

Calculation of invariant rings and their divisor class groups by cutting semi-invariants

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Abstract

Let G be an affine connected algebraic group acting regularly on an affine Krull scheme $X = \text{Spec}(R)$ over an algebraically closed field K of any characteristic. We study on the minimal calculation of the ring R^G of invariants of G in R and their class groups by cutting prime semi-invariants which form free modules over R^G .

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1 Introduction

Let G be an affine algebraic group over an algebraically closed field K of arbitrary characteristic p . Let R be an integral domain containing K as a subfield. We say that (R, G) a K -regular action of G on R , if G acts on R as a rational G -module over K which induces the homomorphism $G \rightarrow \text{Aut}_{K\text{-algebra}}(R)$ (e.g., [12]). Let $U(R)$ denote the group of all units in R and $U_K(R)$ the quotient group of $U(R)$ by the multiplicative group $U(K) = K^\times$ of K . In general $U_K(R)$ is torsion-free, as K is algebraically closed. We say that a non-zero element f of R is said to be a non-zero semi-invariant of R relative to χ , if the map

$$\chi : G \ni \sigma \mapsto \frac{\sigma(f)}{f} \in U(K)$$

is a rational character of G . In order to calculate rings of invariants and their class groups, we can cut some prime semi-invariants and explain this viewpoint in the following example:

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Example 1.1 Let $\mathbf{C}[X_1, X_2, X_3]$ be the 3-dimensional polynomial ring over the complex number field $\mathbf{C}$. Let $\mathbf{G}_m$ be the multiplicative group $\mathbf{C}^\times$ whose action on this algebra is such a way that $\mathbf{G}_m \ni t$ acts on $\{X_1, X_2, X_3\}$ by

$$\text{diag}[t^2, t^{-1}, t^{-1}].$$

Then we have

- (1) $\mathbf{C}[X_1, X_2, X_3]^{\mathbf{G}_m} = \mathbf{C}[X_1 X_2^2, X_1 X_2 X_3, X_1 X_3^2]$.
- (2) The stabilizer $(\mathbf{G}_m)_{X_1} = \langle \text{diag}[1, -1, -1] \rangle$ of $\mathbf{G}_m$ at X_1 on $\{X_1, X_2, X_3\}$.
- (3) $\mathbf{C}[X_1, X_2, X_3]^{(\mathbf{G}_m)_{X_1}} = \mathbf{C}[X_1, X_2^2, X_2 X_3, X_3^2]$.
- (4) The divisor class group $\text{Cl}(\mathbf{C}[X_1, X_2, X_3]^{\mathbf{G}_m}) \cong \mathbf{Z}/2\mathbf{Z}$ which is isomorphic to

$$\text{Hom}((\mathbf{G}_m)_{X_1}, \mathbf{C}^*) \cong \text{Cl}(\mathbf{C}[X_1, X_2, X_3]^{(\mathbf{G}_m)_{X_1}}).$$

- (5) There is the isomorphism

$$\mathbf{C}[X_1, X_2, X_3]^{(\mathbf{G}_m)_{X_1}} / (X_1 - 1) \cong \mathbf{C}[X_1, X_2, X_3]^{\mathbf{G}_m}$$

induced by

$$\psi : \mathbf{C}[X_1, X_2, X_3] \rightarrow \mathbf{C}[X_1, X_2, X_3]$$

$$(\psi(X_1) = 1, \psi(X_2) = X_2, \psi(X_3) = X_3).$$

The purpose of this paper is to generalize the assertion of this example to in the case of factorial (or Krull) domains with affine algebraic group actions in characteristic-free.

