

PROBLEMS ON QUANTUM GROUPS

Reiji TOMATSU
Graduate School of Mathematical Sciences,
The University of Tokyo

1. QUANTUM GROUPS

Throughout this survey, we will mainly treat quantum groups of compact or discrete type. Our standard references are [2, 9, 19, 45]. We denote by $\otimes$ minimal tensor products or spatial tensor products for C^* -algebras or von Neumann algebras, respectively. The leg notations are frequently used. For example, let $T \in B(H \otimes H)$, where H is a Hilbert space. Set the transposition of the i -th and j -th tensor components, $\sigma_{ij} \in B(H \otimes H \otimes H)$ for $i, j = 1, 2, 3$. Then $T_{12} = T \otimes 1$, $T_{13} = \sigma_{23}T_{12}\sigma_{23}$ and so on.

1.1. Compact quantum groups

The following definition of a compact quantum group has been introduced by S. L. Woronowicz [45]:

Definition 1.1 (Woronowicz). A *compact quantum group* (c.q.g.) $\mathbb{G}$ is a pair $(C(\mathbb{G}), \delta)$ that satisfies the following conditions:

- (1) $C(\mathbb{G})$ is a separable unital C^* -algebra;
- (2) (Coproduct) The map $\delta: C(\mathbb{G}) \rightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$ is a coproduct, i.e. it is a faithful unital $*$ -homomorphism satisfying the co-associativity condition,

$$(\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \delta) \circ \delta;$$

- (3) (Cancellation property) The vector spaces $\delta(C(\mathbb{G}))(\mathbb{C} \otimes C(\mathbb{G}))$ and $\delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes \mathbb{C})$ are dense in $C(\mathbb{G}) \otimes C(\mathbb{G})$.

Example 1.2. A compact group $\mathbb{G}$ is regarded as a compact quantum group. Indeed, via the identification $C(\mathbb{G}) \otimes C(\mathbb{G}) = C(\mathbb{G} \times \mathbb{G})$, a coproduct δ is defined by

$$\delta(x)(r, s) := x(rs) \text{ for all } x \in C(\mathbb{G}), r, s \in \mathbb{G}.$$

The cancellation property means $rs = rt$ or $sr = tr$ imply $s = t$ for $r, s, t \in \mathbb{G}$. Note that a compact semigroup with cancellation property is a compact group.

As in a compact group theory, the following state called *Haar state* plays an important role.

Theorem 1.3 (Woronowicz). *There uniquely exists a state $h \in C(\mathbb{G})^*$ such that*

$$(\text{id} \otimes h)(\delta(a)) = h(a)1 = (h \otimes \text{id})(\delta(a)) \text{ for all } a \in C(\mathbb{G}).$$

1.2. Reduced quantum groups

We should note that h may not be faithful in general. For example, the full group C^* -algebra $C^*\mathbb{F}_2$ is a c.q.g. with a coproduct $\delta(r) = r \otimes r$ for $r \in \mathbb{F}_2$. The Haar state is given by $h(r) = 0$ if $r \neq e$. However, h is not faithful because $C^*\mathbb{F}_2 \not\cong C_{\text{red}}^*\mathbb{F}_2$.

Let $N_h := \{a \in C(\mathbb{G}) \mid h(a^*a) = 0\}$. Then it is known that N_h is in fact an ideal of $C(\mathbb{G})$, and we can consider the *reduced compact quantum group* $C(\mathbb{G}_{\text{red}}) := C(\mathbb{G})/N_h$ with a natural coproduct. By definition, h is faithful on $C(\mathbb{G}_{\text{red}})$.

Let $(L^2(\mathbb{G}), \Omega_h)$ be the GNS representation associated with the Haar state h , that is,

- $L^2(\mathbb{G})$ is a Hilbert space;
- $\Omega_h \in L^2(\mathbb{G})$ is the GNS cyclic vector, i.e. we have $L^2(\mathbb{G}) = \overline{C(\mathbb{G})\Omega_h}$ and $h(a) = (a\Omega_h, \Omega_h)$.

Note that N_h is precisely equal to the kernel of the GNS representation.

