

Brauer Group of $\mathbb{R}(X)$ and Eichler Type Theorem

By

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Abstract

The Brauer group of $\mathbb{R}(X)$, the rational function field over the real field, is isomorphic to the continuous direct sum of $\mathbb{Z}/2\mathbb{Z}$. A central division algebra over $\mathbb{R}(X)$ has strong approximation property for $\mathbb{R}[X]$ if and only if it is trivial at the place not corresponding to a prime ideal of $\mathbb{R}[X]$. This is a generalization of Eichler theorem.

We discuss similar problems for algebraic function fields over $\mathbb{R}$ and obtain partial solutions for some cases.

1. Brauer groups of $\mathbb{R}((X))$ and $\mathbb{R}(X)$.

Let $\mathbb{R}((X))$ be the field of formal power series over $\mathbb{R}$. It is a complete valuation field with the residue field $\mathbb{R}$. By J.P.Serre "Corps locaux" Chap 12, we have

$$Br(\mathbb{R}((X))) \simeq Gal(\mathbb{C}/\mathbb{R}) \times Br(\mathbb{R}) \simeq (\mathbb{Z}/2\mathbb{Z})^2.$$

($Br(K)$ denotes the Brauer group of K). We shall determine it more concretely.

Let D be a central division algebra over $\mathbb{R}((X))$. Since $Br(\mathbb{C}((X)))$ is trivial, D splits over $\mathbb{C}((X))$, so that D contains a maximal subfield isomorphic to $\mathbb{C}((X))$. Thus we have

$$D = K + Ki + Kj + Kij, \quad K = \mathbb{R}((X)),$$

$$i^2 = -1, j^2 = f \in K^\times, ji = -ij.$$

We shall denote this D by $\{-1, f\}$.

Since $\{-1, f\} \simeq \{-1, f'\} \iff ff'^{-1} \in N_{K(\sqrt{-1})/K}(K(\sqrt{-1})^\times) = (K^2 + K^2)^\times$, we have $Br(K) \simeq K^\times / (K^2 + K^2)^\times$, whose complete representative system is given by $\{1, -1, X, -X\}$ so that

$$Br(\mathbb{R}((X))) = \{\mathbb{R}((X)), \mathbb{H}((X)), \{-1, X\}, \{-1, -X\}\}.$$

Note that $\mathbb{R}((X)) = \{-1, 1\}$ and $\mathbb{H}((X)) = \{-1, -1\}$ where $\mathbb{H}$ is the usual quaternion algebra over $\mathbb{R}$. $\mathbb{H}((X))$ is unramified over $\mathbb{R}((X))$, while $\{-1, X\}$ and $\{-1, -X\}$ are ramified.

Next, we shall determine the Brauer group of $\mathbb{R}(X)$.

Theorem 1 (1) Every central division algebra over $\mathbb{R}(X)$ has the index ≤ 2 , hence if it is not trivial, it is a quaternion algebra over $\mathbb{R}(X)$,
 (2) $Br(\mathbb{R}(X)) \simeq \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})_0^\mathbb{R} \simeq (\mathbb{Z}/2\mathbb{Z})_0^\mathbb{R} \coprod^{\{\text{sgn}\}}$, where $(\mathbb{Z}/2\mathbb{Z})_0^\mathbb{R}$ denotes the continuous direct sum of $\mathbb{Z}/2\mathbb{Z}$, namely the aggregation of all finite subsets of $\mathbb{R}$ with the group operation: $A \cdot B =$ the symmetric difference of A and B .

Proof Let D be a central division algebra over $\mathbb{R}(X)$. Then by the same reason as before, $\mathbb{C}(X)$ is a splitting field of D . This proves (1), and some maximal subfield of D is isomorphic to $\mathbb{C}(X)$. Thus D is in the form of $D = \{-1, f\}$ for some $f \in K^\times$, $K = \mathbb{R}(X)$, and we have $Br(\mathbb{R}(X)) \simeq K^\times / (K^2 + K^2)^\times$.

If $f = \varphi^2 + \psi^2$, then $f(a) \geq 0$ for $\forall a \in \mathbb{R}$. Conversely, if $f(a) \geq 0$ for $\forall a \in \mathbb{R}$, then f is decomposed into the product $f = \prod_i (X - a_i)^2 \prod_j (X - \alpha_j)(X - \overline{\alpha_j})$, $a_i \in \mathbb{R}, \alpha_j \in \mathbb{C} \setminus \mathbb{R}$. Since $(X - \alpha_j)(X - \overline{\alpha_j}) = N_{K(\sqrt{-1})/K}(X - \alpha_j)$, we have $f \in K^2 + K^2$. Therefore, as a complete representative system of $K^\times / (K^2 + K^2)^\times$, we get $\{\pm(X - a_1) \cdots (X - a_n) \mid a_i \in \mathbb{R}, \text{mutually distinct}\}$.

For $f = \pm(X - a_1) \cdots (X - a_n)$, $D = \{-1, f\}$ is trivial at a such that $f(a) > 0$. It is ramified at a_i and at the non-prime place (which will be denoted by ∞) if the degree of f is odd. Since $\{-1, f_1\} \otimes_{\mathbb{R}(X)} \{-1, f_2\} \sim \{-1, f_1 f_2\}$, the multiplication in $Br(\mathbb{R}(X))$ corresponds to the symmetric difference of the sets of ramified places. Thus we have obtained the desired result (2).

Remark A discrete valuation is called real (or imaginary) if its residue field is $\mathbb{R}$ (or $\mathbb{C}$). The set of all real places will be denoted by $RP(K)$. For $K = \mathbb{R}(X)$, we have $RP(K) = \mathbb{R} \coprod \{\infty\}$.