2 Preliminaries

Let $\mathcal{Q}(A)$ denote the total quotient ring of a ring A and

$$\text{Ht}_1(A) := \{\mathfrak{P} \in \text{Spec}(A) \mid \text{ht}(\mathfrak{P}) = 1\}.$$

For an integral domain A and a subring B of A such that $B = \mathcal{Q}(B) \cap A$ and $\mathcal{Q}(B) \subseteq \mathcal{Q}(A)$, we denote by

$$\text{Ht}_1(A, B) := \{\mathfrak{P} \in \text{Ht}_1(A) \mid \mathfrak{P} \cap B \in \text{Ht}_1(B)\},$$

$$\text{Ht}_1^{(2)}(A, B) := \{\mathfrak{P} \in \text{Ht}_1(A) \mid \text{ht}(\mathfrak{P} \cap B) \geq 2\}$$

and, for $\mathfrak{p} \in \text{Ht}_1(B)$, by

$$\text{Over}_{\mathfrak{p}}(A) := \{\mathfrak{P} \in \text{Ht}_1(A) \mid \mathfrak{P} \cap B = \mathfrak{p}\}.$$

Especially suppose that A is a Krull domain (e.g., [1]). Let $v_{A,\mathfrak{P}}$ be the discrete valuation defined by $\mathfrak{P} \in \text{Ht}_1(A)$ of A . Denote by $\text{Div}(A)$ (resp. $\text{PDiv}(A)$, $\text{Cl}(A)$) the divisor group (resp. the group of principal divisors, the divisor class group) of A . For a subring B of A such that $B = \mathcal{Q}(B) \cap A$, B is a Krull domain (e.g., [1, 3]) and every $\text{Over}_{\mathfrak{p}}(A)$ is non-empty and finite. Let $e(\mathfrak{P}, \mathfrak{p}) = v_{A,\mathfrak{P}}(\mathfrak{p}A)$ be the ramification index of $\mathfrak{P} \in \text{Over}_{\mathfrak{p}}(A)$ for a prime ideal $\mathfrak{p} \in \text{Ht}_1(B)$. If all ramification indices of minimal prime ideals are equal to 1, the extension $B \rightarrow A$ is said to be divisorially unramified (cf. [7]).

Consider an action of a group G on a ring R as automorphisms. For a prime ideal $\mathfrak{P}$ of R , let

$$\mathcal{I}_G(\mathfrak{P}) = \{\sigma \in G \mid \sigma(x) - x \in \mathfrak{P} \ (x \in R)\}$$

which is referred to as the *inertia group* of $\mathfrak{P}$ under this action (for the classical case, see [5]). Let $Z^1(G, \text{U}(R))$ be the group of 1-cocycles of G on the unit group $\text{U}(R)$ of R . For a 1-cocycle χ ,

$$R_\chi := \{x \in R \mid \sigma(x) = \chi(\sigma)x \ (\sigma \in G)\},$$

which is a module over the invariant subring R^G .

The next theorem is a generalization of [11] and is fundamental in this paper:

Theorem 2.1 (cf. [7]) *Let R be a Krull domain acted by a group G as automorphisms. For a cocycle $\chi \in Z^1(G, \text{U}(R))$, R_χ is a free R^G -module if and only if the following conditions are satisfied:*

$$(i) \dim \mathcal{Q}(R^G) \otimes_{R^G} R_\chi = 1$$

(ii) *There is a nonzero element $f \in R_\chi$ satisfying*

$$\forall \mathfrak{p} \in \text{Ht}_1(R^G) \Rightarrow \exists \mathfrak{P} \in \text{Over}_{\mathfrak{p}}(R) \text{ such that } v_{R,\mathfrak{P}}(f) < v_{R,\mathfrak{P}}(\mathfrak{p}R).$$

Here the condition (i) holds, if $R_\chi \cdot R_{-\chi} \neq \{0\}$.

Algebraic groups are affine and defined over a fixed algebraically closed field K of an arbitrary characteristic p . Let $\mathfrak{X}(G)$ be the group of rational characters of an algebraic group G expressed as an additive group with zero. The K -algebras R are not necessarily finite generated as algebras over K .

A subset N of a set M with an action of G is said to be G -invariant, if N is invariant under the action of G on M . In this case $G|_N$ denote the group consisting of the restriction $\sigma|_N$ of all $\sigma \in G$ to N , which is called *the group G on N* .

Pseudo-reflections on finite-dimensional vector spaces are defined in [2] and should be generalized as follows:

Definition 2.2 (Pseudo-reflections of actions) Suppose that R is a Krull K -domain with (R, G) a regular action of an algebraic group G . Define the subgroup

$$\mathfrak{R}(R, G) := \left\langle \bigcup_{\mathfrak{p} \in \text{Ht}_1(R, R^G)} \mathcal{I}_G(\mathfrak{p}) \right\rangle$$

of G which is called the pseudo-reflection group of the action (R, G) .