1.3. Multiplicative unitaries

From the bi-invariance of the state h , the following theorem holds:

Theorem 1.4. *There exist unitary operators $V, W \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ satisfying*

$$V(a\Omega_h \otimes \xi) = \delta(a)(\Omega_h \otimes \xi) \text{ for all } a \in C(\mathbb{G}), \xi \in L^2(\mathbb{G});$$

$$W^*(\xi \otimes a\Omega_h) = \delta(a)(\xi \otimes \Omega_h) \text{ for all } a \in C(\mathbb{G}), \xi \in L^2(\mathbb{G}).$$

Then V and W satisfy the following notable *pentagon equations*:

$$V_{12}V_{13}V_{23} = V_{23}V_{12}, \quad W_{12}W_{13}W_{23} = W_{23}W_{12}. \quad (1.1)$$

By definition, we have the following implementation formula:

$$V(a \otimes 1)V^* = \delta(a) = W^*(1 \otimes a)W \text{ for all } a \in C(\mathbb{G}_{\text{red}}). \quad (1.2)$$

1.4. Von Neumann closures of quantum groups

We denote by $L^\infty(\mathbb{G})$ the weak closure of $C(\mathbb{G}_{\text{red}})$ in $B(L^2(\mathbb{G}))$. The coproduct δ extends to the normal morphism from $L^\infty(\mathbb{G})$ into $L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$ through (1.2). Then the pair $(L^\infty(\mathbb{G}), \delta)$ is called the *von Neumann algebraic compact quantum group*. It is non-trivial that there exists a modular automorphism for h , and the Haar state $h(\cdot) = (\cdot\Omega_h, \Omega_h)$ is faithful on $L^\infty(\mathbb{G})$ [45].

1.5. Kac type quantum groups

Definition 1.5. A compact quantum group is said to be of *Kac type* when the Haar state is tracial, i.e. $h(ab) = h(ba)$ for all $a, b \in C(\mathbb{G})$.

A compact group or C^* -group algebra of a discrete group are typical examples of Kac type quantum groups, but they are not all. As an example, $SU_{-1}(n)$ is of Kac type, which is neither commutative nor co-commutative. Readers should note the first such example discovered by G. I. Kac and V. G. Paljutkin [17].

1.6. Representation theory

Definition 1.6. Let H be a Hilbert space. A unitary $v \in B(H) \otimes L^\infty(\mathbb{G})$ is called a (*right unitary*) *representation* if it satisfies

$$(\text{id} \otimes \delta)(v) = v_{12}v_{13}. \quad (1.3)$$

Similarly we can define a left representation. The above equality is the translation of the equality $v(rs) = v(r)v(s)$, $r, s \in \mathbb{G}$ in terms of a Hopf algebra.

Example 1.7. The multiplicative unitaries $V \in B(L^2(\mathbb{G})) \otimes L^\infty(\mathbb{G})$ and $W \in L^\infty(\mathbb{G}) \otimes B(L^2(\mathbb{G}))$ are right and left representations, respectively. Indeed using the pentagon equation (1.1), we have

$$(\text{id} \otimes \delta)(V) = V_{23}V_{12}V_{23}^* = (V_{12}V_{13}V_{23})V_{23}^* = V_{12}V_{13}.$$

Similarly we obtain $(\delta \otimes \text{id})(W) = W_{13}W_{23}$.

There are the following three operations:

- (**direct sum**)

$$v_1 \oplus v_2 := \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \in B(H_1 \oplus H_2) \otimes L^\infty(\mathbb{G});$$

- (**tensor product**)

$$v_1 \otimes v_2 := (v_1)_{13}(v_2)_{23} \in B(H_1 \otimes H_2) \otimes L^\infty(\mathbb{G});$$

- (**conjugation**) Let $v = (v_{ij})_{i,j \in I}$ be a matrix form of a representation. Consider

$$v^c := (v_{ij}^*)_{i,j \in I}$$

which may not be a unitary, but this still satisfies (1.3). In fact, if the dimension is finite, then it is unitarizable, i.e. there exists a positive invertible operator Q such that

$$\bar{v} := (Q^{1/2} \otimes 1)v^c(Q^{-1/2} \otimes 1)$$

is a unitary matrix, where Q is usually canonically chosen (see §1.9).

The important point is that the Peter-Weyl theory holds even for the quantum setting. Let us introduce the intertwiner space between unitary representations $v_i \in B(H_i) \otimes L^\infty(\mathbb{G})$, $i = 1, 2$,

$$\text{Mor}(v_1, v_2) := \{T \in B(H_1, H_2) \mid (T \otimes 1)v_1 = v_2(T \otimes 1)\}.$$

Definition 1.8. Let $v \in B(H) \otimes L^\infty(\mathbb{G})$ be a unitary representation.

