LECTURES IN MATHEMATICS

Department of Mathematics
KYOTO UNIVERSITY

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ON AUTOMORPHISM GROUP OF $k[x, y]$

BY
MASAYOSHI NAGATA

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it is worthy to know many different proofs of the two variables case. Because of this reason, I am giving here my proof of the two variables case and also some comments on the general case.

Thanks are due to Professor Abhyankar, Professor Zariski for their valuable informations on the problem; to my friends at Purdue University for their discussion on the problem; to Mrs. Hayashi for type-writing the manuscript.

Masayoshi Nagata

Contents

Part 0 Introduction and preliminaries.

Introduction.

By a ring, we shall mean a commutative ring with identity throughout this article.

Let k be a ring and let $k[x] = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables $x_1, \ldots, x_n$ over k. Then the question which we shall deal with is on the structure of the group of automorphisms $\mathrm{Aut}_k\, k[x]$ over k. k and $k[x]$ will maintain these meanings throughout this article.

We shall begin with looking at some special subgroups of $\mathrm{Aut}_k\, k[x]$.

Linear subgroups. Let us consider the general linear group $GL(n, k)$ over k. An arbitrary element $\sigma \in GL(n, k)$ defines a k-automorphism of $k[x]$ so that

$$(\sigma x_1, \ldots, \sigma x_n) = (x_1, \ldots, x_n)\sigma$$

where the right hand side is understood by matric multiplication. Note that if $\sigma, \tau \in GL(n, k)$, then $(\sigma(\tau x_1), \ldots, \sigma(\tau x_n)) = \sigma[(x_1, \ldots, x_n)\tau] = (\sigma x_1, \ldots, \sigma x_n)\tau = (x_1, \ldots, x_n)\sigma\tau$. Thus we may regard that the general linear group $GL(n, k)$ is a subgroup of $\mathrm{Aut}_k\, k[x]$. Furthermore, we can observe affine transformations similarly. Namely, an affine transformation of the n-space is defined by a sequence of

an element, say σ, of $GL(n, k)$ and n elements, say $c_1, \ldots, c_n$, of k. Such an affine transformation defines also a k-automorphism such that x_i is sent to $\sigma x_i + c_i$. Thus the affine transformation group $A(n, k)$ can be regarded as a subgroup of $\mathrm{Aut}_k\, k[x]$.

Jonquières automorphisms. When $a_1, \ldots, a_n$ are units in k and $f_i \in k[x_{i+1}, \ldots, x_n]$ for $i = 1, \ldots, n$ ($f_n \in k$), then there is an element τ of $\mathrm{Aut}_k\, k[x]$ such that $\tau x_i = a_i x_i + f_i$ $(i = 1, \ldots, n)$. This type of automorphism is called a Jonquières automorphism of $k[x]$. The set $J(n, k)$ of all Jonquieres automorphisms of $k[x]$ is a subgroup of $\mathrm{Aut}_k\, k[x]$.

As we shall show later (§ 0.2), it holds that :

Theorem 0.1. Let $g_1, \ldots, g_n$ be nilpotent elements of $k[x]$. Then there is a $\sigma \in \mathrm{Aut}_k\, k[x]$ such that $\sigma x_i = x_i + g_i$ $(i = 1, \ldots, n)$.

The set of all automorphisms of this type forms a subgroup of $\mathrm{Aut}_k\, k[x]$ and is called the subgroup of nilpotency of $\mathrm{Aut}_k\, k[x]$. We shall denote it by $N(n, k)$. As for this subgroup, it holds that :

Proposition 0.2. Let $\sqrt{0}$ be the nil-radical of k and we consider the natural homomorphism $\psi : \mathrm{Aut}_k\, k[x] \longrightarrow \mathrm{Aut}_{k/\sqrt{0}}\, (k/\sqrt{0})[x]$. Then ψ is a surjection and $N(n, k)$ is the kernel of ψ.

The proof will be given later in § 0.2.

Now, one particular problem in our question is :

Qestion 0.3. Is $\mathrm{Aut}_k\, k[x]$ generated by $GL(n, k)$, $J(n, k)$ and $N(n, k)$?

Unfortunately, the answer is negative in the general case as will be shown in Part 2. Part 1 of the article contains a proof of the following theorem, which should be called a theorem of Jung-van der Kulk (cf. § 0.1), and also a structure theorem of $\mathrm{Aut}_k\, k[x]$ due to van der Kulk.

Theorem 0.4. *The question 0.3 is affirmative if* k *is a field and if* $n = 2$.

On the other hand, it is obvious that

Lemma 0.5. If k is the direct sum of rings $k_1, \ldots, k_s$, then $\mathrm{Aut}_k\, k[x]$ is the direct product of $\mathrm{Aut}_{k_i}\, k_i[x]$ $(i = 1, \ldots, s)$.

In view of Proposition 0.2 and Lemma 0.5 above, we have

Corollary 0.6. The question 0.3 is affirmative if $n = 2$ and if k is an Artin ring.

Part 2 contains some results related to the question in case either $n \neq 2$ or k is not an Artin ring.

0.1 Historical remarks on the problem.

The case where $n = 1$ and k is a field is well known and easy ; even if k is not a field, if $n = 1$, then $\mathrm{Aut}_k\, k[x]$ has fairly simple structure (cf. § 0.3).

In the past, mostly the case where k is a field and $n = 2$ has

been treated. In that case, an affirmative answer of the question 0.3 was claimed by Jung [4] in 1941, in the classical case. Some related questions in the classical case were treated also by some authors including Engel [2] and Gutwirth [3]. But I am afraid that their treatments are too difficult to follow. Meanwhile, van der Kulk [6] gave a much better treatment and proved our Question 0.3 for $n = 2$ and k an arbitrary field. Recently, Abhyankar-Moh [1] treated the question from another aspect. Namely, as one will see later in this article, the question has very close relationship with the following classical conjecture :

Conjecture 1.1. Let C be an irreducible curve on an affine plane S defined by $f(x, y) \in k[x, y]$. Assume that C is biregular to an affine line, namely, $k[x, y]/(f) \cong k[t]$. Then there is a polynomial $g(x, y)$ such that $k[x, y] = k[f, g]$.

As was shown by the writer [8], this conjecture is not affirmative in the positive characteristic case, hence the conjecture was modified :

Modified conjecture 1.2. Add one assumption that $\deg f$ is not divisible by the characteristic of the ground field k. Then the assertion in the conjecture holds good.

One of important results in [1] is an affirmative answer of this modified conjecture.

On the other hand, our treatment is very different from those given

by these authors : We are to make use of nice behaviour of birational correspondences of non-singular projective surfaces.

0.2 Some easy results related to the statements in Introduction.

To begin with, we shall prove a well known lemma :

Lemma 2.1. Let M be a k-module and let N be a submodule of M. If I is a nilpotent ideal of k and if $M = N + IM$, then $M = N$.

Proof. $M = N + IM = N + I(N + IM) = N + I^2M$. Similarly, $M = N + I^rM$ for every natural number r. Because of the nilpotency of I, we have $M = N$. q.e.d.

Proposition 2.2. Consider n elements $y_1, \ldots, y_n$ of the polynomial ring $k[x]$. If $k[y] = k[x]$, then $k[y]$ is a polynomial ring, i.e., $F(x) \in k[x]$, $F(y) = 0$ imply $F(x) = 0$. Hence there is a k-automorphism σ of $k[x]$ such that $\sigma x_i = y_i$ $(i = 1, 2, \ldots, n)$.

Proof. Assuming the contrary, let $F(x)$ $(\neq 0)$ be such that $F(y) = 0$. Since $k[y] = k[x]$, there are polynomials $f_1(x), \ldots, f_n(x)$ such that $x_i = f_i(y)$. Let k' be the subring of k generated by the coefficients of $1, F(x), y_1, \ldots, y_n, f_1(x), \ldots, f_n(x)$. Then k' is noetherian, because k' is finitely generated. Therefore, considering k' instead of k, it suffices to prove the assertion under the additional assumption that k is noetherian. Let ϕ be the k-surjection given by $\phi x_i = y_i$ $(i = 1, \ldots, n)$. Then powers ϕ^r are also k-homomorphisms.

Let I_r be the kernel of ϕ^r. Since k is noetherian, there is one r, say s, such that $I_s = I_{s+1} = \cdots$. Let a be an arbitrary element of I_s. Since ϕ^s is a surjection, there is an element b of $k[x]$ such that $a = \phi^s(b)$. Then we see that $0 = \phi^s(a) = \phi^{2s}(b)$. Hence $b \in I_{2s} = I_s$. This implies that $a = \phi^s(b) \in \phi^s(I_s) = \{0\}$. Thus $I_s = \{0\}$ and we see that ϕ is an automorphism. q.e.d.

As a corollary to these two results, we have :

Proposition 2.3. Let $y_1, \cdots, y_n \in k[x]$ be such that they generate $k[x]$ over k <u>modulo</u> the nil-radical, i.e. ($k[y]$ modulo the nilradical) = ($k[x]$ modulo the nil-radical), then $k[x] = k[y]$ and there is a k-automorphism σ such that $\sigma x_i = y_i$ $(i = 1, \cdots, n)$.

Proof. $x_i = t_i + \sum_{j=1}^{s_i} c_{ij} z_{ij}$ ($t_i \in k[y]$, $c_{ij} \in$ (the nilradical of k), $z_{ij} \in k[x]$). Let N be the ideal generated by these c_{ij} $(i = 1, \cdots, n ; j = 1, \cdots, s_i)$. Then N is nilpotent and therefore $k[x] = k[y]$ by Lemma 2.1. Therefore we have the result by Proposition 2.2. q.e.d.

Now, let us prove Theorem 0.1. Setting $y_i = x_i + g_i$, we see that Proposition 2.3.is applied immediately, and we have the theorem.