Then, we have $Br(\mathbb{R}(X)) \simeq (\mathbb{Z}/2\mathbb{Z})_0^{RP(\mathbb{R}(X))}$. The isomorphism is given as follows. Suppose that a central division algebra D over $\mathbb{R}(X)$ corresponds to a finite subset A of $RP(\mathbb{R}(X)) = \mathbb{R} \coprod \{\infty\}$. D is ramified at every $a \in A \setminus \{\infty\}$, and at ∞ if $|A \setminus \{\infty\}|$ is odd. There are two D s which are ramified at no place. They are attributed to $\mathbb{Z}/2\mathbb{Z}$ at ∞ .

Corollary $\mathbb{R}(X)$ satisfies Hasse's principle.

$\{-1, -1\}$ is unramified but non-trivial at every place. All other non-trivial $\{-1, f\}$ are ramified at some places.

2. Brauer group of $\mathbb{R}(X, Y)$.

Let K be a finite extension of $\mathbb{R}(X)$, namely an algebraic function field of one variable over $\mathbb{R}$. In other words, $K = \mathbb{R}(X, Y)$, Y is algebraic over $\mathbb{R}(X)$.

If $\sqrt{-1} \in K$, then K is an algebraic function field of one variable over $\mathbb{C}$, so that $Br(K)$ is trivial.

Hereafter we shall assume that $\sqrt{-1} \notin K$. Since $Br(K(\sqrt{-1}))$ is trivial, a central division algebra D over K , splits over $K(\sqrt{-1})$. This implies that D is a quaternion algebra and $D = \{-1, f\}$ for some $f \in K^\times$. From this we see that $Br(K)$ has the exponent 2, and $Br(K) \simeq K^\times / (K^2 + K^2)^\times$.

A valuation on K which is trivial on $\mathbb{R}^\times$ is called a place. The residue field of a place v is $\mathbb{R}$ or $\mathbb{C}$, according to which v is called real or imaginary. (Note that this terminology

differs from the ones used for algebraic number fields).

For an imaginary place v , D_v is trivial over K_v . For a real place v , D_v is one of four algebras over K_v . The one is trivial, another one is an unramified quaternion, and the other two are ramified quaternions. See the results in §1.

Let $RP(K)$ be the set of all real places. Since the place of $K(\sqrt{-1}) = \mathbb{C}(X, Y)$ are in one-to-one correspondence with points of a compact Riemann surface $\mathfrak{R}$, and since a real place v of K does not decompose in $K(\sqrt{-1})$, $RP(K)$ is identified with a subset of $\mathfrak{R}$.

For a real place v of K , we have $\exists \varphi \in K$, $\text{ord}_v(\varphi) = 1$. Then, $\varphi(z)$ is a local coordinate in a neighbourhood of the corresponding $z_v \in \mathfrak{R}$. Since $z \in RP(K)$ is equivalent to $\varphi(z) \in \mathbb{R}$ in this neighbourhood, $RP(K)$ is a one-dimensional real manifold. Since $\mathfrak{R}$ is compact, $RP(K)$ consists of ν closed curves, where ν is the number of connected components of $RP(K)$.

Theorem 2 We have $Br(K) \simeq (\mathbb{Z}/2\mathbb{Z})_0^{RP(K)}$.

The isomorphism is given as follows: Fix a point z_i ($1 \leq i \leq \nu$) from each connected component of $RP(K)$. Suppose that $Br(K) \ni D$ corresponds to a finite subset A of $RP(K)$. Then, D is ramified at $A \setminus \{z_1, \dots, z_\nu\}$ and possibly at z_i . The ramification at z_i is determined by the rule that D is ramified at even number of places on each connected component of $RP(K)$.

There are 2^ν different division algebras which are ramified at no real place. They are attributed to $(\mathbb{Z}/2\mathbb{Z})^{\{z_1, \dots, z_\nu\}}$.

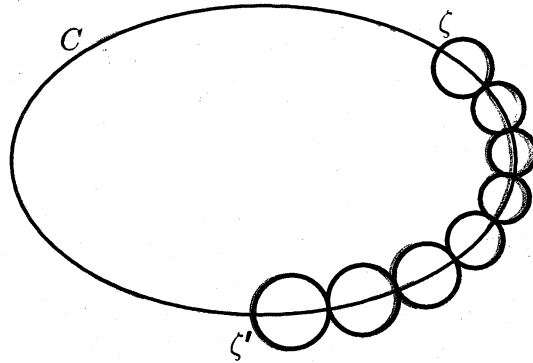
Proof Let $Br_1(K)$ be the group of all division algebras which are ramified at no real place. Then, $D = \{-1, f\} \in Br_1(K)$ is equivalent to that $\text{ord}_z(f)$ is even for every $z \in RP(K)$, namely that $f(z)$ has definite sign on each connected component of $RP(K)$.

As shown later, Hasse's principle holds for $K = \mathbb{R}(X, Y)$. Therefore, $D = \{-1, f\}$ is trivial if and only if f is non-negative on $RP(K)$, so that we have $|Br_1(K)| \leq 2^\nu$. The equality holds if for any connected component C of $RP(K)$, there exists $f \in K^\times$ such that $f(z) \leq 0$ on C but $f(z) \geq 0$ on $RP(K) \setminus C$. Since $RP(K)$ is mapped homeomorphically into $\mathbb{R}^4$ by $z \mapsto (T_i(z))_{1 \leq i \leq 4}$, $T_1(z) = \frac{X(z)}{X(z)^2+1}$, $T_2(z) = \frac{1}{X(z)^2+1}$, $T_3(z) = \frac{Y(z)}{Y(z)^2+1}$, $T_4(z) = \frac{1}{Y(z)^2+1}$, and since the function F defined by $F(z) = -1$ on C and $F(z) = 1$ on $RP(K) \setminus C$ is continuous on $RP(K)$, the polynomial approximation theorem of Weierstrass assures

that there exists a polynomial $P(T_i)$ such that $P(T_i(z)) < 0$ on C but $P(T_i(z)) > 0$ on $RP(K) \setminus C$. This completes the proof of $Br_1(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{\{z_1, \dots, z_\nu\}}$.