Finiteness of pseudo-reflections of regular actions characterize reductivity of algebraic groups. We have

Theorem 2.3 (cf. [8]) Let G^0 be the identity component of an algebraic group G . Then the following conditions are equivalent:

- (i) G^0 is reductive.
- (ii) $\mathfrak{R}(R, G)$ is finite on R for any Krull K -domain R with a regular action of G .

3 The abstract descent of class groups

In this section, suppose that A is Krull. For a subset Γ of $\mathcal{Q}(A)$ satisfying $\gamma \cdot \Gamma \subset A$ for some $\gamma \in A$, let $\text{div}_A(\Gamma)$ be the divisor of ΓA on A . On the other hand, let $\mathcal{I}_A(D)$ be the divisorial fractional ideal of A defined by the divisor D on A . Consider a K -subalgebra B of A satisfying $\mathcal{Q}(B) \cap A = B$. For each $\mathfrak{p} \in \text{Ht}_1(B)$, set

$$d_{\mathfrak{p}} = \sum_{\mathfrak{P} \in \text{Over}_{\mathfrak{p}}(A)} v_{A, \mathfrak{P}}(\mathfrak{p}A) \text{div}_A(\mathfrak{P}) \in \text{Div}(A).$$

Define the subgroup

$$E^*(A, B) := \left(\bigoplus_{\mathfrak{p} \in \text{Ht}_1(B)} \mathbb{Z} d_{\mathfrak{p}} \right) \oplus \text{Bup}(A, B)$$

of $\text{Div}(A)$ where $\text{Bup}(A, B) = \bigoplus_{\mathfrak{p} \in \text{Ht}_1(A), \text{ht}(\mathfrak{p} \cap B) \geq 2} \mathbb{Z} \text{div}_A(\mathfrak{p})$. Let

$$\Phi_{A, B}^* : E^*(A, B) \rightarrow \text{Div}(B)$$

be the homomorphism defined by the composite of the projection

$$E^*(A, B) \rightarrow \bigoplus_{\mathfrak{p} \in \text{Ht}_1(B)} \mathbb{Z} d_{\mathfrak{p}}$$

and the isomorphism

$$\bigoplus_{\mathfrak{p} \in \text{Ht}_1(B)} \mathbb{Z} d_{\mathfrak{p}} \ni \sum_{\mathfrak{p}} a_{\mathfrak{p}} d_{\mathfrak{p}} \mapsto \sum_{\mathfrak{p}} a_{\mathfrak{p}} \text{div}_S(\mathfrak{p}) \in \text{Div}(B)$$

Set $\text{Div}_A(\mathcal{U}(\mathcal{Q}(B))) := \{\text{div}_A(\gamma) \mid \gamma \in \mathcal{U}(\mathcal{Q}(B))\} \subset \text{PDiv}(A)$. Then

$$\text{PDiv}(B) \ni D \mapsto \text{div}_A(\mathbf{I}_B(D)) \in \text{Div}_A(\mathcal{U}(\mathcal{Q}(S)))$$

is an isomorphism whose inverse is the restriction $\Phi_{R,S}^*|_{\text{Div}_A(\mathcal{U}(\mathcal{Q}(B)))}$. Since

$$\text{Div}_A(\mathcal{U}(\mathcal{Q}(B))) \subset E^*(A, B),$$

we define $E(A, B) := E^*(A, B)/\text{Div}_A(\mathcal{U}(\mathcal{Q}(B)))$. Moreover define the subgroup

$$L(A, B) := \{f \in \mathcal{U}(\mathcal{Q}(A)) \mid \text{div}_A(f) \in E^*(A, B)\}$$

of $\mathcal{U}(\mathcal{Q}(A))$. Then:

Theorem 3.1 *Under the circumstances as above, we obtain the sequences*

$$0 \rightarrow (\text{Div}_A(\mathcal{U}(\mathcal{Q}(B))) + \text{Bup}(A, B))/\text{Div}_A(\mathcal{U}(\mathcal{Q}(B))) \rightarrow E(A, B) \rightarrow \text{Cl}(B) \rightarrow 0$$

$$0 \rightarrow \frac{L(A, B)/\mathcal{U}(\mathcal{Q}(B))}{\mathcal{U}(A)/\mathcal{U}(B)} \rightarrow E(A, B) \rightarrow \text{Cl}(A)$$

which are exact.

