

On large time behavior of solutions to the compressible Navier-Stokes equation around a time periodic parallel flow

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1 Introduction

In this article we give a summary of recent results on the stability of time-periodic parallel flows of the compressible Navier-Stokes equation in an infinite layer.

We consider the system of equations

$$\partial_{\tilde{t}}\tilde{\rho} + \operatorname{div}(\tilde{\rho}\tilde{v}) = 0, \quad (1.1)$$

$$\tilde{\rho}(\partial_{\tilde{t}}\tilde{v} + \tilde{v} \cdot \nabla \tilde{v}) - \mu \Delta \tilde{v} - (\mu + \mu') \nabla \operatorname{div} \tilde{v} + \nabla \tilde{P}(\tilde{\rho}) = \tilde{\rho} \tilde{g}, \quad (1.2)$$

in an n dimensional infinite layer $\Omega_\ell = \mathbb{R}^{n-1} \times (0, \ell)$:

$$\Omega_\ell = \{\tilde{x} = {}^T(\tilde{x}', \tilde{x}_n);$$

$$\tilde{x}' = {}^T(\tilde{x}_1, \dots, \tilde{x}_{n-1}) \in \mathbb{R}^{n-1}, 0 < \tilde{x}_n < \ell\}.$$

Here $n \geq 2$; $\tilde{\rho} = \tilde{\rho}(\tilde{x}, \tilde{t})$ and $\tilde{v} = {}^T(\tilde{v}^1(\tilde{x}, \tilde{t}), \dots, \tilde{v}^n(\tilde{x}, \tilde{t}))$ denote the unknown density and velocity at time $\tilde{t} \geq 0$ and position $\tilde{x} \in \Omega_\ell$, respectively; $\tilde{P}$ is the pressure that is assumed to be a smooth function of $\tilde{\rho}$ satisfying $\tilde{P}'(\rho_*) > 0$ for a given constant $\rho_* > 0$; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying $\mu > 0$, $\frac{2}{n}\mu + \mu' \geq 0$; div , ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to $\tilde{x}$, respectively. Here and in what follows T denotes the transposition.

Concerning the external force $\tilde{g}$, we assume that $\tilde{g}$ takes the form

$$\tilde{g} = {}^T(\tilde{g}^1(\tilde{x}_n, \tilde{t}), 0, \dots, 0, \tilde{g}^n(\tilde{x}_n))$$

with $\tilde{g}^1(\tilde{x}_n, \tilde{t})$ being a $\tilde{T}$ -periodic function in $\tilde{t}$, where $\tilde{T} > 0$.

The system (1.1)–(1.2) is considered under the boundary condition

$$\tilde{v}|_{\tilde{x}_n=0} = \tilde{V}^1(\tilde{t})\mathbf{e}_1, \quad \tilde{v}|_{\tilde{x}_n=\ell} = 0, \quad (1.3)$$

and initial condition

$$(\tilde{\rho}, \tilde{v})|_{\tilde{t}=0} = (\tilde{\rho}_0, \tilde{v}_0), \quad (1.4)$$

where $\tilde{V}^1(\tilde{t})$ is a $\tilde{T}$ -periodic function of $\tilde{t}$ and $\mathbf{e}_1 = {}^T(1, 0, \dots, 0) \in \mathbb{R}^n$.

If $\tilde{g}^n$ is suitably small, problem (1.1)–(1.3) has a smooth time-periodic solution $\bar{u}_p = {}^T(\bar{\rho}_p, \bar{v}_p)$, so called time-periodic parallel flow, satisfying

$$\bar{\rho}_p = \bar{\rho}_p(\tilde{x}_n) \geq \underline{\tilde{\rho}}, \quad \frac{1}{\ell} \int_0^\ell \bar{\rho}_p(\tilde{x}_n) d\tilde{x}_n = \rho_*,$$

$$\bar{v}_p = {}^T(\bar{v}_p^1(\tilde{x}_n, \tilde{t}), 0, \dots, 0), \quad \bar{v}_p^1(\tilde{x}_n, \tilde{t} + \tilde{T}) = \bar{v}_p^1(\tilde{x}_n, \tilde{t})$$

for a positive constant $\underline{\tilde{\rho}}$.

