

ON DERIVATIVES OF SET-VALUED MAPS IN SET OPTIMIZATION
集合最適化における集合値写像の導関数について

島根大学総合理工学部 黒岩 大史 (DAISHI KUROIWA)

Interdisciplinary Faculty of Science and Engineering, Shimane University

1 Introduction

Let E be a locally convex topological vector space over $\mathbb{R}$, let K be a closed convex cone in E , and assume K is pointed, that is, $K \cap (-K) = \{\theta\}$ where θ is the null vector of E . We define an order relation $\leq$ on E by

$$x, y \in E, \quad x \leq y \stackrel{\text{def}}{\iff} y - x \in K.$$

For a given set-valued map F from a set X to E , we consider the following set-valued optimization problem:

$$(P) \quad \text{Minimize } F(x) \quad \text{subject to } x \in X.$$

When we consider the set-valued problem (P), there are two types criteria. One is the criterion in vector optimization: the criterion of solutions is based on comparisons of all elements of all values of F , and the solution of vector optimization (VP) is an element $x_0 \in X$ satisfying the condition: there exists $y_0 \in F(x_0)$ such that

$$x \in X, y \in F(x), y \leq y_0 \implies y_0 \leq y, \quad \text{or} \quad (y_0 - K) \cap \bigcup_{x \in X} F(x) = \{y_0\}.$$

The other criterion is called ‘set optimization,’ which introduced by the author, see [3, 6]. A solution of set optimization is an element $x_0 \in X$ satisfying the following condition:

$$x \in X, F(x) \preceq F(x_0) \implies F(x_0) \preceq F(x),$$

where $\preceq$ is a certain binary relation on 2^E . This criterion is based on comparisons of sets, values of F , with respect to $\preceq$. For example, we can consider the following six natural set relations: for given $A, B \subset E$, $A \preceq B$ means,

- (1) $\forall x \in A, \forall y \in B, x \leq y$;
- (2) $\exists x \in A$ such that $\forall y \in B, x \leq y$;
- (3) $\forall y \in B, \exists x \in A$ such that $x \leq y$;
- (4) $\exists y \in B$ such that $\forall x \in A, x \leq y$;
- (5) $\forall x \in A, \exists y \in B$ such that $x \leq y$;
- (6) $\exists x \in A, \exists y \in B$ such that $x \leq y$.

In this paper, set relations $\leq_K^l$ on 2^E are defined as follows: for $A, B \in 2^E$,

$$A \leq_K^l B \stackrel{\text{def}}{\iff} \text{cl}(A + K) \supset B \quad \text{cf. (3)}$$

and we observe the following notions of solutions:

Definition 1. An element $x_0 \in X$ is said to be a minimal solution of (SP) if

$$x \in X, F(x) \leq_K^l F(x_0) \implies F(x_0) \leq_K^l F(x).$$

In this paper, we consider first order optimality conditions of the set-valued optimization problem (SP). To the purpose, we introduce an embedding space into which our minimization problem (SP) is embedded in Section 2, and we define a notion of directional derivative for set-valued maps in Section 3. In Section 4, we give some results about necessary and sufficient optimality conditions by the derivatives.

2 An embedding space

A subset A of E is said to be K -convex if $A + K$ is convex, and A is said to be K^+ -bounded if $\langle y^*, A \rangle$ is bounded from below for any $y^* \in K^+$, where K^+ be the positive polar cone of K , that is

$$K^+ = \{y^* \in E^* \mid \langle y^*, k \rangle \geq 0, \forall k \in K\}.$$

Let $\mathcal{G}$ be the family of all nonempty K -convex and K^+ -bounded subset of E . In this section, we introduce a process of construction of a normed space $\mathcal{V}$ into which $\mathcal{G}$ is embedded. All results in this section, see [5].

At first, we introduce an equivalence relation $\equiv$ on $\mathcal{G}^2$: for each $(A, B), (C, D) \in \mathcal{G}^2$,

$$(A, B) \equiv (C, D) \stackrel{\text{def}}{\iff} \text{cl}(A + D + K) = \text{cl}(B + C + K).$$

We denote the quotient space $\mathcal{G}^2/\equiv$ by $\mathcal{V}$, that is

$$\mathcal{V} = \{[A, B] \mid (A, B) \in \mathcal{G}^2\},$$

where $[A, B] = \{(C, D) \in \mathcal{G}^2 \mid (A, B) \equiv (C, D)\}$. Define addition and scalar multiplication on the quotient space $\mathcal{V}$ as follows:

$$\begin{aligned} [A, B] + [C, D] &= [A + C, B + D], \\ \lambda \cdot [A, B] &= \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0 \\ [(-\lambda)B, (-\lambda)A] & \text{if } \lambda < 0. \end{cases} \end{aligned}$$