We introduce the concept of redundant prime elements which partially generate the subring C of A over B as follows:

Definition 3.2 (Paralleled linear hulls) *Consider an intermediate subring C of A such that $C = \mathcal{Q}(C) \cap A$ and $B \subseteq C$. The pair $(C, \{f_1, \dots, f_m\})$ is defined to be a paralleled linear hull of B with respect to f_i ($1 \leq i \leq m$), if the composite of the inclusion and the canonical epimorphism*

$$\begin{array}{ccc} B & \xrightarrow{\subseteq} & C \\ & \searrow \cong & \downarrow \text{can.} \\ & & C/(\sum_{i=1}^m C(f_i - 1)) \end{array}$$

induces an isomorphism, f_i ($1 \leq i \leq m$) are algebraically independent over $\mathcal{Q}(B)$ and

$$\text{Cl}(B) \cong \text{Cl}(C).$$

Note in general $C \neq B[f_1, \dots, f_m]$.

4 Graded structures and paralleled linear hulls

Let S be an integral domain which is a $\mathbf{Z}^m$ -graded algebra

$$S = \bigoplus_{\mathbf{i} \in \mathbf{Z}^m} S_{\mathbf{i}}$$

over S_0 . Then if S is Krull, so is S_0 , because $S_0 = \mathcal{Q}(S_0) \cap S$.

Definition 4.1 (half primary $\mathbf{Z}^m$ -freeness) *We say that S is half primary $\mathbf{Z}^m$ -free with respect to $\{f_1, \dots, f_m\}$, if*

$$S_{\mathbf{i}} = S_{(i_1, \dots, i_m)} = S_0 \prod_{j=1}^m f_j^{i_j}$$

for any $i_j \geq 0$ and f_j , $1 \leq j \leq m$, is homogeneous prime element in S of degree $(0, \dots, 0, 1, 0, \dots, 0)$ having 1 at the j -th part.

Theorem 4.2 *Suppose that S is a $\mathbf{Z}^m$ -graded Krull domain. If S is half primary $\mathbf{Z}^m$ -free with respect to $\{f_1, \dots, f_m\}$, then $(S, \{f_1, \dots, f_m\})$ is a paralleled linear hull of S_0 .*

Put $\mathbf{Z}_{\leq 0} := \{k \in \mathbf{Z} \mid k \leq 0\}$ and let $\mathbf{Z}_{\leq 0}^m$ be the direct product of k -copies of $\mathbf{Z}_{\leq 0}$. For a subset W of S , let W^{hom} be the set consisting homogenous elements of W in S . Let

$$U_S := \{h \in S^{\text{hom}} \mid h \neq 0, \deg(h) \in \mathbf{Z}_{\leq 0}^m\}.$$

For a subset Ω of $\text{Spec } S$, let Ω^{hom} be the set of all homogeneous prime ideals in Ω . A divisor

$$D = \sum_{\mathfrak{P} \in \text{Ht}_1(S)} a_{\mathfrak{P}} \text{div}_S(\mathfrak{P})$$

of $\text{Div}(S)$ is said to be homogeneous, if all prime ideals in

$$\text{supp}_S(D) := \{\mathfrak{P} \in \text{Ht}_1(S) \mid a_{\mathfrak{P}} \neq 0\}$$

are homogeneous. For a subset of $\mathcal{D}$ of $\text{Div}(S)$, we put

$$\mathcal{D}^{\text{hom}} := \{D \in \mathcal{D} \mid D \text{ is homogeneous}\},$$

$$\text{Ht}_1(S)_0^{\text{hom}} := \text{Ht}_1(S)^{\text{hom}} \setminus \{Sf_1, \dots, Sf_m\}$$

and

$$\text{Div}(S)_0^{\text{hom}} := \{D \in \text{Div}(S)^{\text{hom}} \mid \text{supp}_S(D) \cap \{Sf_1, \dots, Sf_m\} = \emptyset\}.$$