Next, let us prove Proposition 0.2. If $\tau \in \mathrm{Aut}_{k/\sqrt{0}} (k/\sqrt{0})[x]$, then let y_i be an element of $k[x]$ such that (y_i modulo $\sqrt{0}$) = τx_i. Then Proposition 2.3 implies that there is $\sigma \in \mathrm{Aut}_k k[x]$ such

that $\sigma x_i = y_i$. Thus ϕ is a surjection. It is obvious that $N(n, k)$ is the kernel of ϕ. q.e.d.

0.3 The easy case, n = 1.

Proposition 3.1. If $n = 1$, then $\mathrm{Aut}_k\, k[x]$ is generated by $A(1, k)$ and $N(1, k)$. More precisely, there is a one-one correspondence between $\mathrm{Aut}_k\, k[x]$ and the set $M = \{(a, b, f) \mid a = a$ unit in k, $b \in k$, $f = a$ nilpotent element in $k[x]\}$ in such a way that σ corresponds to (a, b, f) if $\sigma x = ax + b + f$.

Proof. In view of Proposition 0.2, we may assume that $\sqrt{0} = \{0\}$. It is obvious that for a given $(a, b, 0) \in M$, there is a $\sigma \in \mathrm{Aut}_k\, k[x]$ such that $ax + b = \sigma x$. Let τ be an arbitrary element of $\mathrm{Aut}_k\, k[x]$ and write $\tau x = c_0 + c_1 x + \ldots + c_s x^s$ $(c_i \in k,\ c_s \neq 0)$. Since there is an $F(x)$ $k[x]$ such that $F(\tau x) = x$, we must have $s = 1$ and $\deg F(x) = 1$ (by our assumption that $\sqrt{0} = \{0\}$). Therefore we see also that c_1 is a unit in k. q.e.d.

0.4 Prerequisites, notation and terminology.

Notation introduced in the introduction above is maintained. When we consider two irreducible algebraic surfaces V, W having the same function field, we identify points P ($\in V$) with Q ($\in W$) if they correspond biregularly under the natural birational correspondence

(i.e., if their local rings over a field K of definition over which they are rational coincides with each other). In this sense, the quadratic dilatation $dil_P V$ of V with center P is well defined. When a successive quadratic dilatation with centers $P_1, \ldots, P_n$ is defined on V, then the transformation is denoted by $dil_{P_1 \ldots P_n}$. Hence $dil_{P_1 \ldots P_n} = dil_{P_n} \cdot dil_{P_1 \ldots P_{n-1}}$

We are assumed to be familiar with basic facts on birational correspondences of non-singular projective surfaces.

A point of V in the usual sense is called an ordinary point of V. By a point of V, we understand that it is either an ordinary point or an infinitely near point V : as for the notion of an infinitely near point, we shall review it in § 0.5.

Let C be an irreducible curve on V. A point P of C (P being an ordinary point of V) is called a one-place point of C if the derived normal ring of the local ring of P on C is local ; this means that if W is a surface birationally dominating V and if ordinary points Q, Q' of W, corresponding to P, lies on the proper transform of C, then $Q = Q'$.

Ω denotes a universal domain which we shall fix, and Ω^2 denotes the affine plane (over Ω). $\mathbb{P}^2$ denotes a projective plane. Using coordinates system (X, Y, Z) on $\mathbb{P}^2$, the line $Z = 0$ is supposed to

be the line at infinity and is denoted by ℓ_∞. The complement of the line is identified with Ω^2.

When V is an irreducible algebraic variety defined over a field K, the function field of V over K is denoted by $K(V)$.

Let V and W are non-singular varieties such that W dominates V. Let T be the correspondence $V \longrightarrow W$. If D is a divisor on V, then the total transform of D is denoted by $T\{D\}$; the proper transform of D is denoted by $T[D]$. Note that $T\{D\}$ is defined by local equations, namely, if Q dominates P under T^{-1}, then (the local equation of $T\{D\}$ at Q) = (the local equation of D at P). Therefore the total transform $T\{\ \}$ preserves linear equivalence.

Linear equivalence of divisors is expressed by the symbol $\sim$.

0.5 Infinitely near points.

One remarkable fact on birational transformations of non-singular projective surfaces is as follows :

Theorem 5.1. Let V and W be non-singular projective surfaces having the same function field. If an ordinary point P is dominated by an ordinary point Q of W, then there is a sequence $P = P_0, \ldots, P_n$ such that (i) $Q = P_n$ and (ii) for each $i > 0$, $\mathrm{dil}_{P_0 \ldots P_{i-1}}$ is well defined on V and P_i is an ordinary point of $\mathrm{dil}_{P_0 \ldots P_{i-1}} V$ which

dominates P_{i-1}.

In this case, Q is called an infinitely near point to P of order n.

Infinitely near points to P of order one correspond to tangential directions around P. Namely, letting t and u be local coordinates at P, we consider the local equation of a curve C at P :

(5.2) $\Sigma_{i+j=m} c_{ij}t^iu^j$ + (terms of higher orders)

with $c_{ij} \in k$, $m > 0$. The term $h(t, u) = \Sigma c_{ij}t^iu^j$ gives (besides that P is an m-ple point of C) tangential directions of branches of C at P ; C has a branch of tangential direction $t = \alpha u$ ($\alpha \in \Omega$) if and only if $t - \alpha u$ is a factor of $h(t, u)$. On $dil_P V$, if $t/u = \alpha$ at a point P' of $dil_P P$, then the local equation of the total transform C* of C at P' is

$$u^m h(t/u, 1) + u^{m+1} g(u, t/u)$$

with some polynomial g. Therefore it is factored to u^m and

(5.3) $h(t/u, 1) + ug(u, t/u)$.

Since u at P' gives $dil_P P$, C* is of the form $m \cdot dil_P P + C_1$ with a curve C_1 whose local equation at P' is (5.3). C_1 goes through P' obviously because $h(t, u)$ is a homogeneous form having $t - \alpha u$ as a factor. Furthermore, if m is the exact multiplicity of P on C,

then, the intersection multiplicity of C_1 with $\mathrm{dil}_P P$, coincides with the multiplicity of the factor $t - \alpha u$ in $h(t, u)$, because, in view of the fact that u defines $\mathrm{dil}_P P$, we see that the former coincides with the intersection multiplicity of the curve $h(t/u, 1) = 0$ with $\mathrm{dil}_P P$ at P'. Thus we obtained also the following :

Proposition 5.4. If a curve C goes through an ordinary point P of V with multiplicity exactly m, then letting C_1 denote the proper transform of C on $\mathrm{dil}_P V$, we see that

(1) The total transform $\mathrm{dil}_P C$ of C is equal to $C_1 + m \cdot \mathrm{dil}_P P$

(2) the intersection number $(C_1, \mathrm{dil}_P P)$ is equal to m.

We note here that the intersection number of two divisors C', C'' coincides with that of total transforms of C', C''.

Proposition 5.5. With the same notation as above, we have

(1) $(\mathrm{dil}_P P, \mathrm{dil}_P P) = -1$.

(2) $(C_1, C_1) = (C, C) - m^2$

Proof. Since $C \sim C' - C''$ (C', C'' are positive divisors) such that $C' \cup C'' \not\ni P$, we see that $(\mathrm{dil}_P C, \mathrm{dil}\ P) = (\mathrm{dil}_P C' - \mathrm{dil}_P C'', \mathrm{dil}_P P) = 0$. Hence $0 = (C_1 + m\ \mathrm{dil}_P P, \mathrm{dil}_P P) = m + m\ (\mathrm{dil}_P P, \mathrm{dil}_P P)$, and we have (1). $(C, C) = (C_1 + m\ \mathrm{dil}_P P, C_1 + m\ \mathrm{dil}_P P) = (C_1, C_1) + 2m(C_1, \mathrm{dil}_P P) + m^2(\mathrm{dil}_P P) = (C_1, C_1) + 2m^2 - m^2 = (C_1, C_1) + m^2$.

q.e.d.

Remark 5.6. Proofs of these two propositions may be done as follows : Proposition 5.4, (1) is proved first. Then considering a special case where $m = 1$, we obtain Proposition 5.5, (1). Then we have Proposition 5.4, (2) by :

$$0 = (C_1 + m\ \mathrm{dil}_P P,\ \mathrm{dil}_P P) = (C_1,\ \mathrm{dil}_P P) + m(\mathrm{dil}_P P,\ \mathrm{dil}_P P)$$

Here we add a remark on the case where m is not the exact multiplicity. We consider the condition on curves C such that C goes through P as an m-ple point and furthermore C goes through certain infinitely near points $Q_1, \ldots, Q_s$ of order one to P, for instance. It is quite natural to understand this condition to be the condition on the local equation of C at P to be such that it is of the form (5.2) and the form $h(t, u) = \Sigma\ c_{ij}t^iu^j$ has factors corresponding to $Q_1, \ldots, Q_s$. Actually, if we look at a linear system of curves on V and consider those members of L satisfying the above condition, then the subset of such members is a well defined linear system under this understanding. If a C goes through P with multiplicity bigger than m, then $h(t, u) = 0$ whence the condition is satisfied by this C. This means that if the multiplicity of P is higher, then the condition stated above do not imply that the curve has branches corresponding to Q_1.

One extreme case is the case where $s > m$. Since h is of

degree m, it cannot have more than m factors unless it is zero. Therefore, as the actual effect, the condition " to go through P with multiplicity m and furthermore goes through $m+1$ distinct infinitely near points of order one to P " is equivalent to the condition " to go through P with multiplicity $m+1$ ".

Therefore we must be careful of the term " to go through ", and therefore we shall define the term after some preliminaries.