Our aim is to study the stability of the time-periodic parallel flow $\bar{u}_p$. We will give a summary of the results on the large time behavior of perturbations to $\bar{u}_p$ when Reynolds and Mach numbers are sufficiently small, which were recently obtained in [1, 2, 3].

To formulate the problem for perturbations, we introduce the following dimensionless variables:

$$\tilde{x} = \ell x, \quad \tilde{t} = \frac{\ell}{V} t, \quad \tilde{v} = V v, \quad \tilde{\rho} = \rho_* \rho, \quad \tilde{P} = \rho_* V^2 P, \quad \tilde{V}^1 = V V^1, \quad \tilde{\mathbf{g}} = \frac{\mu V}{\rho_* \ell^2} \mathbf{g}$$

with $\mathbf{g} = {}^T(g^1(x_n, t), \dots, g^n(x_n))$. Here

$$\gamma = \frac{\sqrt{\tilde{P}'(\rho_*)}}{V}, \quad V = \frac{\rho_* \ell^2}{\mu} \left\{ |\partial_t \tilde{V}^1|_{C(\mathbb{R})} + |\tilde{\mathbf{g}}^1|_{C(\mathbb{R} \times [0, \ell])} \right\} + |\tilde{V}^1|_{C(\mathbb{R})} > 0.$$

Under this change of variables the domain Ω_ℓ is transformed into $\Omega = \mathbb{R}^{n-1} \times (0, 1)$; and $g^1(x_n, t)$ and $V^1(t)$ are periodic in t with period $T > 0$, where T is defined by

$$T = \frac{V}{\ell} \tilde{T}.$$

The time-periodic parallel flow $\bar{u}_p$ is transformed into $u_p = {}^T(\rho_p, v_p)$ satisfying

$$\rho_p = \rho_p(x_n) \geq \underline{\rho}, \quad \int_0^1 \rho_p(x_n) dx_n = 1,$$

for a positive constant $\underline{\rho}$, and

$$v_p = {}^T(v_p^1(x_n, t), 0, \dots, 0), \quad v_p^1(x_n, t + T) = v_p^1(x_n, t)$$

It then follows that the perturbation $u(t) = {}^T(\phi(t), w(t)) := {}^T(\gamma^2(\rho(t) - \rho_p), v(t) - v_p(t))$ is governed by the following system of equations

$$\partial_t \phi + v_p^1 \partial_{x_1} \phi + \gamma^2 \operatorname{div}(\rho_p w) = f^0, \quad (1.5)$$

$$\begin{aligned} \partial_t w - \frac{\nu}{\rho_p} \Delta w - \frac{\tilde{\nu}}{\rho_p} \nabla \operatorname{div} w + \nabla \left(\frac{P'(\rho_p)}{\gamma^2 \rho_p} \phi \right) \\ + \frac{\nu}{\gamma^2 \rho_p^2} (\partial_{x_n}^2 v_p^1) \phi \mathbf{e}_1 + v_p^1 \partial_{x_1} w + (\partial_{x_n} v_p^1) w^n \mathbf{e}_1 = \mathbf{f}, \end{aligned} \quad (1.6)$$

$$w|_{\partial\Omega} = 0, \quad (1.7)$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0). \quad (1.8)$$

Here div , ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x , respectively; ν and $\tilde{\nu}$ are the non-dimensional parameters

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \tilde{\nu} = \nu + \nu', \quad \nu' = \frac{\mu'}{\rho_* \ell V};$$

and f^0 and $\mathbf{f} = {}^T(f^1, \dots, f^n)$ denote the nonlinearities:

$$\begin{aligned} f^0 &= -\operatorname{div}(\phi w), \\ \mathbf{f} &= -w \cdot \nabla w + \frac{\nu \phi}{\gamma^2 \rho_p^2} \left(-\Delta w + \frac{\partial_{x_n}^2 v_p^1}{\gamma^2 \rho_p} \phi \mathbf{e}_1 \right) - \frac{\nu \phi^2}{(\phi + \gamma^2 \rho_p) \gamma^2 \rho_p^2} \left(-\Delta w + \frac{\partial_{x_n}^2 v_p^1}{\gamma^2 \rho_p} \phi \mathbf{e}_1 \right) \\ &\quad - \frac{\tilde{\nu} \phi}{(\phi + \gamma^2 \rho_p) \rho_p} \nabla \operatorname{div} w + \frac{\phi}{\gamma^2 \rho_p} \nabla \left(\frac{P'(\rho_p)}{\gamma^2 \rho_p} \phi \right) - \frac{1}{2\gamma^4 \rho_p} \nabla (P''(\bar{\rho}_p) \phi^2) \\ &\quad + P_2(\rho_p, \phi, \partial_x \phi), \end{aligned}$$