- A unitary representation is said to be *irreducible* when $\text{Mor}(v, v) = \mathbb{C}1_H$.
- Let $w \in B(K) \otimes L^\infty(\mathbb{G})$ be a unitary representation. We say that v and w are *equivalent* if $\text{Mor}(v, w)$ contains a unitary.

Theorem 1.9 (Woronowicz). *The following hold:*

- (1) *An irreducible representation is finite dimensional;*
- (2) *A finite dimensional representation is the direct sum of irreducibles;*

- (3) Let $v \in B(H) \otimes L^\infty(\mathbb{G})$ be a finite dimensional representation. Then $v \in B(H) \otimes C(\mathbb{G})$.

Define the following subspace of $C(\mathbb{G})$ called the *smooth part*:

$$A(\mathbb{G}) := \text{span}\{v_{ij} \mid v \text{ finite dimensional representation}\}.$$

Theorem 1.10 (Woronowicz). *The following hold:*

- (1) $A(\mathbb{G})$ is a unital $*$ -subalgebra that is dense in $C(\mathbb{G})$;
- (2) The set $\{v_{\pi_{ij}}\}_{ij \in I_\pi, \pi \in \text{Irr}(\mathbb{G})}$ is a linear basis of $A(\mathbb{G})$;
- (3) The Haar state h is faithful on $A(\mathbb{G})$.

We denote by $\text{Irr}(\mathbb{G})$ the set of equivalent classes of irreducible representations of $\mathbb{G}$. For each $\pi \in \text{Irr}(\mathbb{G})$, we choose a corresponding representation $v_\pi \in B(H_\pi) \otimes L^\infty(\mathbb{G})$. Note that $\dim(H_\pi) < \infty$ from the previous theorem. The trivial one dimensional representation is denoted by 1 . For $\pi \in \text{Irr}(\mathbb{G})$, the conjugation is denoted by $\bar{\pi}$, which is the unique element of $\text{Irr}(\mathbb{G})$ such that $\text{Mor}(\pi \otimes \bar{\pi}, 1) \neq 0$ or $\text{Mor}(\bar{\pi} \otimes \pi, 1) \neq 0$.

Let us define the $\mathbb{Z}$ -module,

$$\mathcal{R}_\mathbb{G} := \bigoplus_{\pi \in \text{Irr}(\mathbb{G})} \mathbb{Z}\pi.$$

Setting $N_{\pi\rho}^\sigma := \dim \text{Mor}(\pi \otimes \rho, \sigma)$ for $\pi, \rho, \sigma \in \text{Irr}(\mathbb{G})$, we define the following product structure on $\mathcal{R}_\mathbb{G}$:

$$\pi \cdot \rho = \sum_{\sigma \in \text{Irr}(\mathbb{G})} N_{\pi\rho}^\sigma \sigma.$$

The $\mathbb{Z}$ -ring $\mathcal{R}_\mathbb{G}$ is called the *representation ring* of $\mathbb{G}$.

Definition 1.11. We say that a compact quantum group $\mathbb{G}$ has *commutative fusion rules* if $\mathcal{R}_\mathbb{G}$ is commutative.

1.7. Hopf algebra structure

By Theorem 1.10, we can introduce the maps $\varepsilon: A(\mathbb{G}) \rightarrow \mathbb{C}$ and $\kappa: A(\mathbb{G}) \rightarrow A(\mathbb{G})$ defined by

$$\begin{aligned} \varepsilon(v_{\pi_{ij}}) &= \delta_{ij} \quad \text{for } \pi \in \text{Irr}(\mathbb{G}), i, j \in I_\pi, \\ \kappa(v_{\pi_{ij}}) &= v_{\pi_{ji}}^* \quad \text{for } \pi \in \text{Irr}(\mathbb{G}), i, j \in I_\pi. \end{aligned}$$

Theorem 1.12 (Woronowicz). *The following hold:*

- (1) ε is a $*$ -homomorphism satisfying

$$(\varepsilon \otimes \text{id}) \circ \delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \delta \quad \text{on } A(\mathbb{G});$$

- (2) κ is an anti-homomorphism satisfying $\kappa(\kappa(a)^*)^* = a$ for $a \in A(\mathbb{G})$ and

$$m \circ (\kappa \otimes \text{id}) \circ \delta = \varepsilon = m \circ (\text{id} \otimes \kappa) \circ \delta \quad \text{on } A(\mathbb{G}),$$

where $m: A(\mathbb{G}) \otimes A(\mathbb{G}) \rightarrow A(\mathbb{G})$ is the multiplication.