First, we define divisorial cycles in very generalized way. Namely, let $Dc_0(V)$ be the set of point of V (including infinitely near points) and let $Dc_1(V)$ be the set of irreducible curves on V. Let $Dc(V)$ be the free module generated by $Dc_0(V) \cup Dc_1(V)$ over $\mathbb{Z}$. This module $Dc(V)$ is called the <u>divisorial cycle group</u> of V and each member of it is called a <u>divisorial cycle</u> on V.

When P is an ordinary point of V, we define a group homomorphism $Dc(V) \longrightarrow Dc(dil_P V)$ by

(i) if $C \in Dc_1(V)$, then C is mapped to its total transform (in the usual sense),

(ii) P is mapped to its total transform $dil_P P$,

(iii) if $P \neq Q \in Dc_0(V)$, then Q is mapped Q itself.

We shall denote this homomorphism by the same symbol dil_P as the birational transformation. One sees quite easily that dil_P gives an isomorphism.

If W is a nonsingular projective surface having the same function field as V, birational transformation $\tau : V \longrightarrow W$ is factored to the form

$$(dil_{Q_1 \ldots Q_r})^{-1} dil_{P_1 \ldots P_s}$$

with fundamental points $Q_1, \ldots, Q_r$ on W with respect to V and fundamental points $P_1, \ldots, P_s$ on V with respect to W. Therefore by composing isomorphisms dil_{P_i} and $dil_{Q_j}^{-1}$, we obtain an isomorphism $Dc(V) \longrightarrow Dc(W)$, which we shall denote by T again. Then, one sees easily that if the birational transformation T is the composition of two birational transformations T_1, T_2, then the corresponding isomorphism T is also the composition of corresponding isomorphisms to T_1, T_2.

For $c \in Dc(V)$, Tc is called the <u>total transform</u> of c on W.

Secondly, we define positivity of divisorial cycles as follows :

An element $c \in Dc(V)$ is called <u>virtually positive</u> if all of the coefficients of c are non-negative. (Note that element 0 is virtually positive under this definition.) An element $c \in Dc(V)$ is called <u>effectively positive</u> if there is a dilatation $T = dil_{P_1 \ldots P_r}$ of V such that Tc is virtually positive.

Now we define " to go through ". A curve C on V is said to <u>go through</u> $\Sigma_{i=1}^{s} m_i P_i$ ($m_i \in Z$, P_i points of V) if $C - \Sigma m_i P_i$ is effectively positive.

We add here another definition. A point P of V is said to lie on a curve C, if, letting $P_0, \ldots, P_r = P$ be the sequence of points of V such that P_0 is an ordinary point and each P_i $(i > 0)$ is an infinitely near point of order one to P_{i-1}, P lies on the proper transform C^* of C on $\mathrm{dil}_{P_0 \cdots P_{r-1}} V$. The effective multiplicity of P on C in this case is defined by the multiplicity of P on C^*.

We now observe a linear system L on V. L is a set of positive divisors and is defined by a pair of a finite k-submodule of $k(V)$ and a divisor D on V so that $L = \{D + (f) \mid 0 \neq f \in M\}$. Consider the birational correspondence $T : V \longrightarrow W$. We define effective transform $T_{ef}L$ of L on W to be the uniquely determined linear system by the properties that (i) it has no fixed component and (ii) the same module M is associated to it. On the other hand, we define total transform of L. For the purpose, we generalize the notion of a linear system a little. Namely, we consider symbols of type $L - \Sigma\, m_i P_i$, where L is a linear system, $m_i \in Z$ and $P_i \in Dc_0(V)$. To this symbol, we associate $\{D - \Sigma\, m_i P_i \mid D \in L,\ D - \Sigma\, m_i P_i$ effectively positive$\}$, which is a set of effectively positive divisorial cycles. The total transform of $L - \Sigma\, m_i P_i$ is defined to be the set of total transforms of these effectively positive divisorial cycles. Remarks to be given here are :

Remark 5.7. If T is regular, then the total transform TL of

L is of the form $L^* - \Sigma\, m_i P'_i$ (L^* a linear system ; $P'_i \in Dc_0(W)$). $m_\alpha < 0$ implies that the total transform of P'_i is a fixed component of L.

Remark 5.8. In general, if L has no fixed component, then

$$TL = (T_{ef}L + D) - \Sigma\, m_i P'_i$$

with a positive divisor D, $m_i \in \mathbb{Z}$, $P'_i \in Dc_0(W)$, and for every member of $T_{ef}L + D$, the multiplicity of P'_i on it is at least m_i.

Remark 5.9. As for $L - \Sigma\, m_i P_i$ above, if $dil_{Q_1 \cdots Q_r}$ is well defined on V and if every P_i is among these Q_j, then the total transform $dil_{Q_1 \cdots Q_r}(L - \Sigma\, m_i P_i)$ is a linear system.

By virtue of Remark 5.8, we can see the following fact :

Remark 5.10. Let L be a linear system on V. If L has no fixed component and if $T_{ef}L$ has no fixed point, then

$$T_{ef}L = T(L - \Sigma\, m_i P_i)$$

where the P_i are fixed points of L and each m_i is the multiplicity of P_i on a general member of L. Consequently, each P_i is fundamental with respect to W.

As a corollary to this, we have the following remark, which we are going to use later :

Remark 5.11. Let L be a linear system on V. Assume that L has no fixed component. If $T_{ef}L$ consists only of non-singular curves and if $T_{ef}L$ has only one fixed point Q, then

(1) Singular points of a member of L are fundamental with respect to T and are fixed points of L.

(2) A fixed point of L is either a fundamental point with respect to T or Q itself ; if Q is fundamental with respect to T^{-1}, then every fixed point of L is fundamental with respect to T.

Part 1. The case $n = 2$ and k is a field.

1.1. A preliminary step.

We begin with the following lemma.

Lemma 1.1. Let $\sigma \in \mathrm{Aut}_k\, k[x, y]$. If $\sigma x = cx + d$ with $c, d \in k$, then $\sigma \in J$.

Proof. σy is expressed as $yh(x, y) + f(x)$, with $f(x) \in k[x]$, $h(x, y) \in k[x, y]$. Then $k[x, y] = k[x, yh(x, y)]$, therfore there is a k-automorphism of $k[x, y]$ which sends y to $yh(x, y)$. Therefore yh must be irreducible, and h is a unit in $k[x, y]$. Thus $h \in k$, and we see that $\sigma \in J$. q. e. d.

Now we assert that in order to prove the generation of $\mathrm{Aut}_k\, k[x, y]$ by $J = J(2, k)$ and $G = A(2, k)$, we have only to show the following

(1.2) If $\sigma \in \mathrm{Aut}_k\, k[x, y]$, then there is an element τ in the group $J \vee G$ generated by J and G such that $\sigma x = \tau x$.

Because, if this statement is proved, then $\tau^{-1}\sigma$ is in J by Lemma 1.1 and we see that $\sigma \in J \vee G$.

Therefore we are going to prove (1.2) by induction on the degree d of σx. If $d = 1$, then the assertion is obvious and we assume that $d > 1$. We consider the projective plane $\mathbb{P}^2$ with generic point $(tx, ty, t) = (X, Y, Z)$ so that $Z \neq 0$ gives the affine plane Ω^2 with coordinate ring $k[x, y]$. Then we consider the linear system L defined by

$\{x + c \mid c \in k\}$, or more precisely, $L = \{C_a \mid a \in \Omega\} \cup \{d\ell_\infty\}$ where C_a is the plane curve defined by $h(X, Y, Z) + aZ^d$ with homogeneous form h of degree d such that $h(x, y, 1) = \sigma x$. Note that

(1.3) Coefficients of the defining equations for members of L can be chosen to be independent of the member, except for the coefficient of Z^d and also except for the member $d\cdot\ell_\infty$.

Note also that

(1.4) σ defines a birational correspondence $T : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by $(x, y, 1) \longrightarrow (\sigma x, \sigma y, 1)$ and T is biregular on $\Omega^2 = \mathbb{P}^2 - \ell_\infty$. L is then the effective transform of pencil of lines, on the first $\mathbb{P}^2$, going through $(0, 1, 0)$.

Obviously, $\Omega[x, y]/(\sigma x + a) \cong \Omega[t]$ with a variable t over Ω. Therefore (1.3) implies that $d\cdot\ell_\infty$ is the only reducible memebt of L and that $C_a \cap \Omega^2$ is biregular to an affine line. Therefore there is only one ordinary point P common to C_a and ℓ_∞ and P is a one-place point of C_a. Furthermore, if $a \neq a'$, then C_a and $C_{a'}$ do not meet each other in Ω^2. Therefore the point P is common to all C_a. Thus we have

(1.5) There is an ordinary point P on ℓ_∞ such that P is a one-place point of all C_a and $C_a\cdot\ell_\infty = dP$.

Consider one C_a with $a \in k$. Then since $k[x, y]/(\sigma x + a) \cong k[t]$,

we have

$$(1.6)\qquad \begin{cases} x \equiv c_{10} + c_{11}t + \ldots + c_{1r}t^r \\ y \equiv c_{20} + c_{21}t + \ldots + c_{2s}t^s \end{cases} \qquad (\mathrm{mod}\ \sigma x + a)$$

where $c_{ij} \in k$, $c_{1r}c_{2s} \neq 0$.