Take any $f \in K^\times$. If $\text{ord}_{z_0}(f)$ is odd for $z_0 \in RP(K)$, then $f(z)$ changes its sign when z crosses z_0 . Since a connected component C of $RP(K)$ is a closed curve, $f(z)$ must change its sign even times on C , therefore $D = \{-1, f\}$ is ramified at even number of places on C .

Now, we shall show that for any two points ζ and ζ' on C , there exists $f \in K^\times$ such that $D = \{-1, f\}$ is ramified at ζ and ζ' , but not ramified at other real places. Again we shall map $RP(K)$ into $\mathbb{R}^4$ by $z \mapsto (T_i(z))_{1 \leq i \leq 4}$. Since C is a closed analytic curve, there are $\zeta = \zeta_0, \zeta_1, \dots, \zeta_n = \zeta'$ ($\zeta_i \in C$) and small spheres $S_j : \sum_{i=1}^4 (T_i - a_{ij})^2 = r_j^2$ such that $S_j \cap RP(K) = \{\zeta_{j-1}, \zeta_j\}$.



Then, $f = \prod_{j=1}^n \left\{ \sum_{i=1}^4 (T_i(z) - a_{ij})^2 - r_j^2 \right\}$ satisfies $\text{ord}_\zeta(f) = \text{ord}_{\zeta'}(f) = 1$, $\text{ord}_{\zeta_i}(f) = 2$ ($1 \leq i \leq n-1$), and $\text{ord}_z(f) = 0$ for $z \in RP(K) \setminus \{\zeta_i\}$. This f is the desired element of $K^\times$.

Thus we have proved $Br(K)/Br_1(K) \simeq (\mathbb{Z}/2\mathbb{Z})_0^{RP(K) \setminus \{z_1, \dots, z_\nu\}}$, so combining with the result for $Br_1(K)$, we get $Br(K) \simeq (\mathbb{Z}/2\mathbb{Z})_0^{RP(K)}$.

Remark K satisfies Hasse's principle as a result of the following lemma.

Let $\square K$ be the set of all sums of squares, $\square K = \{\sum x_i^2 | x_i \in K\}$.

Lemma Let $K = \mathbb{R}(X, Y)$ be an algebraic function field over $\mathbb{R}$.

(1) For $f \in K^\times$, $f \in \square K$ if and only if $f(z) \geq 0$ for $\forall z \in RP(K)$. Especially, if $RP(K) = \emptyset$ then $\square K = K$.

(2) Every element of $\square K$ can be written as a sum of two squares.

We shall omit the proof here, and refer to [6], Th.3.2, Chap.3 and Th.2.1, Chap.4.

Corollary $K = \mathbb{R}(X, Y)$ satisfies Hasse's principle.

Proof $D = \{-1, f\}$ is locally trivial if and only if $f(z) \geq 0$ for $\forall z \in RP(K)$, which is equivalent to $f \in K^2 + K^2 = N_{K(\sqrt{-1})/K}(K(\sqrt{-1})^\times)$, hence $D = \{-1, f\}$ is trivial.

3. Approximation in idele groups.

Let R be a Dedekind domain, and K be its quotient field. Every prime ideal p of R defines the p -adic valuation on K . This is called a prime valuation. Besides p -adic valuations, we often consider some others, which are called non-prime valuations. For instance, all valuations trivial on $k^\times$ when K is an algebraic function field over the constant field k .

We define the adèle ring R_A of R by $R_A = \prod_p R_p$, where p runs over all prime valuations and R_p denotes the completion of R at the place p . Also we define the adèle ring K_A of K by $K_A = K \otimes_R R_A \simeq \bigcup_S (\prod_{p \in S} K_p \times \prod_{p \notin S} R_p)$ where S runs over all finite set of prime valuations. The idele group $K_A^\times$ is defined as the group of invertible elements of K_A . It is written in the form of $K_A^\times = \bigcup_S (\prod_{p \in S} K_p^\times \times \prod_{p \notin S} R_p^\times)$.

The fundamental system of neighbourhoods of 0 in K_A is given by $\{V(S, n)\}$, where

$$V(S, n) = \prod_{p \in S} p^n R_p \times \prod_{p \notin S} R_p.$$

Similarly, the fundamental system of neighbourhoods of 1 in $K_A^\times$ is given by $\{U(S, n)\}$,

where

$$U(S, n) = \prod_{p \in S} (1 + p^n R_p) \times \prod_{p \notin S} R_p^\times.$$

Let D be a central division algebra over K . A finitely generated R -submodule of D is called an R -lattice, and if it spans D as a K -vector space, it is called a full R -lattice. An R -lattice is called an R -order, if it is a subring including 1 (= the unit element of D).