Lemma 4.3 *Under the circumstances as above we have*

- (i) $\text{Cl}(U_S^{-1}S) = \{0\}$
- (ii) $\text{Div}(S)_0^{\text{homo}} \longrightarrow \text{Cl}(S)$ *is an epimorphism.*
- (iii) $\text{Ht}_1(S)_0^{\text{hom}} \ni \mathfrak{P} \longmapsto \mathfrak{P} \cap S_0 \in \text{Ht}(S_0)$ *is bijective and* $e(\mathfrak{P}, \mathfrak{P} \cap S_0) = 1$.
- (iv) *The composite* $\text{Div}(S)_0^{\text{homo}} \hookrightarrow E^*(S, S_0) \xrightarrow[\Phi_{S, S_0}^*]{} \text{Div}(S_0)$ *is an isomorphism and induces*

$$\text{PDiv}(S) \cap \text{Div}(S)_0^{\text{homo}} \cong \text{PDiv}(S_0).$$

This follows from the idea of M. Nagata on homogeneous localization (e.g., [3]).

By Lemma 4.3 we must have the isomorphism

$$\text{Cl}(S) \cong \text{Cl}(S_0).$$

The remainder of the sketch of the proof of Theorem 4.2 is omitted.

5 Toric quotients

In this section let (R, G) be a regular action of a connected algebraic group G on a Krull domain R containing K as a subring.

Using Nagata's pseudo-geometric rings ([5]) and Rosenlicht's theorem on $U_K(R')$ of affine normal domains R' , we can generalize the result of [4] without the assumption of finite generations of R as follows.

Theorem 5.1 (cf. [10]) *Let f be a nonzero element of $\mathcal{Q}(R)$. If Rf is invariant under the action of G , then Kf is G -invariant and, moreover if $\mathfrak{P} \cap R^G \neq \{0\}$ for any $\mathfrak{P} \in \text{Ht}_1(R)$ such that $v_{R, \mathfrak{P}}(f) < 0$, then*

$$G \ni \sigma \mapsto \frac{\sigma(f)}{f} \in U(K)$$

is a rational character of G .

By this theorem, for a nonzero $f \in R$ satisfying that Rf is G -invariant, the symbol $\delta_{f, G}$ is denoted to the homomorphism

$$\delta_{f, G} : G \ni \sigma \mapsto \frac{\sigma(f)}{f} \in U(K).$$

Lemma 5.2 *We have:*

(i) If the set $\bigcup_{\mathfrak{p} \in \Lambda} \text{Over}_{\mathfrak{p}}(R)$ consists of principal ideals, then it is a finite set, where $\Lambda := \{\mathfrak{p} \in \text{Ht}_1(R^G) \mid |\text{Over}_{\mathfrak{p}}(R)| \geq 2\}$.

(ii) If the set $\text{Ht}_1^{(2)}(R, R^G)$ consists of principal ideals, then it is a finite set.

This finiteness follows from Theorem 2.1 and $\text{rank}(\mathfrak{X}(G)) < \infty$.

Assumption 5.3 Suppose that the both sets of Lemma 5.2 consist of principal ideals of R .

By this there exist non-associated prime elements $f_1, \dots, f_m$ of R such that

$$|\{Rf_1, \dots, Rf_m\} \cap \text{Over}_{\mathfrak{p}}(R)| = |\text{Over}_{\mathfrak{p}}(R)| - 1$$

for every $\mathfrak{p} \in \text{Ht}_1(R^G)$ and

$$\{Rf_1, \dots, Rf_m\} \setminus \left(\bigcup_{\mathfrak{p} \in \text{Ht}_1(R^G)} \text{Over}_{\mathfrak{p}}(R) \right) = \text{Ht}_1^{(2)}(R, R^G).$$

According to Theorem 5.1, the homomorphisms $\delta_{f_i, G}$ are rational characters of G . Let H be the stabilizer

$$\text{Stab}(G : f_1, \dots, f_m) = \bigcap_{i=1}^m G_{f_i} = \bigcap_{i=1}^m \text{Ker}(\delta_{f_i, G})$$

of G at the set $\{f_1, \dots, f_m\}$.