Since the degree d is the number of intersections of C_a and a line at a general position, we see that $d = \max\{r, s\}$. Then, using the transposition of x, y (which is in G) if necessary, we may assume that $s = d$. Then using a linear transformation which fixes y if $r = d$, we may assume that $r < d$. Then r is the number of intersections of $C_a \cap \Omega^2$ with the line $x =$ constant, C_a and the line $x =$ constant must meet at infinity with intersection multiplicity $d - r$. Since C_a has only one point P at infinity and since P is a one-place point such that $C_a \cdot \ell_\infty = dP$, we see that $d - r$ must be the multiplicity of P on C_a. Since L is a linear system which is the effective transform of a pencil of lines, singularities of irreducible members of L must be common to all irreducible members (Part 0, Remark 5.11). Therefore

(1.7) In (1.6), we may assume that $s = d > r$. These r and s are independent of the particular choice of a. Furthermore, $r = d -$ (multiplicity of P on C_a).

Now we consider applications of elements of J. Let $\iota \in J$. Then

the linear system L' to observe is the one defined by $\{\iota\sigma x + a \mid a \in k\}$. Therefore (1.7) implies that if there is an $\iota \in J$ such that the degree of y in t on the new C_a become smaller than d, it implies that the degree of $\iota\sigma x$ is less than d, hence we complete the proof by induction argument on d. Since the above reduction is possible if d is a multiple of r, we have that

(1.8) Assume in (1.6) that $r < s = d \neq 1$. Then in order to prove that $\mathrm{Aut}_k\, k[x, y] = J \vee G$, it is enough to show that d is a multiple of $r = d -$ (the multiplicity of P on C_a).

We are to give two proofs of (1.8); one is geometric (§ 1.2) and the other is rather due to computation of numericals (§ 1.3).

1.2. Geometric proof of (1.8).

We shall make use of some results on rational ruled surfaces. Though we shall refer them to [7] for the detail, we sketch some of basic facts.

By a ruled surface, we mean a projective non-singular surface V from which there is a regular map T onto a curve C such that inverse image of each point of C is a non-singular rational curve, or, equivalently, there is a pencil L of curves on V having no fixed point and such that every member of L is biregular to $\mathbb{P}^1$. Each member of L is called a fibre or a generator. An irreducible curve C on V is

called a section if the intersection number $(C, \ell) = 1$ for a fibre ℓ. A minimal section is a section whose self-intersection number is the smallest among sections. In the case of rational ruled surfaces, the product $\mathbb{P}^1 \times \mathbb{P}^1$ of projective lines is one special example, because it has two ruled surface structures. We take one of its, so that $\mathbb{P}^1 \times P$ $(P \in \mathbb{P}^1)$ are fibres and $Q \times \mathbb{P}^1$ $(Q \in \mathbb{P}^1)$ are minimal sections. Now, take one $Q \times \mathbb{P}^1$ and let $P_1, \ldots, P_n$ be mutually distinct ordinary points lying on the section $Q \times \mathbb{P}^1$. Then we can consider the elementary transformation $\mathrm{elm}_{P_1 \ldots P_n}$ on $\mathbb{P}^1 \times \mathbb{P}^1$, and we obtain a rational ruled surface.

Notation 2.1. A ruled surface which is biregular to $\mathrm{elm}_{P_1 \ldots P_n} \mathbb{P}^1 \times \mathbb{P}^1$ is denoted by F_n in general (including the case where $n = 0$, namely, $\mathbb{P}^1 \times \mathbb{P}^1$ is an F_0).

Of course, we must observe that the definition above does not depend on the particular choice of the points $P_1, \ldots, P_n$, namely two F_n are biregular to each other. Furthermore,

Proposition 2.2. (i) A rational ruled surface is an F_n for some n. Furthermore (ii) if $m \neq n$, then F_m cannot be isomorphic to F_n. Actually, if $m > 0$, then F_m has only one minimal section, whose self-intersection number is $-m$.

Proposition 2.3. Let P be an ordinary point on an F_n.

(i) If P lies on a minimal section, then $\mathrm{elm}_P F_n$ is an F_{n+1}.

(ii) Otherwise, $\mathrm{elm}_P F_n$ is an F_{n-1}.

As for the relationship with $\mathbb{P}^2$,

Proposition 2.4. If P is an ordinary point of $\mathbb{P}^2$, then $\mathrm{dil}_P \mathbb{P}^2$ is an F_1. Fibres are proper transforms of lines going through P and the minimal section is $\mathrm{dil}_P P$.

One preliminary result we need is

Proposition 2.5. Let V be an F_n, ℓ a fibre, b a section and let P be the common point of ℓ and b. Assume that a curve C on V goes through P and P is a one-place point of C. Furthermore, assume that $C \cdot \ell = \alpha P$, $C \cdot b = \beta P$ with natural numbers α and β. Let $W = \mathrm{elm}_P V$, let C', b' be proper transforms of C, b on W, let P' be the point on W corresponding to ℓ, ℓ' the fibre corresponding to P and let Q' be the infinitely near point to P of order one lying on b (hence is the common point of ℓ' and b'). Then

(1) if $\beta > \alpha$, then $C' \cdot b' = (\beta - \alpha)Q'$, $C' \cdot \ell' = \alpha Q'$,

(2) if $\beta < \alpha$, then $C' \cdot \ell' = \alpha P'$,

(3) if $\beta = \alpha$, then C' do not go through any of P', Q'.

In any case, the ordinary point where C' meets ℓ' is a one-place point of C'. $m = \min\{\alpha, \beta\}$ is the multiplicity of P on C.

Proof. Denoting by C'', ℓ'', b'' the proper transforms of

C, ℓ, b, respectively, on $\mathrm{dil}_P V$, we see that $(C'', \mathrm{dil}_P P) = m$ and therefore $(C'', b'') = (C, b) - m$, $(C'', \ell'') = (C, \ell) - m$. From this, we see (1), (2), (3) immediately. The last assertion is immediate from the property of a one-place point. q. e. d.

Before going back to (1.8), we give a remark on fundamental points of the birational correspondence $T : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ given by $(x, y, 1) \longrightarrow (\sigma x, \sigma y, 1)$. For distinction of these projective planes, we denote them by V and W so that $T : V \longrightarrow W$. Assume on the other hand that T is not biregular. Let L^* be the linear system of lines on W and let L^{**} be the effective transform $T_{ef}^{-1} L^*$. Since T is biregular on $\Omega^2 = V - \ell_\infty$, for each irreducible member C of L^{**}, its affine part $C \cap \Omega^2$ is biregular to an affine line. Therefore there is one and only one common ordinary point, say P, of C and ℓ_∞ and P is a one-place point of C. Note that fixed points of L^{**} amounts the same as fundamental points with respect to T. Since L^{**} has no fixed point on Ω^2, P must be the unique fixed point among ordinary points. In view of the fact that P is a one-place point, we have

Proposition 2.6. All of fundamental points with respect to T form a sequence, say, $P = P_0, P_1, \ldots, P_s$ such that each P_i $(i > 0)$ is an infinitely near point to P_{i-1} of order one.

Proof of (1.8). If $r = 1$, or if T defined above is a biregular

map, then $d = 1$ and therefore we assume that $r > 1$ and T is not biregular. Let L be as in §1.1, and we consider fixed points of L. By the same reason as above, they form a sequence $P'_0, P'_1, \ldots, P'_\gamma$ so that each P_i $(i > 0)$ is an infinitely near point of order one to P_{i-1}. Since an irreducible member C of L is rational and of degree $d > 1$, C must have at least one singular point. Hence, by virtue of Part 0, Remark 5.11 P'_0 (even if $\gamma = 0$) and also all P'_i for $i < \gamma$ must be fundamental with respect to T. Therefore Proposition 2.6 shows that $P'_0 = P$ and for $i < \gamma$, $P'_i = P_i$. We consider $\mathrm{dil}_P V$. This is an F_1 with unique minimal section $\mathrm{dil}_P P$ which we shall denote by b_1. Note that V is the unique $\mathbb{P}^2$ birationally dominated by this F_1.

Therefore we have

(2.7) $\quad s \geq 1$.

Starting with this F_1, we consider sequence of F_1, $\mathrm{elm}_{P_1} F_1$, $\ldots$, $\mathrm{elm}_{P_1 \ldots P_\alpha} F_1$ as far as $P_1, \ldots, P_\alpha$ lie on b_1 (for the definition of the term " lie on ", see § 0.5, p.15). Because of the condition, each $V_{\alpha+1} = \mathrm{elm}_{P_1 \ldots P_\alpha} F_1$ is an $F_{\alpha+1}$ (the original $F_1 = \mathrm{dil}_P V$ being V_1). V_β $(\beta \leq \alpha + 1)$ has a special fibre ℓ_β such that ℓ_1 came from ℓ_∞ and ℓ_β $(\beta > 1)$ came from $P_{\beta-1}$. Let b_β be the minimal section of V_β. Note that b_β is the proper

transform of b_1. Let C_β be the proper transform of C on V_β. Since P is a one-place point of C, we see that

(2.8) (i) P_β is the unique common point of b_β and ℓ_β.

(ii) ℓ_β and C_β have only one ordinary point, say P'_β, in common.

(iii) If b_β meets C_β, then $P'_\beta = P_\beta$.

Set $m = d - r$. m is the multiplicity of P on C by (1.7). Therefore $(C_1, b_1) = m$ by Part 0, Proposition 5.4, hence we have $(C_1, \ell_1) = d - m = r$. Let q and r' be non-negative integers such that $m = qr + r'$, $r' < r$. If m is a multiple of r, then d is a multiple of r. Therefore we assume that

(2.9) m is not a multiple of r, i.e., $r' \neq 0$.

We want to show that

(2.10) $\alpha \geq q$.

Assume that $q > \alpha$. $(C_1, \ell_1) = r < m = (C_1, b_1)$ implies that P_1 is an r-ple point of C_1. Then, on V_2, we have $(C_2, \ell_2) = r$, $(C_2, b_2) = m - r$. Thus, step by step, by virtue of our assumption that $q > \alpha$, we see that $(C_{\alpha+1}, b_{\alpha+1}) = m - \alpha r > r = (C_{\alpha+1}, \ell_{\alpha+1})$, and $P'_{\alpha+1}$ is an r-ple point of C and lies on b_1. By Part 0, Remark 5.11, $P'_{\alpha+1}$ is fundamental with respect to W and therefore $P'_{\alpha+1} = P_{\alpha+1}$. Thus (2.10) is proved.