The adèle ring $D_{\mathbf{A}}$ of D is defined by $D_{\mathbf{A}} = D \otimes_K K_{\mathbf{A}}$. It is written in the form of $D_{\mathbf{A}} = \bigcup_S (\prod_{p \in S} D_p \times \prod_{p \notin S} \Gamma_p)$, where Γ is a full R -order of D and $\Gamma_p = \Gamma \otimes_R R_p$. The idele group $D_{\mathbf{A}}^\times$ is defined similarly. The fundamental system of neighbourhoods of 1 in $D_{\mathbf{A}}^\times$ is given by

$$U(S, n) = \prod_{p \in S} (1 + p^n \Gamma_p) \times \prod_{p \notin S} \Gamma_p^\times.$$

D is diagonally imbedded into $D_{\mathbf{A}}$, and $D^\times$ is diagonally imbedded into $D_{\mathbf{A}}^\times$. D is dense in $D_{\mathbf{A}}$ (by chinese remainder theorem), but $D^\times$ is not dense in $D_{\mathbf{A}}^\times$. But $D^\times$ may be dense in some subgroup of $D_{\mathbf{A}}^\times$.

Let $\mathbf{N}_{D/K}$ be the reduced norm $D \rightarrow K$. $\mathbf{N}_{D/K}$ maps $D^\times$ homomorphically into $K^\times$. We shall denote its kernel by $D^{(1)}$. $\mathbf{N}_{D/K}$ is uniquely extended as a $K_{\mathbf{A}}$ -valued polynomial function on $D_{\mathbf{A}}$. This extension is denoted by the same symbol $\mathbf{N}_{D/K}$, and its kernel in $D_{\mathbf{A}}^\times$ is denoted by $D_{\mathbf{A}}^{(1)}$.

Eichler theorem ascertains that for global fields, $D^{(1)}$ is dense in $D_{\mathbf{A}}^{(1)}$ (in the topology of $D_{\mathbf{A}}^\times$) if and only if D_v is not a division algebra for some non-prime v .

For global fields, we have also $D^{(1)} = [D^\times, D^\times]$, the commutator group of $D^\times$. But for a general K , this relation does not hold (For instance, Platonov [3]).

For a general K , in the connection with the cancellation problem of Γ , it seems natural to consider $[D^\times, D^\times]$ rather than $D^{(1)}$. Thus we define the strong approximation property as follows: A central division algebra D is said to have strong approximation property if $[D^\times, D^\times]$ is dense in $[D_{\mathbf{A}}^\times, D_{\mathbf{A}}^\times]$. To find a necessary and sufficient condition for strong approximation property is a generalization of Eichler theorem to a general case.

In the connection with the cancellation problem of Γ , we consider a little weaker approximation property. D is said to have $D^\times$ -approximation, if the closure of $D^\times$ (in the

topology of $D_{\mathbf{A}}^{\times}$) contains $[D_{\mathbf{A}}^{\times}, D_{\mathbf{A}}^{\times}]$. D is said to have $R_{\mathbf{A}}^{\times}D^{\times}$ -approximation, if the closure of $R_{\mathbf{A}}^{\times}D^{\times}$ contains $[D_{\mathbf{A}}^{\times}, D_{\mathbf{A}}^{\times}]$. (Both of $D^{\times}$ and $R_{\mathbf{A}}^{\times}$ are contained in $D_{\mathbf{A}}^{\times}$, so $R_{\mathbf{A}}^{\times}D^{\times} \subset D_{\mathbf{A}}^{\times}$.) The last and weakest approximation property is necessary and sufficient for the cancellation of every full R -order Γ of D (namely $\Gamma \oplus \Gamma \simeq L \oplus \Gamma$ implies $\Gamma \simeq L$, the isomorphism being as Γ -lattices).

4. Eichler theorem for $\mathbb{R}(X)$.

In §1 we have seen that $D = \{-1, f\}$ is trivial at the non-prime place ∞ if and only if f is monic of even degree.

Theorem 3 If D_{∞} is not trivial, then $D^{\times}$ is discrete in $D_{\mathbf{A}}^{\times}$ and $R_{\mathbf{A}}^{\times}D^{\times}$ is closed in $D_{\mathbf{A}}^{\times}$.

Corollary If D_{∞} is not trivial, then $R_{\mathbf{A}}^{\times}D^{\times}$ -approximation property does not hold.

Proof of Corollary It suffices to show $[D_{\mathbf{A}}^{\times}, D_{\mathbf{A}}^{\times}] \not\subset R_{\mathbf{A}}^{\times}D^{\times}$. For a real place a , we shall identify $D_a^{\times}$ with the subgroup $D_a^{\times} \times \prod_{p \neq a} (1)_p$ of $D_{\mathbf{A}}^{\times}$. It is clear that $[D_{\mathbf{A}}^{\times}, D_{\mathbf{A}}^{\times}] \cap D_a^{\times} = [D_a^{\times}, D_a^{\times}]$. Since D_a is a quaternion (or a matrix) algebra over K_a , we have $[D_a^{\times}, D_a^{\times}] = D_a^{(1)}$, so that $[D_a^{\times}, D_a^{\times}] \not\subset K_a^{\times}$.

On the other hand, if $x = (x_p) \in R_{\mathbf{A}}^{\times}D^{\times} \cap D_a^{\times}$, then we have $\exists d \in D^{\times}, \forall p$ (prime place), $\exists r_p \in R_p^{\times}, x_p = r_p d$. For $p \neq a$, we have $x_p = 1$ so that $d = r_p^{-1} \in R_p^{\times} \subset K_p^{\times}$, so that $d \in D^{\times} \cap K_p^{\times} = K^{\times}$, hence $x_a = r_a d \in R_a^{\times}K^{\times} \subset K_a^{\times}$. This assures $R_{\mathbf{A}}^{\times}D^{\times} \cap D_a^{\times} \subset K_a^{\times}$ so that $[D_{\mathbf{A}}^{\times}, D_{\mathbf{A}}^{\times}] \not\subset R_{\mathbf{A}}^{\times}D^{\times}$.