Then $(\mathcal{V}, +, \cdot)$ is a vector space over $\mathbb{R}$. Also let

$$\mu(K) = \left\{ [A, B] \in \mathcal{V} \mid B \leq_K^l A \right\},$$

then $\mu(K)$ is a pointed convex cone in $\mathcal{V}$. Now we define order relation $\preceq_{\mu(K)}$, or simply $\preceq$, on $\mathcal{V}$ as follows:

$$[A, B] \preceq_{\mu(K)} [C, D] \stackrel{\text{def}}{\iff} [C, D] - [A, B] \in \mu(K).$$

Then, $(\mathcal{V}, +, \cdot, \preceq_{\mu(K)})$ is an ordered vector space over $\mathbb{R}$. Let a function φ from $\mathcal{G}$ to $\mathcal{V}$ by

$$\varphi(A) = [A, \{\theta\}] \text{ for all } A \in \mathcal{G},$$

then

$$A \leq_K^l B \iff \varphi(A) \leq_{\mu(K)} \varphi(B),$$

for any $A, B \in \mathcal{G}$. By using this function φ , our set optimization problem (SP) is regarded as a vector optimization problem, that is, when F is a map from X to $\mathcal{G}$, $x_0 \in E$ is a minimal solution of (SP) if and only if

$$\varphi \circ F(X) \cap (\varphi \circ F(x_0) - \mu(K)) = \{\varphi \circ F(x_0)\}.$$

Finally we introduce a norm $|\cdot|$ in $\mathcal{G}^2/\equiv$. Let W be a base of K^+ , that is $\mathbb{R}_+ W = K$, and $\theta^* \notin W$. For each $[A, B] \in \mathcal{V}$,

$$|[A, B]| = \sup_{y^* \in W} |\inf \langle y^*, A \rangle - \inf \langle y^*, B \rangle|,$$

is well-defined, and let $\mathcal{V}(W) = \{[A, B] \in \mathcal{V} \mid |[A, B]| < \infty\}$, then we can see $(\mathcal{V}(W), |\cdot|)$ is a normed vector space, and $\mu(K)$ is closed in $(\mathcal{V}(W), |\cdot|)$.

3 A directional derivative of set-valued maps in set optimization

In the rest of the paper, assume that X is a convex set of a normed space $(Z, \|\cdot\|)$ over $\mathbb{R}$, W is a closed base of K^+ , $\mathcal{V} = \mathcal{V}(W)$, that is $[A, B] \in \mathcal{V}$ when $[A, B] \in \mathcal{V}$, and $F : X \rightarrow \mathcal{G}$. About all results of the rest of the paper, see [7].

Definition 2. Let $x \in X$ and $d \in Z$.

$$CF(x, d) = \left\{ [A, B] \in \mathcal{V} \mid \exists \{\lambda_k\} \downarrow 0 \text{ s.t. } \frac{1}{\lambda_k} [F(x + \lambda_k d), F(x)] \rightarrow [A, B] \right\}$$

is said to be $\mathcal{V}$ -directional derivative clusters of F at x in the direction d . If $CF(x, d)$ is a singleton, then the element is written by $DF(x, d)$ and called $\mathcal{V}$ -directional derivative of F at x in the direction d , and F is said to be $\mathcal{V}$ -directional differentiable at x in the direction d .

Example 1. Let $F : \mathbb{R} \rightarrow 2^{\mathbb{R}^2}$ be a set-valued map defined by

$$F(x) = \text{co} \{(|x|, -|x| + 1), (-|x| + 1, |x|)\}, \quad \forall x \in \mathbb{R},$$

and let $K = \mathbb{R}_+^2 = \{(x_1, x_2) \mid x_1, x_2 \geq 0\}$. Then F is $\mathcal{V}$ -directional differentiable in the any direction $d \in \mathbb{R}$, for example, when $x_0 = 0$,

$$DF(x_0, d) = [\{(0, 0)\}, |d| \text{co}\{(1, -1), (-1, 1)\}], \quad \forall d \in \mathbb{R},$$

when $0 < x_0 < \frac{1}{2}$,

$$DF(x_0, d) = \begin{cases} [\{(0, 0)\}, |d| \text{co}\{(1, -1), (-1, 1)\}], & \text{if } d \geq 0, \\ [|d| \text{co}\{(1, -1), (-1, 1)\}, \{(0, 0)\}], & \text{if } d < 0, \end{cases}$$

and when $x_0 = \frac{1}{2}$,

$$DF(x_0, d) = [|d| \text{co}\{(1, -1), (-1, 1)\}, \{(0, 0)\}], \quad \forall d \in \mathbb{R}.$$

Example 2. A set-valued map $F : X \rightarrow 2^E$ defined by

$$F(x) = g(x) + \sum_{i \in I} r_i(x) A_i, \quad x \in X,$$

where g is a function from X to E which is directional differentiable at $x_0 \in X$, I is a nonempty finite set, and for each $i \in I$, r_i is a function from X to $(0, \infty)$ which is directional differentiable at $x_0 \in X$, and $A_i \in \mathcal{G}$. Then F is $\mathcal{V}$ -directional differentiable at x_0 for each direction $d \in Z$, and we have,

$$\begin{aligned} DF(x_0, d) &= [g'(x_0, d), \{\theta\}] + \sum_{i \in I} r'_i(x_0, d)[A_i, \{\theta\}] \\ &= \left[g'(x_0, d) + \sum_{i \in I_+(d)} r'_i(x_0, d)A_i, - \sum_{i \in I_-(d)} r'_i(x_0, d)A_i \right] \end{aligned}$$

where $I_+(d) = \{i \in I \mid r'_i(x_0, d) > 0\}$ and $I_-(d) = \{i \in I \mid r'_i(x_0, d) < 0\}$.