By the same computation as above, we see that

(2.11) $(C_{q+1}, \ell_{q+1}) = r$, $(C_{q+1}, b_{q+1}) = m - qr = r' > 0$.

Let Q be the common point of ℓ_{q+1} and b_{q+1}. (2.11) shows that r' is the multiplicity of Q on C, and this is independent of the choice of the irreducible member C of L. Hence Q is a fixed point of L. Consider $dil_Q V_{q+1}$, and let b', ℓ', C' be the proper transforms of b_1 (or, b_{q+1}), ℓ_{q+1}, C (or, C_{q+1}) respectively. Since r' is the multiplicity of Q on C_{q+1}, we have $(dil_Q Q, C') = r'$, $(C', \ell') = r - r' > 0$. Thus the common point, say Q', of $dil_Q Q'$ and ℓ' is a point lying on C'. Thus Q' is also a fixed point of L. Hence we have by virtue of Part 0, Remark 5.11 that

(2.11) Q is fundamental with respect to W, hence $Q = P_{q+1}$;

Q' is either P_{q+2} or the point $(0, 1, 0)$ of W.

On $dil_Q V_{q+1}$, irreducible curves such that (i) self-intersection numbers are ne⌣ative and (ii) lies completely out side of Ω^2 are b', ℓ' and $dil_Q Q$ only. Therefore on $dil_{P_{q+1}\cdots P_s} V_{q+1}$, those irreducible curves having selfintersection number -1 and lying completely outside of Ω^2 are only some curves of the form $dil_{P_i} P_i$. therefore, in order to obtain W by successive contractions of irreducible exceptional curves of the first kind, we must contrant some of $dil_{P_i} P_i$, which contradicts the assumption that P_i is fundamental.

Thus r' cannot be positive, and we complete the proof of (1.8).

1.3. Remarks to the proof.

Let us look at the proof of the last section again. The first place we should change now is (2.9), where r' was assumed to be positive, but we know now that $r' = 0$. (2.10) is still good and (2.11) becomes

(3.11) $\quad (C_{q+1}, \ell_{q+1}) = r, \quad (C_{q+1}, b_{q+1}) = 0.$

Since L is the effective transform of a complete linear system with a base condition, L itself must be a complete linear system with a certain base condition. On the other hand, members of L are of degree $d = m + r$ (r is assumed to be at least 2) and P_0 is an m-ple point, $P_1, \ldots, P_{q+1}$ are r-ple points. Let L_d be the linear system of curves of degree d on V, then we see, by virtue of the fact that $m = qr$, that

$$\dim [\, L_d - (mP_0 + \Sigma_{i=1}^{q} rP_i)] \geqq \frac{1}{2}[d^2 + 3d - m(m+1) - qr(r+1)]$$

$$= \frac{1}{2}[m^2 + r^2 + 2mr + 3m + 3r - m^2 - m - mr - m]$$

$$= \frac{1}{2}[r^2 + mr + m + 3r] = \frac{1}{2}(dr + d + 2r) > d + r \geq d + 2.$$

But, $\dim L = 1$, hence L must have either at least one more singular fixed point or at least two more fixed point. Therefore, by virtue of

Remark 5.11 in Part 0, P_{q+1} is a fixed point of L. If P_{q+1} lies on ℓ_q, then P_{q+1} must be the common point of $dil_{P_q} P_q$ and the proper transform of ℓ_q on $dil_{P_q} V_a$. Then, by the same argument as at the end of the last section, we have a contradiction. Hence P_{q+1} is an ordinary point of ℓ_{q+1}. Since $(C_{q+1}, b_{q+1}) = 0$, we see that $P_{q+1} \notin b_{q+1}$. Thus P_{q+1} is an ordinary point of ℓ_{q+1} and is neither the common point of ℓ_{q+1} and b_{q+1} nor the point which is the proper transform of ℓ_q. Now we consider the sequence $W_0 = V_{q+1}$, $W_1 = elm_{P_{q+1}} W_0$, ..., $W_\beta = elm_{P_{q+\beta}} W_{\beta-1}$, as far as possible but with restriction that $\beta \leq q$. Let ℓ' be the fibre on W_i corresponding to P_{q+i} and let b_i' be the minimal section of W_i. Note that the common point of ℓ_i' and b_i' is the proper transform of ℓ_{i-1}'. By this fact and by the same argument as at the end of the last section, we see that if $q + i + 1 \leq s$, then P_{q+i+1} is an ordinary point lying on ℓ_i' which is not the common point of ℓ_i' and b_i'. Therefore we see that W_i is an F_{q-i}, hence $q + i + 1 \geq s$ unless $i = q$. Therefore $\beta = q$ and W_q is an F_1. W_q dominates a $\mathbb{P}^2$, say W^*. Let P^* be the transform on W^* of the minimal section of W_q.

By the nature of the transform, we see that

(1) Either $W^* = W$ or $s > 2q$ and $P_{2q+1}, \ldots, P_s$ are all of the fundamental points of W^* with respect to W.

(2) The transformation $j = \mathrm{cont}_{b_q} \cdot \mathrm{elm}_{P_1 \ldots P_{2q}} \cdot \mathrm{dil}_{P_0}$ is induces by some element τ of $\mathrm{Aut}_k\, k[x, y]$.

We shall show now

Proposition 3.2. Above j is a special type of Jonquières transformation and τ is an element of $J = J(2, k)$ under suitable choice of coordinates systems on V and W^*. Conversely, every element of J is obtained in this manner.

Proof. We begin with the last assertion. Let τ be an element of J. The linear system L^* corresponding to this birational transformation is defined by the module generated by $1, x, y + f(x)$ with $f(x) \in k[x]$. Therefore the module of defining homogeneous forms of members of L^* is generated by Z^q, XZ^{q-1} and $YZ^{q-1} + f^*(X, Z)$, where $q = \deg f(x)$ and f^* is the form of degree q such that $f^*(x, 1) = f(x)$. Then one sees easily that $P = (0, 1, 0)$ is a $(q-1)$-ple point of members of L^*, hence τ is a Jonquières transformation. By a similar argument as we made above in this section, we see that the birational transformation defined by τ is of the form as j above. This completes the proof of the last assertion. Let us prove the first assertion. We use the notation of §1.2. By virtue of the generation of $\mathrm{Aut}_k\, k[x, y]$ by $G = A(2,k)$ and J, we see that the fundamental points $P = P_0, P_1, \ldots, P_s$ are all k-rational. On the other hand, we see that j is a Jonquières

transformation such that the module of the corresponding linear system contains Z^q and XZ^{q-1}. Indeed, because of the special position of the points $P_1, \ldots, P_q$ such that they lie on $\mathrm{dil}_P P$, we see easily that $\ell + (q-1)\ell_\infty$ goes through $(q-1)P + \Sigma_{i=1}^{2q} P_i$ if ℓ is a curve going through P. Since $P, P_1, \ldots, P_{2q}$ are all k-rational, we see that j induces an element of $\mathrm{Aut}_k\, k[x, y]$. Since Z^q and XZ^{q-1} are in the module, we may assume that x is invariant under the automorphism induced by j. Then, by Lemma 1.1, we see that j is in J. Thus the proof is completed.

We shall prove one more result:

Theorem 3.3. $\mathrm{Aut}_k\, k[x, y]$ is so-called amalgamated product of $G = A(2, k)$ and $J = J(2, k)$. Namely, $\sigma_i \in G$, $\sigma_i \notin J$, $\tau_i \in J$, $\tau_i \notin G$ $(i = 1, 2, \ldots, n \geq 1)$ imply that $\tau_1\sigma_1 \cdots \tau_{n-1}\sigma_{n-1}\tau_n \notin G$. (cf. [6])

Proof. Note that the set of elements of G which fixes the point $(0, 1, 0)$ is exactly the subgroup $G \cap J$. Each element of J which is not in G defines a Jonquières transformation having $(0, 1, 0)$ as the unique ordinary fundamental point and also having $(0, 1, 0)$ of the new plane as the unique ordinary fundamental point with respect to the inverse transformation. Therefore, if we look at the multiplication of $\tau_i\sigma_i$ to $\tau_{i+1}\sigma_{i+1} \cdots \sigma_{n-1}\tau_n$, we see that no fundamental points of these factors are cancelled out by the multiplication because σ_i changes

$(0, 1, 0)$, hence the factor τ_i gives fundamental points which are infinitely near points to the last fundamental point with $\tau_{i+1}\sigma_{i+1}\cdots\tau_n$. Thus the number of fundamental points with respect to $\tau_1\sigma_1\cdots\sigma_{n-1}\tau^n$ is exactly the sum of the number of fundamental points with respect to τ_i $(i = 1, \ldots, n)$, hence the product cannot be in G. This completes the proof of Theorem 3.3.

1.4. Another proof.

We shall prove (1.8) in a stronger form. Namely

Theorem 4.1. Assume that C is an irreducible rational curve of degree d on $\mathbb{P}^2$ and that there is an ordinary point P such that

(1) P is a one-place point of C,

(2) $C \cdot \ell_\infty = dP$, and

(3) the affien part $C \cap \Omega^2$ is biregular to an affine line.

Let $P = P_1, P_2, \ldots, P_n$ be all the singularities of C (P_i being an infinitely near point if P_{i-1} for each $i = 2, \ldots, n$) with respective multiplicities $m = m_1, m_2, \ldots, m_n$. Then either d is a multiple of $d - m_1$ or $\Sigma m_i^2 > d^2 + 3$.