So, $A(\mathbb{G})$ has a Hopf $*$ -algebra structure.

1.8. Modular objects

We introduce the *Woronowicz characters* $\{f_z\}_{z \in \mathbb{C}}$ on $A(\mathbb{G})$ [45, Theorem 2.4]. The multiplicative functional $f_z: A(\mathbb{G}) \rightarrow \mathbb{C}$ is uniquely determined by the following properties:

- (1) $f_0 = \varepsilon$;
- (2) For any $a \in A(\mathbb{G})$, the function $\mathbb{C} \ni z \mapsto f_z(a) \in \mathbb{C}$ is entirely holomorphic;
- (3) $(f_{z_1} \otimes f_{z_2}) \circ \delta = f_{z_1+z_2}$ for any $z_1, z_2 \in \mathbb{C}$;
- (4) For any $z \in \mathbb{C}$ and $a \in A(\mathbb{G})$, $f_z(\kappa(a)) = f_{-z}(a)$, $f_z(a^*) = \overline{f_{-\bar{z}}(a)}$;
- (5) For any $a \in A(\mathbb{G})$, $\kappa^2(a) = (f_1 \otimes \text{id} \otimes f_{-1})((\delta \otimes \text{id})(\delta(a)))$;
- (6) For any $a, b \in A(\mathbb{G})$, $h(ab) = h(b(f_1 \otimes \text{id} \otimes f_1)((\delta \otimes \text{id})(\delta(a))))$.

The modular automorphism group $\{\sigma_t^h\}_{t \in \mathbb{R}}$ is given by

$$\sigma_t^h(x) = (f_{it} \otimes \text{id} \otimes f_{it})((\delta \otimes \text{id})(\delta(x))) \quad \text{for all } t \in \mathbb{R}, x \in A(\mathbb{G}).$$

We define the following map $\tau_t: A(\mathbb{G}) \rightarrow A(\mathbb{G})$ by

$$\tau_t(x) = (f_{it} \otimes \text{id} \otimes f_{-it})((\delta \otimes \text{id})(\delta(x))) \quad \text{for all } t \in \mathbb{R}, x \in A(\mathbb{G}).$$

Then $\{\tau_t\}_{t \in \mathbb{R}}$ is a one-parameter automorphism group on $A(\mathbb{G})$ and it is called the *scaling automorphism group*. Since the Haar state h is invariant under the $*$ -preserving maps σ_t^h and τ_t , we can extend them to the maps on $C(\mathbb{G}_{\text{red}})$, and on $L^\infty(\mathbb{G})$.

1.9. Quantum Peter-Weyl theorem

Let $v \in B(H) \otimes A(\mathbb{G})$ be a finite dimensional representation. We set

$$Q_v := (\text{id} \otimes f_1)(v),$$

which is an invertible positive operator on H . For $\pi \in \text{Irr}(\mathbb{G})$, we write Q_π instead of Q_{v_π} . When $\mathbb{G}$ is of Kac type, then $Q_v = 1$ for any representation v .

Definition 1.13. The value $\text{Tr}_H(Q_v)$ is called the *quantum dimension* of v , and denoted by $\dim_q(v)$.

Since it can be shown that $\text{Tr}(Q_\pi^{-1}) = \text{Tr}(Q_\pi)$, we have $\dim_q(v) \geq \dim H$.

Theorem 1.14. The Haar state h satisfies the following generalized orthogonality:

$$\begin{aligned} (\text{id} \otimes h)(v_\pi(\xi \eta^* \otimes 1)v_\rho^*) &= \delta_{\pi,\rho} \dim_q(\pi)(Q_\pi \xi, \eta), \\ (\text{id} \otimes h)(v_\pi^*(\xi \eta^* \otimes 1)v_\rho) &= \delta_{\pi,\rho} \dim_q(\pi)(Q_\pi^{-1} \xi, \eta), \end{aligned}$$

where $\pi, \rho \in \text{Irr}(\mathbb{G})$, $\xi \in H_\pi$ and $\eta \in H_\rho$.

