)/2$.
- (B_n) $S \cong \mathbf{P}^2$, $c_1(\mathcal{E}) = \mathcal{O}(n+1)$, $c_2(\mathcal{E}) = (n+1)(n+2)/2$, $d = n(n+1)/2$.
- (M_n) S is a K3 surface, $c_1(\mathcal{E})^2 = 2n^2 - 4$, $c_2(\mathcal{E}) = n^2 - 4$, $d = n^2$.
- ($?_n$) S is a surface of general type.

The existence of scrolls of the above type (S_n) is classical and is due to Segre. The case (B_n) is shown to exist for every n in [IT]. The existence of the type (M_n) is proved by Mukai ([Mu]). On the other hand, no example of type ($?_n$) is found: [IT] suspect rather that there is no such scroll. Any way, such scrolls must satisfy several numerical relations among their invariants. Here I propose a conjecture concerning further classifications derived from these relations, which is verified for $n \leq 11$ by hand and for $n \leq 1100$ by a computer. In particular, a scroll of the type ($?_n$) does not exist unless $n = 6, 10, 11, 12, 16, 18, 20, 24, 30, \dots$. On the other hand, no example is known and the existence problem is unsettled for these n (and also for $n > 1100$, of course).

Mathematical tools used here are almost the same as [IT], but we review them here for the convenience of the reader. When I started this study, I was not aware of this paper [IT]. I would like to express my hearty thanks to Professors Ottaviani who informed me of the result in [IT]. I thank Professor Mukai, who communicated to me the existence of scrolls of the type (M_n). I also thank Professors Lanteri and Schneider for their cooperations in e-mail correspondence

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§1. Computing Chern classes

(1.1) Throughout this paper let M be a scroll in $\mathbf{P}^{2n-1}$ over S as in §0. Let $\mathcal{E}$ be the vector bundle on S of rank $r = n - s + 1$, $s = \dim S$, such that $M \cong \mathbf{P}_S(\mathcal{E})$ and the tautological bundle $H(\mathcal{E})$ is the hyperplane section bundle $\mathcal{O}_M(1)$, which will be denoted simply by H from now on. We put $d = \deg M = H^n\{M\}$.

(1.2) *Fact.* Via the ring homomorphism $\pi^* : H^*(S) \rightarrow H^*(M)$, the cohomology ring of M becomes a free $H^*(S)$ -module generated by $1, h, h^2, \dots, h^{r-1}$, where $h = c_1(H) \in H^2(M)$. Moreover $\sum_{i=0}^r (-h)^i e_{r-i} = 0$ in $H^*(M)$, where $e_j = \pi^* c_j(\mathcal{E})$.

This is well known for general vector bundle $\mathcal{E}$ of rank r . Thus, the ring structure of $H^*(M)$ is determined by $H^*(S)$ and Chern classes of $\mathcal{E}$.

(1.3) *Fact.* Put $s_i(\mathcal{E}) = \pi_* h^{r-1+i} \in H^{2i}(S)$ and $s(\mathcal{E}) = \sum_{i=0}^{\infty} s_i(\mathcal{E}) \in H^*(S)$. Then $s(\mathcal{E})c(\mathcal{E}^\vee) = 1$, where $c(\mathcal{E}^\vee)$ is the total Chern class of the dual bundle $\mathcal{E}^\vee$ of $\mathcal{E}$.

This is also standard. $s_i(\mathcal{E})$ is called the i -th Segre class of $\mathcal{E}$ and $s(\mathcal{E})$ is called the total Segre class of $\mathcal{E}$. Moreover the following formulas are well known:

$$\begin{aligned} s_1(\mathcal{E}) &= c_1(\mathcal{E}), \\ s_2(\mathcal{E}) &= c_1(\mathcal{E})^2 - c_2(\mathcal{E}), \\ &\dots \end{aligned}$$

(1.4) *Corollary.* For any vector bundle E on X with $\text{rank } E = r$ and for any line bundle L on X with $c_1(L) = \ell$, we have $s_i(E \otimes L) = \sum_j \binom{r-1+i}{j} s_{i-j}(E) \ell^j$.

Indeed, $s_i(E \otimes L) = \pi_* ((h + \ell)^{(r-1+i)}) = \sum_j \binom{r-1+i}{j} \pi_* h^{r-1+i-j} \cdot \ell^j$ for $\pi : \mathbf{P}(E) \rightarrow X$.