In particular, if the linear system L of curves of degree d going through $\Sigma m_i p_i$ is of positive dimension, then $d - m_1$ divides d.

Before proving the theorem, we introduce a notion which we shall call a (d, r)-sequence.

When two natural numbers d and r such that $d > r$ are given, the sequence $r_1, \ldots, r_q$ defined as follows is called the (d, r)-sequence :

Start with $d_0 = d$ and $d_1 = r$. When $d_0, \ldots, d_j$ are defined and if $d_j > 0$, let q_i and d_{j+1} be such that $d_{j-1} = q_j d_j + d_{j+1}$ $(0 \leq d_{j+1} < d_j)$. Then for every k such that $\left(\sum_{i<j} q_i\right) + 1 \leq k \leq \sum_{i<j} q_i$, r_k is defined to be d_j.

Lemma 4.2. Under the notation, we have

$$q = \Sigma_{i=1}^{\alpha} q_i, \quad d_\alpha = \text{G.C.M.}(d, r) \quad \text{and}$$

$$\Sigma r_i = d + r - d_\alpha, \quad \Sigma r_i^2 = dr.$$

Prood. We have

$$d_0 = q_1 d_1 + d_2 : \qquad d_0 d_1 = q_1 d_1^2 + d_1 d_2.$$

$$d_1 = q_2 d_2 + d_3 ; \qquad d_1 d_2 = q_2 d_2^2 + d_2 d_3,$$

$$\cdots\cdots \qquad\qquad \cdots\cdots$$

$$\cdots\cdots \qquad\qquad \cdots\cdots$$

$$d_{\alpha-2} = q_{\alpha-1} d_{\alpha-1} + d_\alpha ; \qquad d_{\alpha-2} d_{\alpha-1} = q_{\alpha-1} d_{\alpha-1}^2 + d_{\alpha-1} d_\alpha$$

$$d_{\alpha-1} = q_\alpha d_\alpha ; \qquad d_{\alpha-1} d_\alpha = q_\alpha d_\alpha^2 .$$

Summing up these equalities respectively, we have $d_0 + d_1 = \Sigma\, q_i d_i + d_\alpha$; $d_0 d_1 = \Sigma\, q_i d_i^2$ and we have the required result.

Proposition 4.3. Let C be an irreducible curve on a non-singular surface F and let P be a one-place point of C. Let m be the multiplicity of P on C. Let D be another irreducible curve on F which goes through P as a simple point. Let d be the intersection multiplicity of C with D at P, and let c be the G.C.M. (d, m). Let the (d, m)-sequence be $r_1, \ldots, r_q$. Then there is a sequence of points $P_1 = P, P_2, \ldots, P_q$ which is determined uniquely by d/c, r/c and D such that (i) each P_{i+1} is an infinitely near point of P_i of order one and (ii) effective multiplicity of P_i on C is r_i. (The way of determination of P_i is shown by the proof below.)

Proof. We use an induction argument on d. If $d = m$, then $q = 1$, $r_1 = m$ and the assertion is obvious. Assume that $d > m$. Consider the quadratic dilatation $\mathrm{dil}_P F$, the proper transforms C', D' of C, D and also the intersection number $(\mathrm{dil}_P P, C')$. Since P is an m-ple point of C, we have $(\mathrm{dil}_P P, C') = m$ by Proposition 5.4 in Part 0. Consider the unique common point P_2 of $\mathrm{dil}_P P$ and D'. By our assumption on P, P_2 is the unique common ordinary point of $\mathrm{dil}_P P$ and C'. On the other hand, since the intersection multiplicity at P of C and D is d and since P is m-ple on C, the intersection multiplicity

at P_2 of C' and D' is $d - m$. Therefore the multiplicity of P_2 on C' is the minimum of m and $d - m$. Now, if $d - m > m$, then considering C' and D' instead of C and D respectively, we have a case with less d, and the proof is completed by our induction argument. On the other hand, if $m > d - m$, then considering dil_P P and C' instead of D and C respectively, we complete the proof similary.

The following lemma is obvious.

Lemma 4.4. Let $m_1, \ldots, m_n$ be a sequence of natural numbers, such that $m_1 \geq m_2 \geq \cdots \geq m_n$ (≥ 1). Then for any $\alpha < n$

$$\sum m_i^2 < \sum_{i \neq \alpha, n} m_i^2 + (m_\alpha + 1)^2 + (m_n - 1)^2.$$

Corollary 4.5. Let $m_1, \ldots, m_n$ and $s_1, \ldots, s_\nu$ be positive integers, such that $\sum s_i \geq \sum m_j$, $m_1 \geq \cdots \geq m_n$ and $s_1 \geq \cdots \geq s_\nu$. If $s_i \geq m_i$ for $i = 1, \ldots, \nu - 1$, then

$$\sum s_i^2 \geq \sum m_j^2, \qquad \sum s_i(s_i - 1) \geq \sum m_j(m_j - 1).$$

Now we are going to prove Theorem 4.1.

Consider C, d, m_i, P_i and so on as in the theorem. Let (d, m_1)-sequence be $m_1 = r_1, r_2, \ldots, r_q$.

(1) Assume that G.C.M. $(d, m) = 1$. Then we see by virtue of

Proposition 4.3 that $m_i = r_i$ for any $i \leq n$ and $r_{n+1} = r_{n+1} = \cdots = r_q = 1$. This means that $2\cdot(\text{genus of } C) = d^2 - 3d + 2 - \Sigma\, r_i^2 + \Sigma\, r_i = d(d - m_1 - 2) + m_1 + 1$ by Lemma 4.2. Therefore, by the fact that C is rational, we have $d - m_1 - 2 < 0$, whence $m_1 \geq d - 1$, and we see that $m_1 = d - 1$, and therefore $1 = d - m_1$ divides d in this case.

(2) Assume now that $\delta = (d, m_1) \neq 1$ and that $d - m_1$ does not divide d. Then $n \geq q$ and $m_i = r_i$ for any $i \leq q$ and $m_j \leq \delta$ for any $j > q$. On the other hand,

$$0 = 2(\text{genus of } C) = d^2 - 3d + 2 - \Sigma\, m_i^2 + \Sigma\, m_i$$

$$= d^2 - 3d + 2 - \sum_{i\leq q} m_i^2 + \sum_{i\leq q} m_i - \sum_{j>q} m_j^2 + \sum_{j>q} m_j$$

$$= d(d - m_1) - 2d + m_1 + 2 - \delta - \sum_{j>q} m_j^2 + \sum_{j>q} m_j .$$

Let the $(d, d - m_1)$-sequence be $s_1, \ldots, s_q$. Then $\Sigma\, s_i^2 = d(d - m_1)$, $\Sigma\, s_i = d + (d - m_1) - \delta$. Therefore

$$(4.6) \qquad \sum_{j>q} m_j^2 - \sum_{j>q} m_j = \Sigma\, s_i^2 - \Sigma\, s_i + 2 - 2\delta .$$

Since $d - m_1$ does not divide d, $d - m_1$ is a proper multiple of δ ; $d - m_1 = u\delta$ $(u \geq 2)$. On the other hand, let β and γ be integers such that $\Sigma_{j>q}\, m_j = \beta\delta + \gamma$, $0 < \gamma < \delta$. Set $\delta_1 = \cdots = \delta_\beta = \delta$, $\delta_{\beta+1} = \gamma$. Then $\Sigma\, \delta_i = \Sigma_{j>q}\, m_j$ and, by Corollary 4.5, and by the fact

that $m_i \leq \delta$

$$\sum \delta_i^2 - \sum \delta_i \geq \sum_{j>q} m_j^2 - \sum_{j>q} m_j .$$

Assume for a moment that $\sum_{j>q} m_j \leq \sum s_i + 2\delta$. Then, since $s_i \geq \delta$, we have similarly

$$\sum_{i\geq 2} s_i^2 - \sum_{i\geq 2} s_i \geq \sum_{i>u+2} \delta_i^2 - \sum_{i>u+2} \delta_i .$$

Therefore

$$\begin{aligned}
\sum s_i^2 - \sum s_i &\geq s_1^2 - s_1 + \sum_{i>u+2} \delta_i^2 - \sum_{i>u+2} \delta_i \\
&= u^2\delta^2 - u\delta + \sum_{i>u+2} \delta_i^2 - \sum_{i>u+2} \delta_i \\
&= (u^2 - u - 2)\delta^2 + 2\delta + \sum \delta_i^2 - \sum \delta_i \\
&\geq 2\delta + \sum_{j>q} m_j^2 - \sum_{j>q} m_j \\
&= 2\delta + \sum s_i^2 - \sum s_i + 2 - 2\delta \qquad \text{(by (4.6).)}
\end{aligned}$$

This implies $2 \leq 0$, which is impossible. Therefore we must have

$$\sum_{j>q} m_j > \sum s_i + 2\delta .$$

Then, since $\sum_{i\leq q} m_i + \sum s_i = d + m_1 - \delta + d + (d - m_1) - \delta = 3d - 2\delta$ (by Lemma 4.2), we have

$$\Sigma\, m_i > 3d \ .$$

Since $0 = d^2 - 3d + 2 - \Sigma\, m_i^2 + \Sigma\, m_i$, we have $\Sigma\, m_i^2 = d^2 - 3d + 2 + \Sigma\, m_i$ $> d^2 + 2$. This proves the main part of Theorem 4.1. As for L, this implies that two members of L have intersection number bigger than $d^2 + 2$ which cannot happen unless they have common components. Since L has an irreducible member C, we see that $\dim L = 0$.

By these (1) and (2), we completes the proof of the Theorem 4.1.