Proof of Theorem 3 $D = \{-1, f\}$ means that

$$D = K + Ki + Kj + Kij$$

$$i^2 = -1, j^2 = f, ji = -ij.$$

Then $\Gamma = R + Ri + Rj + Rij$ is a full R -order of D ($K = \mathbb{R}(X)$, $R = \mathbb{R}[X]$).

A fundamental neighbourhood of 1 in $D_{\mathbf{A}}^{\times}$ is $U(g) = \prod_p (1 + g\Gamma_p) \cap \prod_p \Gamma_p^{\times}$ for $g \in R$ and we have $U(g) \cap D^{\times} = (1 + g\Gamma) \cap \Gamma^{\times}$, so the first half of Theorem 3 is proved if

$$(1 + g\Gamma) \cap \Gamma^\times = (1).$$

Suppose that $d = \varphi_1 + \varphi_2 i + \varphi_3 j + \varphi_4 ij \in (1 + g\Gamma) \cap \Gamma^\times$, $\varphi_i \in R$. This means that $\varphi_1 \equiv 1 \pmod{g}$, $\varphi_i \equiv 0 \pmod{g}$ for $i \geq 2$, and $\prod_{D/K}(d) = \varphi_1^2 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2) \in R^\times = \mathbb{R}^\times$. If $g \in R \setminus R^\times$, substituting a zero of g , we see that $\prod_{D/K}(d) = 1$.

Since each φ_i^2 has, if not zero, a positive coefficient of the highest degree term, such terms of φ_1^2 and φ_2^2 (resp. φ_3^2 and φ_4^2) do not cancel.

From $\varphi_1^2 + \varphi_2^2 - 1 = f(\varphi_3^2 + \varphi_4^2)$, if f is of odd degree, both hand sides should be zero. This implies that $\varphi_3 = \varphi_4 = 0$ and $\varphi_1, \varphi_2 \in \mathbb{R}$, which implies $\varphi_2 = 0$ and $\varphi_1 = 1$ because φ_2 is a multiple of g .

If f is of even degree with a negative coefficient of the highest degree term, then the highest degree terms of $\varphi_1^2 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2)$ do not cancel, so that we have $\forall i, \varphi_i \in \mathbb{R}$. This again implies $\varphi_i = 0$ for $i \geq 2$, and so $\varphi_1 = 1$.

Thus the first half of Theorem 3 has been proved. Similar discussions show that $(R + g\Gamma)^\times = R^\times$, if $g \in R \setminus R^\times$.

$\Gamma_g = R + g\Gamma$ is a full R -order of D , and $(\Gamma_g)_\mathbb{A}^\times = \prod_p (R_p + g\Gamma_p)^\times$ is an open subgroup of $D_\mathbb{A}^\times$, so $(\Gamma_g)_\mathbb{A}^\times D^\times$ is open and closed, hence $\bigcap_g (\Gamma_g)_\mathbb{A}^\times D^\times$ is a closed subgroup of $D_\mathbb{A}^\times$, containing $R_\mathbb{A}^\times D^\times$.

We shall show the inverse inclusion. Take any $x \in \bigcap_g (\Gamma_g)_\mathbb{A}^\times D^\times$, then $\forall g, \exists \gamma_g \in (\Gamma_g)_\mathbb{A}^\times, \exists d_g \in D^\times, x = \gamma_g d_g$. Since $(\Gamma_g)_\mathbb{A}^\times \cap D^\times = (R + g\Gamma)^\times = R^\times$, d_g is determined modulo $R^\times$, so if g_1 is a multiple of g , then d_{g_1} differs from d_g only modulo $R^\times$. This implies that we can choose d_g independently of g , thus $\exists d \in D^\times, xd^{-1} \in \bigcap_g (\Gamma_g)_\mathbb{A}^\times$.

But we have $R_\mathbb{A}^\times = \bigcap_g (\Gamma_g)_\mathbb{A}^\times$, because $\forall p, \bigcap_g (R_p + g\Gamma_p)^\times = R_p^\times$. Thus the proof of the second half of Theorem 3 is completed.

Theorem 4 If D_∞ is trivial, hence if f is monic of even degree, then $[D^\times, D^\times] = D^{(1)}$ is dense in $[D_\mathbb{A}^\times, D_\mathbb{A}^\times]$.

This theorem is divided into the following two parts.

Theorem 4.1 If f is monic of even degree, then for $g, h \in R$ such that $(h, gf) = 1$, we

have

$$(1 + g\Gamma) \cap (i + h\Gamma) \cap \Gamma^\times \neq \phi.$$

Theorem 4.2 The conclusion part of Theorem 4.1 is equivalent to strong approximation property.

Proof of Theorem 4.1 It suffices to show the existence of $\varphi_i \in R, 1 \leq i \leq 4$ such that

$$\begin{aligned} \varphi_1 &\equiv 1 \pmod{g}, & \varphi_i &\equiv 0 \pmod{g}, & 2 \leq i \leq 4 \\ \varphi_2 &\equiv 1 \pmod{h}, & \varphi_i &\equiv 0 \pmod{h}, & i = 1, 3, 4, \text{ and} \end{aligned}$$

$$(1) \quad \varphi_1^2 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2) = 1.$$

Put $\varphi_1 = 1 + g^2 fu_1, \varphi_2 = g^2 fu_2, \varphi_i = gu_i (i = 3, 4)$, then the required congruence modulo g is automatically satisfied. Substituting them into (1) and dividing both sides by $g^2 f$, we get

$$(2) \quad 2u_1 + g^2 fu_1^2 + g^2 fu_2^2 - (u_3^2 + u_4^2) = 0.$$

Since $(h^2, g^2 f) = 1$, $g^2 f$ is invertible in $R/h^2 R$, so there exist $\psi, \psi' \in R$ such that

$$g^2 f\psi = 1 + h^2 \psi'.$$

Put $u_1 = -\psi + h^2 v_1, u_2 = \psi + h^2 v_2, u_i = hv_i (i = 3, 4)$, then the required congruence modulo h is automatically satisfied. Substituting them into (2), we get

$$-2\psi + 2h^2 v_1 + g^2 f\{2\psi^2 + 2h^2 \psi(v_2 - v_1) + h^4(v_1^2 + v_2^2)\} = h^2(v_3^2 + v_4^2).$$