(1.5) Let $\pi : M = \mathbf{P}(\mathcal{E}) \rightarrow S$ be the projection. Let $\mathcal{A}$ be the kernel of the natural surjection $\pi^* \mathcal{E} \rightarrow \mathcal{O}[H]$. Then we have an exact sequence $0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}^\vee \otimes H \rightarrow \mathcal{A}^\vee \otimes H \rightarrow 0$. This is identified with the relative Euler sequence, and so we have the exact sequence $0 \rightarrow \mathcal{A}^\vee \otimes H \rightarrow \Theta_M \rightarrow \pi^* \Theta_S \rightarrow 0$, where Θ_X denotes the tangent bundle of X .

From these exact sequences we obtain the following relation $c(\Theta_M) = \pi^* c(\Theta_S) c(\pi^* \mathcal{E}^\vee \otimes H)$ of total Chern classes.

(1.6) Let $\mathcal{N}$ be the normal bundle of M in $\mathbf{P}^{2n-1}$ and let Θ be the restriction of the tangent bundle of $\mathbf{P}^{2n-1}$ to M . Then we have $c(\Theta_M) c(\mathcal{N}) = c(\Theta) = (1 + h)^{2n}$, where $h = c_1(H)$.

Combining (1.5), we get $c(\mathcal{N}) = (1 + h)^{2n} c(\pi^* \mathcal{E}^\vee \otimes H)^{-1} \pi^* c(\Theta_S)^{-1} = (1 + h)^{2n} s(\pi^* \mathcal{E} \otimes [-H]) \pi^* s(\Omega_S)$.

(1.7) In the following computation, $\pi^* \alpha$ will be denoted simply by α for $\alpha \in H^*(S)$.

By (1.4), we have $s(\mathcal{E}_M \otimes [-H]) = \sum_{i=0}^{\infty} (\sum_{j=0}^{r-1+i} \binom{r-1+i}{j} s_{i-j}(\mathcal{E}) (-h)^j)$. Therefore the component of $(1 + h)^{2n} s(\mathcal{E}_M \otimes [-H])$ of degree $2k$ is $\sum_{i,j} \binom{2n}{k-i} \binom{r-1+i}{j} (-1)^j h^{k-i+j} s_{i-j}(\mathcal{E}) = \sum_{\ell} (\sum_j \binom{2n}{k-j-\ell} \binom{r-1+j+\ell}{j} (-1)^j) h^{k-\ell} s_{\ell}(\mathcal{E}) = \sum_{\ell} \binom{2n-r-\ell}{k-\ell} h^{k-\ell} s_{\ell}(\mathcal{E})$ by the following

$$(1.8) \text{ Claim. } \sum_{(a,b)|a+b=c} \binom{m-1+a}{a} (-1)^a \binom{p}{b} = \binom{p-m}{c}.$$

To see this, use the Taylor expansion $(1 + T)^{-m} = \sum_{a \geq 0} \binom{m-1+a}{a} (-T)^a$ and compute the coefficients of T^c in $(1 + T)^{-m} (1 + T)^p = (1 + T)^{p-m}$.

(1.9) Combining (1.6) and (1.7), we get a formula for $c(\mathcal{N})$. On the other hand, we have $c_n(\mathcal{N}) = 0$ and $c_{n-1}(\mathcal{N}) = dh^{n-1}$ since M is of codimension $n - 1$ in $\mathbf{P}^N$. This gives non-trivial relations among Chern classes of $\mathcal{E}$ and Θ_S . In the next section we analyse them precisely in case $s = \dim S = 2$ and $r = n - 1$.

§2. Over a surface

From now on we assume $s = \dim S = 2$ and set $e_j = c_j(\mathcal{E})$, $\gamma_i = c_i(\Theta_S)$.

(2.1) From (1.6) and (1.7) we obtain

$$c_{n-1}(\mathcal{N}) = \binom{n+1}{n-1} h^{n-1} + \binom{n}{n-2} h^{n-2} e_1 + \binom{n-1}{n-3} h^{n-3} (e_1^2 - e_2) - \binom{n+1}{n-2} h^{n-2} \gamma_1 - \binom{n}{n-3} h^{n-3} e_1 \gamma_1 + \binom{n+1}{n-3} h^{n-3} (\gamma_1^2 - \gamma_2).$$

This is dh^{n-1} , while $h^{n-1} - h^{n-2}e_1 + h^{n-3}e_2 = 0$ is the unique relation in $H^*(M)$ (cf. (1.2)). Hence, substituting $h^{n-1} = h^{n-2}e_1 - h^{n-3}e_2$ and comparing the coefficients of h^{n-2} and h^{n-3} , we get the following relations in $H^*(S)$:

$$(i) \quad (n^2 - d)e_1 = (n+1)n(n-1)\gamma_1/6, \quad \text{and}$$

$$(ii) \quad \frac{(n-1)(n-2)}{2} e_1^2 + (d - n^2 + n - 1)e_2 - \frac{n(n-1)(n-2)}{6} e_1 \gamma_1 + \frac{(n+1)n(n-1)(n-2)}{24} (\gamma_1^2 - \gamma_2) = 0.$$