1.5. One question on rational curves.

As we stated in Part 0 as Conjecture 1.1, it has been questioned that when C is a rational curve in $\mathbb{P}^2$ of degree d such that (i) there is only one ordinary point P which is common to C and ℓ_∞, (ii) $C - \{P\}$ (i.e., $C \cap \Omega^2$) is biregular to an affine line, then whether or not d is a multiple of d -(the multiplicity of P on C).

In the characteristic $p \neq 0$ case, one have a counter-example as we shall see later in this section.

On the other hand, Abhyankar and Moh [1] proved affirmatively the question in the classical case (i.e., in the case of ground field of characteristic zero), and also the following : If d is not a multiple of the characteristic of the ground field, then d is a multiple of d - (the multiplicity of P on C).

Note that if this conjecture is affirmative, then one can see easily by induction on d that such a C (with d not multiple of the characteristic of the ground field k) is defined by an $f(x, y) \in k[x, y]$, for which there is an element $\sigma \in \mathrm{Aut}_k\, k[x, y]$ so that $\sigma x = f(x, y)$.

Now we shall give a family of proposed counter-examples. Assume that the ground field k is of characteristic $p \neq 0$. For each $b \in k$, let C_b be the curve on Ω^2 defined by the following representation by a parameter t :

$$\text{(5.1)} \qquad \begin{cases} x = t^{p^2} \\ y = t^{\alpha p} + t + b \end{cases}$$

where α is a fixed natural number, > 1 and prime to p.

One sees quite easily that

$$k[t^{p^2},\ t^{\alpha p} + t + b] = k[t].$$

Therefore C_b is biregular to an affine line. Let C_b' be the completion of C_b in $\mathbb{P}^2$. Then $\{C_b' \mid b \in k\}$ spans a pencil L of curves of degree $d = \max\{p^2, \alpha p\}$. Every C_b' satisfies the conditions (i) and (ii) above with $P = (1, 0, 0)$ or $(0, 1, 0)$ according to $p > \alpha$ or $p < \alpha$. But the multiplicity of P on C_b' is the difference of p^2 and αp

hence every C'_b is a counter-example which we required. We give one more remark :

Proposition 5.2. The linear system L has variable singularities among infinitely near points.

Proof is immediate from Theorem 4.1.

Part 2 Questions in the general case.

2.1 An example.

In Part 1, we observed the case of a polynomial ring of two variables over a field. As we discussed in Part 0, these results can be generalized easily to the case of a polynomial ring of two variables over an Artin ring.

But, if we deal with a more general case, we shall meet with many difficulties. In this Part 2, we shall discuss some of such difficulties.

Let us begin with an example.

Let K be a field and consider the polynomial ring $R = K[x, y, z]$ of three variables. Set $k = K[z]$. Then R is a polynomial ring of two variables x, y over k. We are going to define an element σ of $\mathrm{Aut}_k R$ by :

$$(1.1) \qquad \begin{cases} \sigma x = x - 2y(zx + y^2) - z(zx + y^2)^2 \\ \sigma y = y + z(zx + y^2). \end{cases}$$

It is obvious that (1.1) defines a k-homomorphism $\sigma : R \longrightarrow R$. It is also obvious that σ is an injection. As for this σ, we have

$$(1.2) \qquad \sigma(zx + y^2) = zx + y^2 .$$

Indeed, $\sigma(zx + y^2) = z(x - 2y(zx + y^2) - z(zx + y^2)^2) + (y + z(zx + y^2))^2$

$$= zx + y^2 .$$

Let τ be a k-injection $R \longrightarrow R$ defined by

$$(1.3) \qquad \begin{cases} \tau x = x + 2y(zx + y^2) - z(zx + y^2)^2 \\ \tau y = y - z(zx + y^2) . \end{cases}$$

Then, by virtue of (1.2), we have

$$\sigma\tau x = \sigma(x + 2y(zx + y^2) - z(zx + y^2)^2)$$

$$= x - 2y(zx+y^2) - z(zx+y^2)^2 + 2(y + z(zx+y^2))(zx+y^2) - z(zx+y^2)^2$$

$$= x ,$$

$$\sigma\tau y = \sigma(y - z(zx + y^2)) = y + z(zx + y^2) - z(zx + y^2)$$

$$= y .$$

Thus $\sigma\tau = 1$. Since σ and τ are injective, we see that σ is an automorphism and that $\tau = \sigma^{-1}$. Now we assert :

Theorem 1.4. The automorphism σ defined above is not in the group $J(2, k) \vee A(2, k)$.

Proof. Let k^* be the field of quotients of k. Then σ, as an element of $\mathrm{Aut}_{k^*} k^*[x, y]$, is equal to $\sigma_1\sigma_2\sigma_1^{-1}$, where

$$\sigma_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + z^{-1}y^2 \\ y \end{pmatrix}, \qquad \sigma_1^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - z^{-1}y^2 \\ y \end{pmatrix},$$

$$\sigma_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + z^2x \end{pmatrix}.$$

Note that $\sigma_1 \in J(2, k^*)$ and that $\pi\sigma_2\pi \in J(2, k)$ with $\pi \in SL(2, k)$ such that $\pi x = y$, $\pi y = x$. Assume that $\sigma \in J(2, k) \vee A(2, k)$, i.e., there are $\tau_i \in J(2, k)$, $\lambda_i \in A(2, k)$ such that

$$\begin{cases} \sigma = \lambda_1\tau_1\lambda_2 \cdots \tau_2\lambda_{n+1} \\ \tau_i \notin A(2, k) \quad \text{for every} \quad i, \\ \lambda_i \notin J(2, k) \quad \text{unless} \quad \lambda_i = 1, \ i = 1 \quad \text{or} \quad n + 1. \end{cases}$$

Then we have

$$\lambda_1\tau_1\lambda_2 \cdots \tau_n\lambda_{n+1}\sigma_1\pi\sigma_2\pi\sigma_1^{-1} = 1.$$

By the uniqueness of the factorization proved at Part 1, Theorem 3.3, we see that $\lambda_{n+1} = 1$ and $\tau_n\sigma_1 \in A(2, k^*)$. We write

$$\begin{cases} \tau_n x = a_1 x + f(y) \\ \tau_n y = a_2 y \ . \end{cases}$$

Then $\tau_n\sigma_1 x = a_1 x + f(y) + z^{-1}a_2^2 y^2$. Since $\tau_n \in J(2, k)$, a_1 and a_2 are units in k and $f(y) \in k[y]$. Therefore $f(y)$ has no term to cancell out the term $z^{-1}a_2^2 y^2$. Hence $\tau_n\sigma_1 x$ cannot be of degree ≤ 1 in y. Thus $\tau_n\sigma_1 \notin A(2, k^*)$. q. e. d.

k in the above example is a polynomial ring in one variable over a field, hence is an Euclid ring. Thus our example shows that even if the coefficient ring k is so good as an Euclid ring and $n = 2$, the group $\mathrm{Aut}_k\, k[x, y]$ is not generated by $J(2, k)$ and $A(2, k)$ (cf. Exercise 1.6 below). Therefore we ask :

Question 1.5. Find a good structure theorem for $\mathrm{Aut}_k\, k[x, y]$ in the case where $k = K[z]$, K a field and x, y, z algebraically independent over K, or more generally in the case where k is reasonably good integral domain and x, y algebraically in dependent over k.

Excercise 1.6. Let k be an integral domain with a non-unit z. Take $\sigma \in \mathrm{Aut}_k\, k[x, y]$ defined by the same equality as (1.1). Prove that σ is not in $J(2, k) \vee A(2, k)$.

2.2 Another example.

In § 2.1, we gave an example of $\sigma \in \mathrm{Aut}_k\, k[x, y]$ which is not in $J(2, k) \vee A(2, k)$. Letting k^* be the field of quotients of k, the example σ as an element of $\mathrm{Aut}_{k^*}\, k^*[x, y]$ is the product of three automorphisms each of which induces a Jonquieres transformation of $\mathbb{P}^2$.

If we condsider worse k, then we have much simpler example, and we have the following

Theorem 2.1. Assume that k is an integral domain which is not a principal ideal ring. Then there is an element σ of $\mathrm{Aut}_k\, k[x, y]$ (x, y being algebraically independent elements over k) such that

(i) σ is not in $J(2, k) \vee A(2, k)$, and

(ii) σ induces a *Jonquières transformation* of $\mathbb{P}^2$ over the field of quotients k^* of k.

Let us construct a required example.

Let a, b be elements of k such that $ak + bk$ is not a principal ideal ; such pair exists because of our assumption on k. Let $f(X)$ be a polynomial in one variable X with coefficients in k. Let σ be the k-homomorphism $k[x, y] \longrightarrow k[x, y]$ defined by

$$(2.2) \qquad \begin{cases} \sigma x = x - bf(ax + by) \\ \sigma y = y + af(ax + by) \end{cases}$$

We are going to prove that this σ is the example.

Consider k-homomorphism τ defined by

$$(2.3) \qquad \begin{cases} \tau x = x + bf(ax + by) \\ \tau y = y - af(ax + by) \ . \end{cases}$$

Then we have

(2.4) $$\tau(ax + by) = ax + by.$$

Indeed, $\tau(ax + by) = a(x + bf(ax + by)) + b(y - af(ax + by))$

$= ax + by.$

Therefore we see that

$$\tau\sigma x = \tau(x - bf(ax + by)) = x + bf(ax + by) - bf(ax + by) = x,$$

and similarly, we have $\tau\sigma y = y$. Thus we see that σ is an element of $\mathrm{Aut}_k\, k[x, y]$ and that $\tau = \sigma^{-1}$.

Since $\sigma(ax + by) = ax + by$, we see by virtue of Lemma 1.1 in Part I that σ induces a Jonquières transformation of $\mathbb{P}^2$ over k^*.