From the choice of f_i and Theorem 2.1, we must have

$$R_{\sum_i a_i \delta_{f_i, G}} = R^G \prod_i f_i^{a_i} \quad (5.1)$$

for any integer $a_i \geq 0$ ($1 \leq i \leq m$) and put

$$R^{\mathbf{f}} = \sum_{a_1, \dots, a_m \in \mathbf{Z}} R_{\sum_i a_i \delta_{f_i, G}} \subset R$$

which is a K -subalgebra of R^H . Clearly $R^H = R^{\mathbf{f}}$ in the case where the ground field K is of characteristic $p = 0$. The equalities (5.1) imply that the subgroup $\langle \delta_{f_1, G}, \dots, \delta_{f_m, G} \rangle$ of $\mathfrak{X}(G)$ is free of rank m . On the other hand

$$R^{\mathbf{f}} = \mathcal{Q}(R^{\mathbf{f}}) \cap R$$

and hence the K -subalgebra $R^{\mathbf{f}}$ is a Krull domain with the $\mathbf{Z}^m$ -graded structure defined by the homogeneous part

$$R_{\mathbf{a}}^{\mathbf{f}} = R_{\sum_i a_i \delta_{f_i, G}}$$

of degree $\mathbf{a} = (a_1, \dots, a_m) \in \mathbf{Z}^m$. Consequently, from (5.1) we infer that, for $S = R^{\mathbf{f}}$ and $S_0 = R^G$, S is half primary $\mathbf{Z}^m$ -free with respect to $\{f_1, \dots, f_m\}$.

Theorem 5.4 *Under the circumstances as above, $(R^f, \{f_1, \dots, f_m\})$ is a paralleled linear hull of R^G .*

This theorem follows from Theorem 4.2.

Next, the class group $\text{Cl}(R^f) \cong \text{Cl}(R^G)$ shall be studied by the abstract descent method. For this purpose we introduce the notation as bellow: Consider a K -subalegba M of R such that $M \supset \{f_1, \dots, f_m\}$ and $\mathcal{Q}(M) \cap R = M$ which is invariant under the action of G . Since M is a Krull domain, for a subset $\mathcal{D}$ of the divisor group $\text{Div}(M)$ of M , let us define the subset

$$\mathcal{D}_{f(M)} := \{D \in \mathcal{D} \mid \text{supp}_M(D) \cap \{Mf_1, \dots, Mf_m\} = \emptyset\}$$

without prime elements f_i as supports of divisors. The group G acts on $\text{Div}(M)$ naturally. If $\mathcal{D}$ is an G -invariant subset, let $\mathcal{D}^G$ denote the set consisting G -invariant divisors of $\mathcal{D}$ and, for a simplicity, denote $\mathcal{D}_{f(M)}^G$ by the set $\mathcal{D}^G \cap \mathcal{D}_{f(M)}$.

As R^f is invariant under the action of G on R , we see $\text{Ht}_1(R^f)^{\text{homo}} = \text{Ht}_1(R^f)^G$ and

$$\text{Div}(R^f)_0^{\text{homo}} = \text{Div}(R^f)_{f(R^f)}^G. \quad (5.2)$$

Recalling $\mathcal{Q}(R^f) \cap R = R^f$, we have

$$\Phi_{R, R^f}^* : E^*(R, R^f) \rightarrow \text{Div}(R^f)$$

which is an isomorphism, since $\text{Bup}(R, R^f) = \{0\}$ follows from Assumption 5.3. For any $\mathfrak{p} \in \text{Ht}_1(R^f)_0^{\text{homo}}$, $\text{ht}(\mathfrak{p} \cap R^G) = 1$ and $\text{Over}_{\mathfrak{p} \cap R^G}(R^f) = \{\mathfrak{p}\}$, which shows the set $\text{Over}_{\mathfrak{p}}(R)$ consists of a unique prime ideal and is G -invariant and $\text{Over}_{\mathfrak{p}}(R) = \text{Over}_{\mathfrak{p} \cap R^G}(R)$. Thus we have the commutative diagram