where

$$\begin{aligned} P_2 &= \frac{\phi^3}{(\phi + \gamma^2 \rho_p) \gamma^4 \rho_p^3} \nabla P(\rho_p) - \frac{1}{2\gamma^4 (\phi + \gamma^2 \rho_p)} \nabla (\phi^3 P_3(\rho_p, \phi)) \\ &\quad + \frac{\phi \nabla (P''(\rho_p) \phi^2)}{2\gamma^4 \rho_p (\phi + \gamma^2 \rho_p)} - \frac{\phi^2 \nabla (P'(\rho_p) \phi)}{(\phi + \gamma^2 \rho_p) \gamma^4 \rho_p^2} \end{aligned}$$

with

$$P_3(\bar{\rho}_p, \phi) = \int_0^1 (1 - \theta)^2 P'''(\theta \gamma^{-2} \phi + \rho_p) d\theta.$$

We note that the Reynolds number Re and Mach number Ma are given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively.

As for the stability of parallel flows of the compressible Navier-Stokes equations, Iooss and Padula ([4]) studied the linearized stability of a stationary parallel flow in a cylindrical domain under the perturbations periodic in the unbounded direction of the domain. It was shown that the linearized operator generates a C_0 -semigroup in L^2 -space on the basic period cell with

zero mean value condition for the density-component. Using the Fourier series expansion, the authors of [4] showed that the linearized semigroup is written as a direct sum of an analytic semigroup and an exponentially decaying C_0 -semigroup, which correspond to low and high frequency parts of the semigroup, respectively. It was also proved that the essential spectrum of the linearized operator lies in the left-half plane strictly away from the imaginary axis and the part of the spectrum lying in the right-half to the line $\{\operatorname{Re} \lambda = -c\}$ for some number $c > 0$ consists of finite number of eigenvalues with finite multiplicities. In particular, if the Reynolds number is suitably small, then the semigroup decays exponentially.

On the other hand, the stability of a stationary parallel flow in the infinite layer Ω were considered in [5, 6, 7, 8] under the perturbations in some L^2 -Sobolev space on Ω . It was shown in [5, 8] that the asymptotic leading part of the low frequency part of the linearized semigroup is given by an $n - 1$ dimensional heat kernel and the high frequency part decays exponentially as $t \rightarrow \infty$, if the Reynolds and Mach numbers are sufficiently small and the density of the parallel flow is sufficiently close to a positive constant. As for the nonlinear problem, it was proved in [5, 6, 7] that the stationary parallel flow is asymptotically stable under sufficiently small initial perturbations in $H^m(\Omega) \cap L^1(\Omega)$ with $m \geq [n/2] + 1$. Furthermore, the asymptotic leading part of the perturbation is given by the same $n - 1$ dimensional heat kernel as in the case of the linearized problem when $n \geq 3$. In the case of $n = 2$, the asymptotic leading part is no longer described by linear heat equations but by a one-dimensional viscous Burgers equation ([7]).

These results on stationary parallel flows were extended to the time-periodic case in [1, 2, 3]. In section 2 we will give assumptions on the given data $\tilde{g}$ and $\tilde{V}^1$ and state some properties of time-periodic parallel flow. In section 3 we will consider the linearized problem and give a summary of the results obtained in [2, 3]. We will give a Floquet representation for a part of low frequency part of the linearized evolution operator, which plays an important role in the analysis of the nonlinear problem. In section 4 we will consider the nonlinear problem and state the results on the global existence and asymptotic behavior obtained by J. Brezina ([1]).

2 Time-periodic parallel flow

We assume the following regularity for $\tilde{g}$, $\tilde{V}^1$ and $\tilde{P}$.