Now let us prove that σ is not in $J(2, k) \vee A(2, k)$. Since σ induces a Jonquières transformation over k^*, we see by virtue of Theorem 3.3 that if σ is in $J(2, k) \vee A(2, k)$ then $\sigma = \lambda'\sigma'\lambda$ with $\sigma' \in J(2, k)$, $\lambda, \lambda' \in A(2, k)$. (2.4) shows that a polynomial $g(x, y)$ of degree ≤ 1 is mapped to one of degree ≤ 1 by σ if and only if g is of the form $d(ax + by) + c$ $(c \in k\ ;\ d \in k^*)$. $\lambda^{-1}y$ is such a polynomial. This means that by the linear automorphism λ^{-1}, y is mapped to $d(ax + by) + c$ $(da, db, c \in k)$. Since $ak + bk$ is not a principal ideal, we see that $dak + dbk$ is not equal to the unit ideal k.

Hence, modulo a maximal ideal m of k containing $dak + dbk$, we have that $\lambda^{-1}y$ is congruent to an element of k, and λ^{-1} is not an element of $\mathrm{Aut}_k\, k[x, y]$. Thus we obtained a contradiction, and therefore $\sigma \notin J(2, k) \vee A(2, k)$. q. e. d.

2.3 The case of three or more variables.

Consider the case where $n = 3$ and k is a field. I am quite certain that $\mathrm{Aut}_k\, k[x] \neq J(3, k) \vee A(3, k)$ $(= J(3, k) \vee GL(3, k))$. Actually, look at the automorphism σ defined in § 2.1. This is an element of $\mathrm{Aut}_K\, K[x, y, z]$, K being a field.

Conjecture 3.1. This σ is not in $J(3, K) \vee A(3, K)$.

I do not have any rigorous proof of this conjecture yet. But I have some reasoning for this, and I am going to explain it later. (Theorem 1.4 in § 2.1 is a part of it.)

One important fact used in Part 1 is that, in the case of non-singular surfaces, birational transformations are very well described by fundamental points with help of the notion of infinitely near points. But, in the case of three dimensional non-singular varieties, we do not have satisfactory theory of infinitely near points ; relationship between monoidal dilatations whose centers have common points is complicated and therefore description of a birational transformation making use of monoidal transformations with nice centers is difficult.

Let us look at elements of $J(3, k) \vee A(3, k)$, or more generally, those of $J(n, k) \vee A(n, k)$ with $n \geq 3$: let us assume that k is a field. In order to look at fundamental points of induced birational correspondences, it may be better to look at a smaller group $J^*(n, k)$ defined by

$$(3.2) \qquad J^*(n, k) = \{\sigma \in J(n, k) \mid \sigma x_i = x_i \text{ for every } i \geq 2\}.$$

It is easy to see

Lemma 3.3. $J(n, k) \vee A(n, k) = J^*(n, k) \vee A(n, k) = J^*(n, k) \vee GL(n, k)$.

Now, we take coordinates of $\mathbb{P}^n$ so that $\tau \in \mathrm{Aut}_k\, k[x]$ induces birational transformation given by $(x_1, \ldots, x_n, 1) \longrightarrow (x_1, \ldots, \tau x_n, 1)$. Let $\tau \in J^*(n, k)$, $\tau \notin A(n, k)$. Then

$$(3.4) \qquad \tau x_1 = ax_1 + f(x_2, \ldots, x_n), \quad \tau x_i = x_i \text{ for } i \geq 2.$$

Therefore

$$(3.5) \qquad \tau^{-1} x_1 = a^{-1}(x_1 - f(x_2, \ldots, x_n)), \quad \tau^{-1} x_i = x_i \text{ for } i \geq 2.$$

Thus we have

Lemma 3.6. Under the birational correspondence induced by τ, (i) the hyperplane at infinity of the first $\mathbb{P}^n$ corresponds to the point $(1, 0, \ldots, 0)$ of the latter $\mathbb{P}^n$ (as the proper transform),

(ii) the point $(1, 0, \ldots, 0)$ of the first $\mathbb{P}^n$ corresponds to the hyperplane at infinity of the latter $\mathbb{P}^n$.

Now we ask

Question 3.7. Assume that $\tau_1, \ldots, \tau_m \in J^*(n, k)$, $\lambda_1, \ldots, \lambda_{m-1} \in A(n, k)$, $\lambda_i \notin A(n, k)$ for any i and that every λ_j do not fix the point $(1, 0, \ldots, 0)$. Let T_i be the graph of the birational correspondence induced by $\tau_i \lambda_i \tau_{n+1} \cdots \lambda_{m-1} \tau_m$. Does T_{i-1} dominates T_i for every $i = 2, \ldots, m$?

If this question, or something similar, has an affirmative answer, we would be able to prove not only an adaption of Theorem 3.3 in Part 1 for the group $J(n, k) \vee A(n, k)$ (cf. Lemma 3.3), but also Conjecture 3.1.

Another base is the following conjecture :*)

Conjecture 3.8. Assuem that $\nu \in J(3, k) \vee A(3, k)$, $\nu \notin A(3, k)$. Let the highest degree parts of νx_1, νx_2, νx_3 are g_1, g_2, g_3 respectively. Then, either there is one of the g_i which is a polynomial of the other two, or, there are two of these g_i which are powers of the same form.

Note that our example σ in § 2.1 as an element of $\mathrm{Aut}_K K[x, y, z]$ does not satisfy the condition in the conclusion of the conjecture. Therefore, if this Conjecture 3.8 has an affirmative answer, then so

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*) This was proposed by Professor S. Abhyankar.

does Conjecture 3.1.

One difficulty in proving Conjecture 3.8 lies on the fact that there is an injection $\eta : k[x] \longrightarrow k[x]$ $(n = 3)$ such that the highest degree parts of ηx_i $(i = 1, 2, 3)$ satisfy the condition in the conjecture, and on the other hand, for a suitable choice of $\tau \in J(3, k)$, the highest degree parts of $\tau\eta x_i$ $(i = 1, 2, 3)$ do not satisfy the condition.

By the way, we prove :

Proposition 3.9. Assume that k is a field, $n = 3$, $\sigma \in \mathrm{Aut}_k\, k[x]$, and that $\deg \sigma x_i \leq 2$, $\deg \sigma^{-1} x_i \leq 2$ for every $i = 1, 2, 3$. Then $\sigma = \lambda\tau\lambda'$ with $\lambda, \lambda' \in A(3, k)$, $\tau \in J(3, k)$.

Proof. Considering elements of the form $\lambda\sigma$ with $\lambda \in A(3, k)$, we may assume that $\sigma x_i = x_i + f_i$ for each i, where f_i is a homogeneous form of degree 2. Then $\sigma^{-1} x_i = x_i + g_i$ with a homogeneous form g_i of degree 2, by virtue of our assumption.

$$x_i = \sigma^{-1}\sigma x_i = \sigma^{-1}(x_i + f_i) = x_i + g_i + f_i(x_1 + g_1, x_2 + g_2, x_3 + g_3).$$

Compairing degree 2 parts, we have $g_i = -f_i(x_1, x_2, x_3)$;

From degree 4 parts, we have $0 = f_i(g_1, g_2, g_3)$.

If the locus of (g_1, g_2, g_3) in the projective plane $\mathbb{P}^2$ is dense in $\mathbb{P}^2$, then $f_i = 0$ by the second equality, hence $g_i = 0$ by the first

equality, which is a contradiction. If the locus of (g_1, g_2, g_3) is a curve C, then all of f_1, f_2, f_3 must be a multiple of the defining polynomial F for the curve C. Hence, either C is a conic and $f_i = c_iF$ with $c_i \in k$, or C is a line and $f_i = h_iF$ with linear forms h_i. The former case gives us a contradiction because of the equality $g_i = -f_i$. Thus the locus C is a line, which means that g_1, g_2, g_3 are linearly dependent. Therefore, using a linear transformation, we may assume that $g_3 = 0$. In this case g_1, g_2 are linearly independent and $F = X_3$. Then, considering the degree 3 parts, we have $0 = x_3h_i(g_1, g_2, 0)$ (for $i = 1, 2$). Since the locus of $(g_1, g_2, 0)$ is the line C defined by $x_3 = 0$, we see that h_i is a multiple of x_3. Thus $f_1 = c_1x_3^2$, $f_2 = c_2x_3^2$, which contradicts our assumption that g_1, g_2 are linearly independent. Thus the locus of (g_1, g_2, g_3) must be a point, which means that $g_1 = c_1G$ $(c_i \in k)$ with a fixed quadratic form G. Then using a linear transformation, we may assume that $g_2 = g_3 = 0$. Then the remainder of the proof is obtained by the following

Lemma 3.10. Assume that $\sigma \in \mathrm{Aut}_k\, k[x]$. If $\sigma x_i = x_i$ for every $i \leq 2$, then $\sigma \in J(n, k)$.

Proof is similar to the one for Lemma 1.1 in Part 1.

2.4 One more question.

We like ask here a question. Consider the case where k is a field

and $n \geq 1$. If $n = 1$ or 2, then we see the following proposition by our observation made in Part 0, Part 1 :

Proposition 4.1. If K is a field and if $n \leq 2$, then $\mathrm{Aut}_k\, k[x]$ is generated by $A(n, k)$ and $\mathrm{Aut}_{k[x_1]}\, k[x]$.

Therefore we ask :

Question 4.2. Assume that k is a field and $n \geq 3$. Is $\mathrm{Aut}_k\, k[x]$ generated by $A(n, k)$ and $\mathrm{Aut}_{k[x_1]}\, k[x]$?

We note here that our assumption that k is a field is important. Indeed, our example in § 2.1 shows that the question has negative answer if k is an integral domain which is not a field (in the case $n = 2$).

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