Since $-2\psi + 2g^2 f\psi^2 = -2\psi(1 - g^2 f\psi) = 2h^2 \psi\psi'$, we have

$$(3) \quad 2\psi\psi' + 2(1 - g^2 f\psi)v_1 + 2g^2 f\psi v_2 + g^2 fh^2(v_1^2 + v_2^2) = v_3^2 + v_4^2.$$

Put $v_1 = (1 - g^2 f\psi)w$ and $v_2 = g^2 f\psi w$, then we get

$$(4) \quad 2\psi\psi' + \{(1 - g^2 f\psi)^2 + (g^2 f\psi)^2\}(2w + g^2 fh^2 w^2) = v_3^2 + v_4^2.$$

A polynomial $P \in R = \mathbb{R}[X]$ belongs to $R^2 + R^2$, if and only if $P(a) \geq 0$ for $\forall a \in \mathbb{R}$, as shown in the proof of Theorem 1. So it suffices to show that the left hand side of (4) is everywhere non-negative for some $w \in R$.

Put $2\psi\psi' = F$ and $g^2fh^2 = G$, then $(1 - g^2f\psi)^2 + (g^2f\psi)^2 = 1 - 2g^2f\psi(1 - g^2f\psi) = 1 + 2g^2f\psi h^2\psi' = 1 + FG$, so we have

$$(5) \quad F + (1 + FG)(2w + Gw^2) \geq 0.$$

The above calculation also shows $1 + FG \geq \frac{1}{2}$, namely $FG \geq -\frac{1}{2}$. Since f is monic of even degree, we have $\lim_{t \rightarrow \pm\infty} G(t) = \infty$ so that $\exists M > 0, \forall t \in \mathbb{R}, G(t) \geq -M$. Since $\{t \mid G(t) \leq 0\}$ is compact, F is bounded there, so $\exists N > 0, |F(t)| \leq N$ for $G(t) \leq 0$.

The left hand side of (5) is zero for

$$w = \frac{1}{G} \left\{ -1 \pm (1 + FG)^{-\frac{1}{2}} \right\}.$$

Since $(1+t)^{-\frac{1}{2}} \leq 1 - \frac{t}{2} + \frac{3}{\sqrt{2}}t^2$ for $t \geq -\frac{1}{2}$, if we set $w = -\frac{F}{2} + \frac{3}{\sqrt{2}}F^2G$, then (5) is satisfied for $G \geq 0$. Let P be an everywhere positive polynomial of two variables s and t , then $w = -\frac{F}{2} + \frac{3}{\sqrt{2}}F^2G + P(G, FG)$ satisfies (5) for $G \geq 0$.

The condition (5) is satisfied also for $G < 0$, if

$$(6) \quad -1 \leq -\frac{t}{2} + \frac{3}{\sqrt{2}}t^2 + sP(s, t) \leq -1 + (1+t)^{-\frac{1}{2}}$$

on $\Delta = \{(s, t) \mid -M \leq s \leq 0, t \geq -\frac{1}{2}, |t| \leq N|s|\}$. The condition (6) is satisfied if

$$\varepsilon \geq \frac{1}{s} \left\{ 1 - (1+t)^{-\frac{1}{2}} - \frac{t}{2} + \frac{3}{\sqrt{2}}t^2 \right\} + P(s, t) \geq 0$$

on Δ , where $\varepsilon \leq (1 + NM)^{-\frac{1}{2}}/M$. Since $\alpha(s, t) = \frac{1}{s} \left\{ 1 - (1+t)^{-\frac{1}{2}} - \frac{t}{2} + \frac{3}{\sqrt{2}}t^2 \right\}$ is non-positive and continuous on Δ (it is continuous at $(0, 0)$ because of $|t| \leq N|s|$), such a polynomial $P(s, t)$ exists by virtue of polynomial approximation theorem of Weierstrass. $P(s, t)$ can be assumed everywhere positive, because we can put $P = Q^2 + \frac{\varepsilon}{2}$, Q being an approximating polynomial of $\sqrt{|\alpha(s, t)|}$. Thus Theorem 4.1 has been proved.

Proof of Theorem 4.2 Let H be the closure of $[D^\times, D^\times] = D^{(1)}$ in $D_{\mathbb{A}}^\times$. Let p_0 be a prime place where D is unramified, and let $i_{p_0} = (1, \dots, 1, i, 1, \dots) \in D_{\mathbb{A}}^\times$ be the element of $D_{\mathbb{A}}^\times$ whose p_0 -coordinate is i , while other coordinates are 1.

The proof is completed by the following steps, which are slight modifications of ones given in [1] §51.

Step 1 The conclusion part of Theorem 4.1 is equivalent to that $\forall p_0$ (where D_{p_0} is unramified), $i_{p_0} \in H$ (note that $i_{p_0} \in D_{p_0}^{(1)} = [D_{p_0}^\times, D_{p_0}^\times] \subset [D_{\mathbb{A}}^\times, D_{\mathbb{A}}^\times]$).

Step 2 Identify $D_{p_0}^{(1)}$ with a subgroup $D_{p_0}^{(1)} \times \prod_{p \neq p_0} (1)_p$ of $D_{\mathbb{A}}^\times$, then $H \cap D_{p_0}^{(1)}$ is a closed normal subgroup of $D_{p_0}^{(1)}$.