Assumption 2.1 *Let m be an integer satisfying $m \geq 2$. We assume that*

$\tilde{\mathbf{g}} = {}^T(\tilde{g}^1(\tilde{x}_n, \tilde{t}), 0, \dots, 0, \tilde{g}^n(\tilde{x}_n)), \tilde{V}^1(\tilde{t})$ and $\tilde{P}$ belong to the spaces

$$\tilde{g}^1 \in \bigcap_{j=0}^{\lfloor \frac{m}{2} \rfloor} C_{per}^j([0, \tau]; H^{m-2j}(0, \ell)), \quad \tilde{g}^n \in C^m[0, \ell],$$

$$\tilde{V}^1 \in C_{per}^{\lfloor \frac{m+1}{2} \rfloor}([0, \tilde{T}]),$$

and

$$\tilde{P} \in C^{m+1}(\mathbb{R}).$$

It is easily verified that $\mathbf{g}$, V^1 and P belong to similar spaces as $\tilde{\mathbf{g}}$, $\tilde{V}^1$ and $\tilde{P}$.

Let us consider the time-periodic parallel flow. The dimensionless form of problem (1.1)–(1.3) is written as

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad (2.1)$$

$$\rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - \tilde{\nu} \nabla \operatorname{div} v + \nabla P(\rho) = \nu \rho \mathbf{g}, \quad (2.2)$$

$$v|_{x_n=0} = V^1(t) \mathbf{e}_1, \quad v|_{x_n=1} = 0. \quad (2.3)$$

The following result was shown in [2].

Proposition 2.2 ([2]) *There exists $\delta_0 > 0$ such that if*

$$\nu |g^n|_{C^m([0,1])} \leq \delta_0,$$

then the following assertions hold true.

There exists a time-periodic solution $u_p = {}^T(\rho_p(x_n), v_p(x_n, t))$ of (2.1)–(2.3) that satisfies

$$v_p \in \bigcap_{j=0}^{\lfloor \frac{m+2}{2} \rfloor} C_{per}^j(J_T; H^{m+2-2j}(0, 1)), \quad \rho_p \in C^{m+1}[0, 1],$$

and

$$0 < \underline{\rho} \leq \rho_p(x_n) \leq \bar{\rho}, \quad \int_0^1 \rho_p(x_n) dx_n = 1, \quad v_p(x_n, t) = v_p^1(x_n, t) \mathbf{e}_1$$

with

$$P'(\rho) > 0 \text{ for } \underline{\rho} \leq \rho \leq \bar{\rho},$$

$$|\rho_p - 1|_{C^{m+1}([0,1])} \leq \frac{C}{\gamma^2} \nu (|P''|_{C^{m-1}(\underline{\rho}, \bar{\rho})} + |g^n|_{C^m([0,1])}),$$

$$|P'(\rho_p) - \gamma^2|_{C([0,1])} \leq \frac{C}{\gamma^2} \nu |g^n|_{C([0,1])},$$

and

$$\frac{\rho_p P'(\rho_p)}{\gamma^2} \geq a_0 \quad (2.4)$$

for some constants $0 < \underline{\rho} < 1 < \bar{\rho}$ and $a_0 > 0$.

3 The linearized problem

In this section we consider the linearized problem

$$\partial_t u + L(t)u = 0, \quad t > s, \quad w|_{\partial\Omega} = 0, \quad u|_{t=s} = u_0. \quad (3.1)$$

Here $L(t)$ is the operator given by

$$L(t) = \begin{pmatrix} v_p^1(t) \partial_{x_1} & \gamma^2 \operatorname{div}(\rho_p \cdot) \\ \nabla \left(\frac{P'(\rho_p)}{\gamma^2 \rho_p} \cdot \right) & -\frac{\nu}{\rho_p} \Delta I_n - \frac{\tilde{\nu}}{\rho_p} \nabla \operatorname{div} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{\nu}{\gamma^2 \rho_p^2} \partial_{x_n}^2 v_p^1(t) \mathbf{e}_1 & v_p^1(t) \partial_{x_1} I_n + (\partial_{x_n} v_p^1(t)) \mathbf{e}_1^T \mathbf{e}_n \end{pmatrix}.$$

Note that $L(t)$ satisfies $L(t+T) = L(t)$.