$$\begin{array}{ccc} \text{Div}(R)_{f(R)}^G \cap E^*(R, R^f) & \xrightarrow{\subset} & E^*(R, R^f) \\ \downarrow & & \cong \downarrow \Phi_{R, R^f}^* \\ \text{Div}(R^f)_{f(R^f)}^G & \xrightarrow[\subset]{} & \text{Div}(R^f) \end{array}$$

and $\text{Div}(R)_{f(R)}^G \cap E^*(R, R^f) \cong \text{Div}(R^f)_{f(R^f)}^G$. Putting

$$L(R, R^f)_f := \{g \in L(R, R^f) \mid \text{div}_R(g) \in \text{Div}(R)_{f(R)}\},$$

we have the exact sequence

$$0 \rightarrow L(R, R^f)_f / (U(R) \cap L(R, R^f)_f) \rightarrow \text{Div}(R)_{f(R)}^G \cap E^*(R, R^f) \rightarrow \text{Cl}(R).$$

Moreover putting

$$L(R^f)_f := \{h \in U(\mathcal{Q}(R^f)) \mid \text{div}_{R^f}(h) \in \text{Div}(R^f)_{f(R^f)}^G\},$$

by Lemma 4.3 and (5.2) we have the exact sequence

$$0 \rightarrow L(R^f)_f / (U(R^f) \cap L(R^f)_f) \rightarrow \text{Div}(R^f)_{f(R^f)}^G \rightarrow \text{Cl}(R^f) \rightarrow 0$$

and $L(R^f)_f / (U(R^f) \cap L(R^f)_f) \cong U(\mathcal{Q}(R^G)) / U(R^G)$ whose isomorphism demoted to $\tilde{\Phi}_{R^f, R^G}^*$.

Consequently under the circumstances as above, we see

Theorem 5.5 *If R is factorial, then*

$$\begin{aligned} \text{Cl}(R^G) &\cong \text{Cl}(R^f) \cong \frac{L(R, R^f)_f / (U(R) \cap L(R, R^f)_f)}{L(R^f)_f / (U(R^f) \cap L(R^f)_f)} \\ &= \frac{L(R, R^f)_f / (U(R) \cap L(R, R^f)_f)}{\tilde{\Phi}_{R^f, R^G}^{*-1}(U(\mathcal{Q}(R^G)) / U(R^G))}. \end{aligned}$$

For any $g \in L(R, R^f)_f$, as $\text{div}_R(g)$ is G -invariant and

$$\text{supp}_R(\text{div}_R(g)) \subset \{\mathfrak{P} \in \text{Ht}^1(R) \mid \mathfrak{P} \cap R^G \neq \{0\}\},$$

the subspace Kg is G -invariant and $\delta_{g,G} \in \mathfrak{X}(G)$. Suppose that

$$U(R) \cap L(R, R^f)_f \subset R^f. \quad (5.3)$$

Then $\text{Cl}(R^G) \cong L(R, R^f)_f / L(R^f)_f$. Put

$$\mathfrak{X}(H)_{R,f} := \{\delta_{g,G}|_H \mid g \in L(R, R^f)_f\}.$$

In case of $p = 0$ we see $R^H = R^f$ and obtain

Corollary 5.6 *Suppose that R is factorial and the condition (5.3) holds. If $p = 0$, then*

$$\text{Cl}(R^G) \cong \mathfrak{X}(H)_{R,f}.$$

Moreover by [6, 8, 12] we have

Corollary 5.7 *Suppose that R is affine factorial K -domain with trivial units. Let (R, G) be a stable regular action of an algebraic torus G (i.e., $\text{Spec}(R)$ contains a non-empty open subset consisting of closed G -orbits, see [12]). If $p = 0$, then $\text{Cl}(R^G) \cong \mathfrak{X}(H/\mathfrak{R}(R, H))$.*

In this case, the extension $R^H \rightarrow R^{\mathfrak{R}(R, H)}$ is divisorially unramified and $R^{\mathfrak{R}(R, H)}$ is factorial. Thus this follows from Corollary 5.6 for $R = R^{\mathfrak{R}(R, H)}$.

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