Step 3 If D is unramified at p_0 , then $i_{p_0} \in H$ implies $D_{p_0}^{(1)} \subset H$.

If D_{p_0} is a matrix algebra, the assertion is a result of simplicity of $PSL(2, K_{p_0})$. If D_{p_0} is an unramified quaternion algebra, since $x = a + bi + cj + dij \in D_{p_0}^{(1)}$ satisfies $x^2 - 2ax + 1 = 0$, the condition $x \in H$ depends only on a . (Here we identify $x \in D_{p_0}^{(1)}$ with $x_{p_0} = (1, \dots, 1, x, 1, \dots) \in D_{\mathbb{A}}^\times$.)

Take any $x = a + bi + cj + dij \in D_{p_0}^{(1)}$. Since $b^2 + c^2 + d^2$ has a root in K_{p_0} , we have $\exists e \in K_{p_0}, b^2 + c^2 + d^2 = e^2$. If $i \in H$, then $-ai + ej \in H$, therefore $i(-ai + ej) = a + eij \in H$, hence $x \in H$. This means $D_{p_0}^{(1)} \subset H$.

Step 4 Assume the conclusion part of Theorem 4.1. For a finite set S of prime places, we have $\prod_{p \in S} D_p^{(1)} \times \prod_{p \notin S} (1)_p \subset H$.

If D is unramified on S , the assertion is a consequence of Step 3.

Let S_0 be the set of all prime places where D is ramified. The assertion for $S = S_0$ follows from the fact that $D^{(1)}$ is dense in $\prod_{p \in S_0} D_p^{(1)}$ in the product topology of $D_p^\times$.

Step 5 $\bigcup_S \left(\prod_{p \in S} D_p^{(1)} \times \prod_{p \notin S} (1)_p \right)$ is dense in $[D_{\mathbb{A}}^\times, D_{\mathbb{A}}^\times]$.

Combining the five assertions above, we complete the proof of Theorem 4.2.

5. Eichler theorem for $\mathbb{R}(X, Y)$.

For an algebraic function field $K = \mathbb{R}(X, Y)$, we shall fix a set P of valuations (which are trivial on $\mathbb{R}^\times$). A valuation $v \in P$ is called a prime place and $v \notin P$ is called a non-

prime place. We assume that there exists a non-prime place. Then, $R_P = \{x \in K \mid \forall v \in P, v(x) \leq 1\}$ is a Dedekind domain and K is its quotient field. A prime ideal of R_P is given by $p_v = \{x \in R_P \mid v(x) < 1\}$ for $v \in P$.

The adèle ring and the idele group are constructed using prime places only. We shall write R instead of R_P .

We consider the following property(E):

(E) A central division algebra D over K has strong approximation property, if D is trivial at some non-prime place.

The converse of the property(E) holds always as shown below.

Theorem 5 If a central division algebra D is non-trivial at every non-prime place, then D does not have $R_A^\times D^\times$ -approximation property.

Remark Before proving this theorem, we shall mention about the product formula. The formula is expressed as follows using ord_v ; $v(x) = \theta^{\text{ord}_v(x)}$ ($0 < \theta < 1$).

$$\forall x \in K^\times, \sum_{v:\text{real}} \text{ord}_v(x) + 2 \sum_{v:\text{imag.}} \text{ord}_v(x) = 0,$$

where the sum is taken over all places, prime or not.

Proof Similar discussions as the proof of Theorem 3 show that it suffices to prove that

$$(R + g\Gamma)^\times = R^\times \text{ for } g \in R \setminus R^\times.$$

Let $D = \{-1, f\}$, $f \in R$. The assumption of Theorem 5 means that all non-prime places are real and that for every non-prime place v , $\text{ord}_v(f)$ is odd or $\text{ord}_v(f)$ is even with a negative coefficient of the lowest degree term with respect to the prime element π_v .

Suppose that $\varphi_1 + \varphi_2 i + \varphi_3 j + \varphi_4 ij \in (R + g\Gamma)^\times$, then $\varphi_1 \in R$, $\varphi_i \in gR$ ($2 \leq i \leq 4$), and $\varphi_1^2 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2) \in R^\times$. Put $\varphi = \varphi_1^2 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2)$, then $\varphi \in R^\times$ implies $\text{ord}_v(\varphi) = 0$ for every prime place v . As for a non-prime place v , the assumption on f implies that the lowest degree terms do not cancel, so that $\text{ord}_v(\varphi) = \text{Min}(2\text{ord}_v(\varphi_1), 2\text{ord}_v(\varphi_2), \text{ord}_v(f) + 2\text{ord}_v(\varphi_3), \text{ord}_v(f) + 2\text{ord}_v(\varphi_4))$, if $\varphi_i \neq 0$.

Combining this with the product formula, we have

$$\sum_{\text{non-prime}} \text{ord}_v(\varphi_i) \geq 0 \quad (i = 1, 2),$$

$$\sum_{\text{non-prime}} \text{ord}_v(\varphi_i) \geq \frac{1}{2} \sum_{\text{prime}} \alpha_v \text{ord}_v(f) \quad (i = 3, 4)$$

where $\alpha_v = 1$ for a real v and $\alpha_v = 2$ for an imaginary v . Since $\varphi_i \in R$ and $f \in R$, we have $\text{ord}_v(\varphi_i) \geq 0$ and $\text{ord}_v(f) \geq 0$ for a prime place v , hence again from the product formula, we must have $\text{ord}_v(\varphi_i) = 0$ for every prime place v . This means $\varphi_i \in R^\times$. For $i \geq 2$, this contradicts with $\varphi_i \in gR$, so we must have $\varphi_i = 0$, which in turn implies $\varphi_1 \in R^\times$. This completes the proof of $(R + g\Gamma)^\times = R^\times$.