We introduce the space Z_s defined by

$$Z_s = \{u = {}^T(\phi, w); \phi \in C_{loc}([s, \infty); H^1(\Omega)),$$

$$\partial_{x'}^{\alpha'} w \in C_{loc}([s, \infty); L^2(\Omega)) \cap L_{loc}^2([s, \infty); H_0^1(\Omega)) \quad (|\alpha'| \leq 1),$$

$$w \in C_{loc}((s, \infty); H_0^1(\Omega))\}.$$

It was shown in [2] that for any initial data $u_0 = {}^T(\phi_0, w_0)$ satisfying $u_0 \in (H^1 \cap L^2)(\Omega)$ with $\partial_{x'} w_0 \in L^2(\Omega)$ there exists a unique solution $u(t)$ of linear problem (3.1) in Z_s . We denote $U(t, s)$ the solution operator for (3.1) given by

$$u(t) = U(t, s)u_0.$$

To investigate problem (3.1) we consider the Fourier transform of (3.1) with respect to $x' \in \mathbb{R}^{n-1}$

$$\frac{d}{dt} \hat{u} + \hat{L}_{\xi'}(t) \hat{u} = 0, \quad t > s, \quad \hat{u}|_{t=s} = \hat{u}_0. \quad (3.2)$$

Here $\widehat{\phi} = \widehat{\phi}(\xi', x_n, t)$ and $\widehat{w} = \widehat{w}(\xi', x_n, t)$ are the Fourier transforms of $\phi = \phi(x', x_n, t)$ and $w = w(x', x_n, t)$ in $x' \in \mathbb{R}^{n-1}$ with $\xi' = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ being the dual variable; $\widehat{L}_{\xi'}(t)$ is the operator on $(H^1 \times L^2)(0, 1)$ defined as

$$\begin{aligned} D(\widehat{L}_{\xi'}(t)) &= (H^1 \times [H^2 \cap H_0^1])(0, 1), \\ \widehat{L}_{\xi'}(t) &= \begin{pmatrix} i\xi_1 v_p^1(t) & i\gamma^2 \rho_p^T \xi' & \gamma^2 \partial_{x_n}(\rho_p \cdot) \\ i\xi' \frac{P'(\rho_p)}{\gamma^2 \rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)I_{n-1} + \frac{\tilde{\nu}}{\rho_p} \xi'^T \xi' & -i \frac{\tilde{\nu}}{\rho_p} \xi' \partial_{x_n} \\ \partial_{x_n} \left(\frac{P'(\rho_p)}{\gamma^2 \rho_p} \cdot \right) & -i \frac{\tilde{\nu}}{\rho_p} \xi'^T \partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\tilde{\nu}}{\rho_p} \partial_{x_n}^2 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 & 0 \\ \frac{\nu}{\gamma^2 \rho_p^2} (\partial_{x_n}^2 v_p^1(t)) e'_1 & i\xi_1 v_p^1(t) I_{n-1} & \partial_{x_n}(v_p^1(t)) e'_1 \\ 0 & 0 & i\xi_1 v_p^1(t) \end{pmatrix}. \end{aligned}$$

For each $t \in \mathbb{R}$ and $\xi' \in \mathbb{R}^{n-1}$, $\widehat{L}_{\xi'}(t)$ is sectorial on $(H^1 \times L^2)(0, 1)$. We denote the solution operator for (3.2) by $\widehat{U}_{\xi'}(t, s)$. We note that it holds that

$$U(t, s)u_0 = \mathcal{F}^{-1} \left[\widehat{U}_{\xi'}(t, s) \widehat{u}_0 \right]$$

for $u_0 \in (H^1 \cap L^2)(\Omega)$ with $\partial_{x'} w_0 \in L^2(\Omega)$.