The key point of the proof of this result is to observe that $(\text{id} \otimes h)(v_\pi(\xi \eta^* \otimes 1)v_\rho^*) \in \text{Mor}(\rho, \pi)$. The matrix form is sometimes useful. Take an ONB $\{\varepsilon_{\pi_i}\}_{i \in I_\pi}$. Then we have

$$h(v_{\pi_{i,j}} v_{\rho_{k,\ell}}^*) = \delta_{\pi,\rho} \dim_q(\pi)^{-1} Q_{\pi_{\ell,j}} \delta_{i,k}, \quad h_\mathbb{G}(v_{\pi_{i,j}}^* v_{\rho_{k,\ell}}) = \delta_{\pi,\rho} \dim_q(\pi)^{-1} (F_\pi^{-1})_{k,i} \delta_{j,\ell}.$$

In particular, $h(v_{\pi_{ij}}) = \delta_{\pi,1} \delta_{ij}$.

We see that the matrix $(Q_\pi^{1/2} \otimes 1)v_\pi^c(Q_\pi^{-1/2} \otimes 1)$ is a unitary representation equivalent to $v_{\bar{\pi}}$.

Now note that $\dim \text{Mor}(\pi \otimes \bar{\pi}, 1) = 1$, and take $t_\pi \in \text{Mor}(\pi \otimes \bar{\pi}, 1)$ such that $t_\pi^* t_\pi = \dim_q(\pi)$. The following theorem is due to Woronowicz, but our notation is slightly different because we do not use special ONB (see [8]).

Theorem 1.15 (Quantum Peter-Weyl theorem). *One has the following unitary isomorphism:*

$$L^2(\mathbb{G}) \rightarrow \bigoplus_{\pi \in \text{Irr}(\mathbb{G})} H_{\bar{\pi}} \otimes H_\pi,$$

which maps $(\omega_{\xi, \eta} \otimes \text{id})(v_\pi)\Omega_h$ to $(1 \otimes \eta^*) \circ t_{\bar{\pi}} \otimes \xi$. *intertwines the left and right $\mathbb{G}$ -actions.*

1.10. Non-trivial examples 1

For a classical compact Lie group $\mathbb{G}$, we can construct the q -deformation $\mathbb{G}_q$ [18], where $q \in [-1, 1] \setminus \{0\}$. If $q = 1$, $\mathbb{G}_1$ is nothing but the original $\mathbb{G}$. The object corresponding to $q = 0$ is considered as a *quantum semigroup* which is not a quantum group because that does not have a Haar state.

Now we explain the simplest and the most important example $SU_q(2)$ [44]. The continuous function algebra $C(SU_q(2))$ is the universal C^* -algebra generated by four elements x, u, v and y with the following relations:

$$\begin{aligned} ux &= qxu, & vx &= qxv, & yu &= quy, & yv &= qvy, & uv &= vu, \\ xy - q^{-1}uv &= 1 = yx - quv, \\ x^* &= y, & u^* &= -q^{-1}v. \end{aligned}$$

To introduce a coproduct δ , the following 2 by 2 unitary matrix is useful:

$$v(\pi_{1/2}) := \begin{pmatrix} x & u \\ v & y \end{pmatrix}.$$

Then the coproduct δ is given by

$$\begin{pmatrix} \delta(x) & \delta(u) \\ \delta(v) & \delta(y) \end{pmatrix} := \begin{pmatrix} x \otimes 1 & u \otimes 1 \\ v \otimes 1 & y \otimes 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \otimes x & 1 \otimes u \\ 1 \otimes v & 1 \otimes y \end{pmatrix},$$

This means $v(\pi_{1/2})$ is a representation, which is in fact irreducible.

It is known that $\text{Irr}(SU_q(2))$ is naturally identified with the positive half integers $(1/2)\mathbb{Z}_{\geq 0}$. Each $\nu \in \text{Irr}(SU_q(2))$ is called the *spin* and the dimension of $v(\pi_\nu)$ is $2\nu + 1$. The quantum dimension of π_ν is given by the q -integer $[2\nu + 1]_q := (q^{-2\nu-1} - q^{2\nu+1})/(q^{-1} - q)$ [20].

On tensor products, we have the same formula (Clebsch-Gordan rule) as that of $SU(2)$,

$$\pi_\mu \otimes \pi_\nu = \pi_{|\mu-\nu|} \oplus \pi_{|\mu-\nu|+1} \oplus \cdots \oplus \pi_{\mu+\nu-1} \oplus \pi_{\mu+\nu}.$$

In particular, $SU_q(2)$ has commutative fusion rules [20, 44].