4 Optimality conditions of a set optimization problem

In this section we observe necessary and sufficient optimality conditions of solutions of (SP). At first, we define weak minimal solutions of (SP). To the purpose, we introduce a binary relation $<^l_K$ on $\mathcal{G}$: for $A, B \in \mathcal{G}$,

$$A <^l_K B \stackrel{\text{def}}{\iff} \exists V \subset E : \text{a neighborhood of } \theta \text{ such that } A + K \supset B + V.$$

Proposition 1. For any $A, B \in \mathcal{G}$, $[A, B] \in \text{Int}\mu(K)$ implies $B <^l_K A$, where $\text{Int}\mu(K)$ is the set of all interior points with respect to $(\mathcal{V}, |\cdot|)$.

Definition 3. An element $x_0 \in X$ is said to be a weak minimal solution of (SP) if

$$\nexists x \in X \text{ s.t. } F(x) <^l_K F(x_0).$$

Also we define local solutions of (SP).

Definition 4. An element $x_0 \in X$ is said to be, a local minimal solution of (SP) if there exists N a neighborhood of x_0 such that

$$x \in N \cap X, F(x) \leq^l_K F(x_0) \implies F(x_0) \leq^l_K F(x),$$

a local weak minimal solution of (SP) if there exists N a neighborhood of x_0 such that

$$\nexists x \in N \cap X \text{ s.t. } F(x) <^l_K F(x_0).$$

Now we have a result of a necessary condition of local weak optimality of (SP).

Theorem 1. If x_0 be a local weak minimal solution of (SP) of F , then we have

$$CF(x_0, x - x_0) \cap (-\text{Int}\mu(K)) = \emptyset, \quad \forall x \in X.$$

Also we have a result of a sufficient condition of local optimality of (SP).

Theorem 2. Assume that Z is a finite dimensional space, and F is $\mathcal{V}$ -directional derivative at $x_0 \in X$ in each direction. Moreover, we assume that

$$DF(x_0, d) = \lim_{t \downarrow 0} \frac{1}{\lambda} [F(x_0 + \lambda d), F(x_0)]$$

converges uniformly and continuous with respect to d on the unit ball. If

$$DF(x_0, d) \notin -\mu(K), \quad \forall d \in T_X(x_0) \setminus \{\theta\},$$

then x_0 is a local minimal solution of (SP).

Example 3. We discuss Example 1, in the standpoint of the above optimality conditions. At first, we can check easily that no (global) minimal, and no (global) weak minimal solution of (SP), 0 is the only local minimal solution of (SP), and for each $x \in \mathbb{R}$, x is a local weak minimal solution of (SP).

When $x_0 = 0$, $DF(x_0, d) \in -\mu(K)$ holds if and only if

$$\mathbb{R}_+^2 \supset |d| \operatorname{co}\{(1, -1), (-1, 1)\},$$

but it does not hold when $d \neq 0$. From Theorem 2, we have x_0 is a local minimal solution of (SP). Also when $0 < x_0 < \frac{1}{2}$, $DF(x_0, d) \in -\mu(K)$ holds if and only if

$$\begin{cases} \mathbb{R}_+^2 \supset |d| \operatorname{co}\{(1, -1), (-1, 1)\}, & \text{if } d \geq 0, \\ \mathbb{R}_+^2 + |d| \operatorname{co}\{(1, -1), (-1, 1)\} \ni (0, 0), & \text{if } d < 0. \end{cases}$$

and $\mathbb{R}_+^2 + |d| \operatorname{co}\{(1, -1), (-1, 1)\} \ni (0, 0)$ is always true when $d < 0$. This is consistent with Theorem 2, since x_0 is not local minimal solution. Moreover, $DF(x_0, d) \in -\operatorname{Int}\mu(K)$ holds if and only if

$$\begin{cases} (0, \infty)^2 \supset |d| \operatorname{co}\{(1, -1), (-1, 1)\}, & \text{if } d \geq 0 \\ (0, \infty)^2 + |d| \operatorname{co}\{(1, -1), (-1, 1)\} \ni (0, 0), & \text{if } d < 0 \end{cases}$$

does not hold for each $d \in \mathbb{R}$. This is also consistent with Theorem 1, since x_0 is a local weak minimal solution of (SP).

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