We also need to investigate the *adjoint problem*

$$-\partial_s u + \widehat{L}_{\xi'}^*(s)u = 0, \quad s < t, \quad u|_{s=t} = u_0.$$

Here $\widehat{L}_{\xi'}^*(s)$ is a formal adjoint operator defined by

$$\begin{aligned} D(\widehat{L}_{\xi'}^*(s)) &= (H^1 \times [H^2 \cap H_0^1])(0, 1), \\ \widehat{L}_{\xi'}^*(s) &= \begin{pmatrix} -i\xi_1 v_p^1(s) & -i\gamma^2 \rho_p^T \xi' & -\gamma^2 \partial_{x_n}(\rho_p \cdot) \\ -i\xi' \frac{P'(\rho_p)}{\gamma^2 \rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)I_{n-1} + \frac{\tilde{\nu}}{\rho_p} \xi'^T \xi' & -i \frac{\tilde{\nu}}{\rho_p} \xi' \partial_{x_n} \\ -\partial_{x_n} \left(\frac{P'(\rho_p)}{\gamma^2 \rho_p} \cdot \right) & -i \frac{\tilde{\nu}}{\rho_p} \xi'^T \partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\tilde{\nu}}{\rho_p} \partial_{x_n}^2 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & \frac{\nu \gamma^2}{P'(\rho_p)} (\partial_{x_n}^2 v_p^1(s))^T e'_1 & 0 \\ 0 & -i\xi_1 v_p^1(s) I_{n-1} & 0 \\ 0 & \partial_{x_n}(v_p^1(s))^T e'_1 & -i\xi_1 v_p^1(s) \end{pmatrix}. \end{aligned}$$

We denote the solution operator for the adjoint problem by $\widehat{U}_{\xi'}^*(s, t)$.

It holds that $\widehat{U}_{\xi'}(t, s)$ and $\widehat{U}_{\xi'}^*(s, t)$ are defined for all $t \geq s$ and

$$\widehat{U}_{\xi'}(t+T, s+T) = \widehat{U}_{\xi'}(t, s), \quad \widehat{U}_{\xi'}^*(s+T, t+T) = \widehat{U}_{\xi'}^*(s, t).$$

Since $\widehat{L}_{\xi'}(t)$ is T -periodic in t , the spectrum of $\widehat{U}_{\xi'}(T, 0)$ plays an important role in the study of the large time behavior. The following results were established in [2].

We set

$$X_0 = (H^1 \times L^2)(0, 1).$$

Theorem 3.1 ([2]) *There exist positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \widehat{\nu}) \geq \gamma_0^2$ then there exists $r_0 > 0$ such that for each ξ' with $|\xi'| \leq r_0$ there hold the following assertions.*

(i) *The spectrum of operator $\widehat{U}_{\xi'}(T, 0)$ on $(H^1 \times H_0^1)(0, 1)$ satisfies*

$$\sigma(\widehat{U}_{\xi'}(T, 0)) \subset \{\mu_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\} \quad (3.3)$$

for a constant $q_0 > 0$ with $\frac{3}{2}q_0 < \operatorname{Re} \mu_{\xi'} < 1$. Here $\mu_{\xi'} = e^{\lambda_{\xi'} T}$ is a simple eigenvalue of $\widehat{U}_{\xi'}(T, 0)$ and $\lambda_{\xi'}$ has an expansion

$$\lambda_{\xi'} = -i\kappa_0\xi_1 - \kappa_1\xi_1^2 - \kappa''|\xi''|^2 + O(|\xi'|^3), \quad (3.4)$$

where $\kappa_0 \in \mathbb{R}$ and $\kappa_1 > 0$, $\kappa'' > 0$.

Let $\widehat{\Pi}_{\xi'}$ be the eigenprojection for the eigenvalue $\mu_{\xi'}$. Then there holds

$$|\widehat{U}_{\xi'}(t, s)(I - \widehat{\Pi}_{\xi'})u|_{H^1} \leq Ce^{-d(t-s)}|(I - \widehat{\Pi}_{\xi'})u|_{X_0}$$

for $u \in X_0$ and $t - s \geq T$. Here d is a positive constant depending on r_0 .

(ii) *The spectrum of operator $\widehat{U}_{\xi'}^*(0, T)$ on $H^1 \times H_0^1$ satisfies*

$$\sigma(\widehat{U}_{\xi'}^*(0, T)) \subset \{\bar{\mu}_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\}.$$

Here $\bar{\mu}_{\xi'}$ is a simple eigenvalue of $\widehat{U}_{\xi'}^(0, T)$.*

Let $\widehat{\Pi}_{\xi'}^$ be the eigenprojection for the eigenvalue $\bar{\mu}_{\xi'}$. Then there holds*

$$\langle \widehat{\Pi}_{\xi'}^* u, v \rangle = \langle u, \widehat{\Pi}_{\xi'} v \rangle$$

for $u, v \in X_0$.