Next from $c_n(\mathcal{N}) = 0$ we get

$$(iii) \quad 3e_1^2 - 2e_2 - ne_1\gamma_1 + \frac{(n+1)(n-1)}{6} (\gamma_1^2 - \gamma_2) = 0.$$

(2.2) Eliminating γ_2 from (ii) and (iii), we get $-3(n^2 - 4)e_1^2 + 6(2d - n^2 - 2)e_2 + n(n-2)(n+2)e_1\gamma_1 = 0$. Eliminating γ_1 further using (i) and noting $d = (e_1^2 - e_2)\{S\}$, we obtain

$$(*) \quad \{2(q+2)d - q(q+5)\} \{(q+2)(q-4)d - (q+2)^2 e_2 + (q-1)(q-4)\} = -q(q-1)(q-4)(q+5)$$

for $q = n^2$.

(2.3) Thus, $2(q+2)d - q(q+5)$ is one of the finitely many divisors of the right hand side, so there are only finitely many numerical possibilities for d and e_2 . Note that they are positive integers since $\mathcal{E}$ is ample.

(2.4) For each (d, e_2) satisfying $(*)$, we examine the relation (i). By the result [IT], it suffices to consider the case where the canonical bundle of S is ample, or equivalently, $d > q = n^2$. Let $6(d - n^2)/(n+1)n(n-1) = a/b$ for coprime integers a, b . Then $ae_1 \sim -b\gamma_1$, so $e_1 \sim bA$ and $K_S \sim -\gamma_1 \sim aA$ for some ample line bundle A on S . Therefore

$$e_1^2 = d + e_2 \quad \text{is divided by} \quad b^2$$

(2.5) In the above case, we have $\gamma_1^2 = a^2(d + e_2)/b^2$ and $e_1\gamma_1 = -a(d + e_2)/b$. Using (iii), we solve γ_2 . The result has to satisfy the Noether relation:

$$\gamma_1^2 + \gamma_2 \equiv 0 \text{ modulo } 12.$$

(2.6) It is easy to produce a computer program to enumerate pairs (n, d) satisfying the numerical conditions (2.2), (2.4) and (2.5). In view of the result of our experiment, we make the following

Conjecture. Any pair (n, d) with $n > 3$ as above is one of the following types:

- (1) $n \equiv 0, 2, 6, 12$ or 16 modulo 18 and $d = q(q+5)/6$ for $q = n^2$. Moreover $e_2 = (q-4)(q+3)/6$, $K_S \sim ne_1$, $K_S^2 = q(q^2 + 2q - 6)/3$, $\gamma_2 = e(S) = (q^3 + 8q^2 + 24q + 36)/3$ and $\chi(\mathcal{O}_S) = (q^3 + 5q^2 + 9q + 18)/18$ for the corresponding scrolls.
- (2) $n = 10$ and $d = 595$. Moreover $e_2 = 561$, $K_S \sim 3e_1$, $K_S^2 = 10404$, $e(S) = 12648$ and $\chi(\mathcal{O}_S) = 1921$.
- (3) $n = 11$ and $d = 231$. Moreover $e_2 = 221$, $2K_S \sim e_1$, $K_S^2 = 113$, $e(S) = 283$ and $\chi(\mathcal{O}_S) = 33$.

(2.7) *Remarks.*

- (1) The above conjecture is verified for $n \leq 11$ by my hand, and for $n \leq 1100$ by a personal

computer. In particular, the case with smallest n is $(n, d) = (6, 246)$. [IT] apparently claims that such a case can be ruled out by “divisibility manipulations”, but I cannot understand the reasoning.

(2) The above condition for n of the type (1) is equivalent to $q \equiv 0$ or 4 modulo 18 .

(3) It is perhaps a delicate problem whether scrolls of the type $(?_n)$ exist or not for a pair (n, d) in (2.6). I find no example at present. To settle the problem, we need some more geometric observations. I feel that the case (2.6.3) might be of particular interest, since this is the unique case with odd n and moreover the invariants are relatively small.

(4) The sectional genus $g = g(M, \mathcal{O}(1))$ can be computed by using the relation $2g - 2 = (K_S + e_1)e_1\{S\}$, and g is bounded by the Castelnuovo inequality. But it turns out that no pair (n, d) in (2.6) is thus ruled out.

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