Remark Property(E) depends not only on K , but also on R , or equivalently on the choice of non-prime places. However:

Theorem 6 (1) Suppose that property(E) holds whenever R has only one non-prime place, then it holds for any R .

(2) For the rational function field $K = \mathbb{R}(X)$, property(E) holds for any R .

Proof of (1) Let $P(R)$ be the set of all prime places for the Dedekind domain R . Then $P(R') \subset P(R)$ implies $R \subset R'$. We shall denote the idele group of D with respect to R by $D_A^\times(R)$. Then $P(R) = P(R') \amalg P(R_1)$ implies that $D_A^\times(R)$ is the product topological group of $D_A^\times(R')$ and $D_A^\times(R_1)$, because of $D_A^\times(R) = \bigcup_S \left(\prod_{v \in S} D_v^\times \times \prod_{v \in P(R) \setminus S} \Gamma_v^\times \right)$ where S runs over all finite subsets of $P(R)$.

$D^\times$ is imbedded diagonally in $D_A^\times$, and strong approximation property means precisely that the image $i_R(D^{(1)})$ is dense in $[D_A^\times(R), D_A^\times(R)]$.

If $P(R') \subset P(R)$, then the projection $D_A^\times(R) \rightarrow D_A^\times(R')$ maps $i_R(D^{(1)})$ onto $i_{R'}(D^{(1)})$ and $[D_A^\times(R), D_A^\times(R)]$ onto $[D_A^\times(R'), D_A^\times(R')]$. Therefore, if $i_R(D^{(1)})$ is dense in $[D_A^\times(R), D_A^\times(R)]$, then $i_{R'}(D^{(1)})$ is dense in $[D_A^\times(R'), D_A^\times(R')]$.

Now suppose that D is trivial at some non-prime place v of a given R . Let P_0 be the set of all places other than v , and suppose that property(E) holds for R_0 corresponding to P_0 , then $i_{R_0}(D^{(1)})$ is dense in $[D_A^\times(R_0), D_A^\times(R_0)]$, hence $i_R(D^{(1)})$ is dense in $[D_A^\times(R), D_A^\times(R)]$, so property(E) holds for R .

Remark The proof of Theorem 4.2 does work for a general algebraic function field $K = \mathbb{R}(X, Y)$ and its Dedekind domain R . So, strong approximation property holds for $D = \{-1, f\}$, if $(1 + g\Gamma) \cap (i + h\Gamma) \cap \Gamma^\times \neq \emptyset$ for $\forall g, h \in R$ such that $(gf, h) = 1$.

Also the proof of Theorem 4.1 works partially. For $\psi, \psi' \in R$ such that $g^2 f \psi = 1 + h^2 \psi'$, put $F = 2\psi\psi'$ and $G = g^2 f h^2$. Then, we have $(1 + g\Gamma) \cap (i + h\Gamma) \cap \Gamma^\times \neq \emptyset$ if $\exists w \in R$, $F + (1 + FG)(2w + Gw^2) \in R^2 + R^2$.

Suppose that R has only one non-prime place v , then $f \in R$ means that f does not have a pole other than v . If v is real and D_v is trivial, then $\text{ord}_v(f)$ is even and $f(z)$ is positive near v . Since $RP(K)$ is compact, this implies that f , hence G , is bounded from below on $RP(K)$, and that F is bounded on $\{z \in RP(K) | G(z) \leq 0\}$. If v is imaginary, then both F and G are bounded on $RP(K)$. So, similar discussions as the proof of Theorem 4.1 show that $\exists w \in R$, $F + (1 + FG)(2w + Gw^2) \geq 0$ on $RP(K)$.

The proof for general K fails only because the condition " $\varphi \in R$ and $\varphi \geq 0$ on $RP(K)$ " does not imply $\varphi \in R^2 + R^2$. Since Hasse's principle is satisfied, $\varphi \in K^2 + K^2$ is assured, but $\varphi \in R^2 + R^2$ is not concluded. We shall give a counter example for an elliptic function field $K = \mathbb{R}(X, Y)$, $Y^2 = (X - a)(X - b)(X - c)$. If $\alpha \in \mathbb{R}$ is smaller than $\min(a, b, c)$, then we have $X - \alpha > 0$ on $RP(K)$. $X - \alpha$ has a double pole at the non-prime place v , while an element of $R^2 + R^2 = N_{K(\sqrt{-1})/K}(R + \sqrt{-1}R)$ should have $\text{ord}_v \leq -4$.

Proof of Theorem 6 (2)

Let $K = \mathbb{R}(X)$ and suppose that R has only one non-prime place v .

If $R \neq \mathbb{R}[X]$, then v corresponds to an irreducible polynomial p , and $\varphi \in R$ is equivalent to $\varphi = g/p^\nu$, $g \in \mathbb{R}[X]$ and $\deg g \leq \nu \deg p$. Here we can assume that ν is even. Then $\varphi \geq 0$ on $RP(K)$ implies $g \geq 0$ on $RP(K)$, so g is of even degree and can be written as $g = g_1^2 + g_2^2$, $g_i \in \mathbb{R}[X]$, $\deg g_i \leq \frac{1}{2} \deg g$. Therefore $\varphi = (g_1/p^{\nu/2})^2 + (g_2/p^{\nu/2})^2$ and $\deg g_i \leq \frac{\nu}{2} \deg p$, so that $\varphi \in R^2 + R^2$.

From the remark above, this completes the proof of Theorem 6 (2).

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