Theorem 3.1 can be proved by a perturbation argument from the case $\xi' = 0$. See [2] for details.

Based on Theorem 3.1 we can obtain a Floquet representation of a part of $U(t, s)$.

Let ν_0 , γ_0 and r_0 are the numbers given by Theorem 3.1. In the rest of this section we assume that $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$.

We set

$$u^{(0)}(t) = \widehat{U}_0(t, 0)u_0^{(0)}. \quad (3.5)$$

Here $u_0^{(0)}$ is an eigenfunction of the operator $\widehat{U}_0(T, 0)$ for the eigenvalue $e^{\lambda_0 T} = 1$. Observe that

$$u^{(0)}(t + T) = u^{(0)}(t).$$

We also define the multiplier $\Lambda : L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\mathbb{R}^{n-1})$ by

$$\Lambda\sigma = \mathcal{F}^{-1}[\widehat{\chi}_1 \lambda_{\xi'} \widehat{\sigma}].$$

Here $\widehat{\chi}_1$ is defined by

$$\widehat{\chi}_1(\xi') = \begin{cases} 1, & |\xi'| < r_0, \\ 0, & |\xi'| \geq r_0 \end{cases}$$

for $\xi' \in \mathbb{R}^{n-1}$.

Clearly, Λ is a bounded linear operator on $L^2(\mathbb{R}^{n-1})$. It then follows that Λ generates a uniformly continuous group $\{e^{t\Lambda}\}_{t \in \mathbb{R}}$. Furthermore, it holds that

$$\|\partial_{x'}^k e^{t\Lambda} \sigma\|_{L^2(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \|\sigma\|_{L^p(\mathbb{R}^{n-1})}, \quad k = 0, 1, \dots, \quad 1 \leq p \leq 2.$$

We have the following Floquet representation for $U(t, s)$.

Theorem 3.2 ([3])

(i) *There exist time periodic operators*

$$\mathcal{Q}(t) : L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\Omega), \quad \mathcal{Q}(t + T) = \mathcal{Q}(t),$$

$$\mathcal{P}(t) : L^2(\Omega) \rightarrow L^2(\mathbb{R}^{n-1}), \quad \mathcal{P}(t + T) = \mathcal{P}(t)$$

such that the operator $\mathbb{P}(t) := \mathcal{Q}(t)\mathcal{P}(t) : L^2(\Omega) \rightarrow L^2(\Omega)$ satisfies

$$\mathbb{P}(t)^2 = \mathbb{P}(t), \quad \mathbb{P}(t + t) = \mathbb{P}(t),$$

$$\mathbb{P}(t)(\partial_t + L(t))u(t) = (\partial_t + L(t))(\mathbb{P}(t)u(t)) = \mathcal{Q}(t)[(\partial_t - \Lambda)(\mathcal{P}(t)u(t))]$$

for $u \in L^2(0, T; (H^1 \times [H^2 \cap H_0^1])(\Omega)) \cap H^1(0, T; L^2(\Omega))$.

(ii) It holds that

$$\mathbb{P}(t)U(t, s) = U(t, s)\mathbb{P}(s) = \mathcal{Q}(t)e^{(t-s)\Lambda}\mathcal{P}(s).$$

Furthermore,

$$\|\partial_t^j \partial_x^k \partial_{x_n}^l \mathbb{P}(t)U(t, s)u\|_{L^2(\Omega)} \leq C(1+t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u\|_{L^1(\Omega)}$$

for $0 \leq 2j + l \leq m$, $k = 0, 1, \dots$.

(iii) Let $\mathcal{H}(t)$ be a heat semigroup defined by

$$\mathcal{H}(t) = \mathcal{F}^{-1}e^{-(i\kappa_0\xi_1 + \kappa_1\xi_1^2 + \kappa''|\xi''|^2)t}\mathcal{F}.$$

Suppose that $1 \leq p \leq 2$. Then it holds that

$$\begin{aligned} & \|\partial_x^k \partial_{x_n}^l (\mathbb{P}(t)U(t, s)u - [\mathcal{H}(t-s)\sigma]u^{(0)}(t))\|_{L^2(\Omega)} \\ & \leq C(1+t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|u\|_{L^p(\Omega)} \end{aligned}$$

for $u = {}^T(\phi, w)$, $k = 0, 1, \dots$, and $0 \leq l \leq m$. Here $u^{(0)}(t)$ is the function given in (3.5) and $\sigma = \int_0^1 \phi(x', x_n) dx_n$.

(iv) $(I - \mathbb{P}(t))U(t, s) = U(t, s)(I - \mathbb{P}(s))$ satisfies

$$\|(I - \mathbb{P}(t))U(t, s)u\|_{H^1(\Omega)} \leq Ce^{-d(t-s)}(\|u\|_{(H^1 \times L^2)(\Omega)} + \|\partial_{x'} w\|_{L^2(\Omega)})$$

for $t - s \geq T$. Here d is a positive constant.

4 The nonlinear problem

In this section we consider the nonlinear problem (1.5)–(1.8).

Brezina ([1]) recently proved the global existence and the asymptotic behavior for (1.5)–(1.8) when the Reynolds and Mach numbers are sufficiently small.

Theorem 4.1 ([1]) *Let $n \geq 2$ and let m be an integer satisfying $m \geq [n/2] + 1$. Suppose that $\tilde{g}$, $\tilde{V}^1$ and $\tilde{P}$ satisfy Assumption 2.1 for m replaced by $m+1$. Then there are positive numbers ν_1 and γ_1 such that the following assertions hold true, provided that $\nu \geq \nu_1$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_1^2$.*

There is a positive number ε_0 such that if $u_0 \in {}^T(\phi_0, w_0) \in H^m \cap L^1(\Omega)$ satisfies a suitable compatibility condition and $\|u_0\|_{H^m \cap L^1(\Omega)} \leq \varepsilon_0$, then there exists a global solution $u(t)$ of (1.5)–(1.8) in $C([0, \infty); H^m(\Omega))$ and $u(t)$ satisfies

$$\|\partial_{x'}^k u(t)\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{k}{2}}), \quad k = 0, 1,$$

as $t \rightarrow \infty$.

Furthermore, there holds

$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{1}{2}} \eta_n(t))$$

as $t \rightarrow \infty$. Here $\eta_n(t) = 1$ for $n \geq 4$, $\eta_n(t) = \log t$ for $n = 3$ and $\eta_n(t) = t^\delta$ for $n = 2$, where δ is an arbitrarily positive number; $u^{(0)} = u^{(0)}(x_n, t)$ is the function given in (3.5); and $\sigma = \sigma(x', t)$ satisfies

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + \kappa_0 \partial_{x_1} \sigma = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n) dx_n$$

if $n \geq 3$, and

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma + \kappa_0 \partial_{x_1} \sigma + a_0 \partial_{x_1}(\sigma^2) = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n) dx_n$$

if $n = 2$, where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$ for $n \geq 3$, and a_0 is a constant.

Remark 4.2 A result similar to Theorem 4.1 also holds for the case of stationary parallel flows ([7]).

Theorem 4.1 is proved by the decomposition method based on the spectral analysis in section 3. We write problem (1.5)–(1.8) as

$$\partial_t u + L(t)u = \mathbf{F}(u), \quad u(0) = u_0.$$

We decompose the solution $u(t)$ of (1.5)–(1.8) into

$$u(t) = u_1(t) + u_\infty(t),$$

where

$$u_1(t) = \mathbb{P}(t)u(t), \quad u_\infty(t) = (I - \mathbb{P}(t))u(t).$$

It then follows from Theorem 3.2 that

$$u_1(t) = \mathcal{Q}(t) \left[e^{t\Lambda} \mathcal{P}(0)u_0 + \int_0^t e^{(t-s)\Lambda} \mathcal{P}(s) \mathbf{F}(u(s)) ds \right],$$

$$\partial_t u_\infty + L(t)u_\infty = (I - \mathbb{P}(t)) \mathbf{F}(u), \quad u_\infty(0) = (I - \mathbb{P}(0))u_0.$$

To estimate u_1 , we use the estimates obtained in Theorem 3.2, while u_∞ is estimated by a variant of the Matsumura-Nishida energy method ([9, 6, 7]). See [1] for details.

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