This phenomena hold for every q -deformation of a classical compact Lie group, that is, the fusion rule is invariant under the q -deformation, and it is commutative.

1.11. Non-trivial example 2

Our second interesting example is a *universal quantum group*. There are a lot of variants, and we explain the only original examples here [35].

Definition 1.16. Let $F \in GL(n, \mathbb{C})$.

- (Universal orthogonal quantum group $A_o(F)$) Assume that $F\bar{F} = \pm 1$. The function algebra $C(A_o(F))$ is the universal C^* -algebra generated by u_{ij} , $i, j = 1, \dots, n$, which satisfy

$$u = (F \otimes 1)u^c(F^{-1} \otimes 1),$$

where $u = (u_{ij})_{ij}$ and $u^c = (u_{ij}^*)_{ij}$.

- (Universal quantum group $A_u(F)$) The function algebra $C(A_u(F))$ is the universal C^* -algebra generated by u_{ij} , $i, j = 1, \dots, n$ such that u and $(F \otimes 1)u^c(F^{-1} \otimes 1)$ are unitary.

The both coproducts are given by

$$\delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

This means the matrix u is a representation, which is in fact irreducible.

By definition, we obtain the surjective morphism $r: C(A_u(F)) \rightarrow C(A_o(F))$ as quantum groups. Hence $A_o(F)$ is a quantum subgroup of $A_u(F)$.

The fusion rules of $A_o(F)$ and $A_u(F)$ are computed by T. Banica [3, 4]. The c.q.g. $A_o(F)$ has the same fusion rule as $SU(2)$, and in fact, it is monoidally equivalent to $SU_q(2)$ for some q [8] (see §3.4). We can regard $\text{Irr}(A_u(F))$ as the free monoid $\mathbb{N} \star \mathbb{N}$ whose product is written like xy for $x, y \in \mathbb{N} \star \mathbb{N}$. Let α and β are the generators. We define the conjugation on $\mathbb{N} \star \mathbb{N}$ such that $\bar{\alpha} = \beta$. The representation ring is $\mathcal{R} = \mathbb{Z}[\mathbb{N} \star \mathbb{N}]$ as a $\mathbb{Z}$ -module. The product structure $x \cdot y$ for $x, y \in \mathbb{N} \star \mathbb{N}$ is given by

$$x \cdot y := \sum_{\{a \in \mathbb{N} \star \mathbb{N} \mid x = x_0 a, y = \bar{a} y_0\}} x_0 y_0.$$

For example, $\alpha \cdot \alpha = \alpha^2$ and $\alpha \cdot \beta = \alpha\beta + 1$. So, the fusion rule does not depend on F . In particular, $\dim \text{Mor}(x \otimes y, z) = 0$ or 1 for all $x, y, z \in \text{Irr}(A_u(F))$.

Recall the definition of $A_u(F)$ where we have taken a unitary matrix $u \in B(\mathbb{C}^n) \otimes C(A_u(F))$. In fact, u and $\bar{u}$ are irreducible representations corresponding to α and β , respectively.

2. DISCRETE QUANTUM GROUPS

Let $\mathbb{G}$ be a c.q.g. In this section, we study basic properties of the dual $\widehat{\mathbb{G}}$.

2.1. Right and left group algebras

Recall the multiplicative unitaries V, W , which is right and left representations of $\mathbb{G}$ on $L^2(\mathbb{G})$. We introduce the following subspaces:

$$R(\mathbb{G}) := \overline{\text{span}}^w \{(\text{id} \otimes \omega)(V) \mid \omega \in L^\infty(\mathbb{G})_*\},$$

$$L(\mathbb{G}) := \overline{\text{span}}^w \{(\omega \otimes \text{id})(W) \mid \omega \in L^\infty(\mathbb{G})_*\}.$$

We call them *right, left group algebras*, respectively.

Define the maps $\beta, \beta^\ell: B(L^2(\mathbb{G})) \rightarrow B(L^2(\mathbb{G})) \otimes B(L^2(\mathbb{G}))$,

$$\beta(x) := V^*(1 \otimes x)V \text{ for } x \in B(L^2(\mathbb{G})),$$

$$\beta^\ell(x) := W(x \otimes 1)W^* \text{ for } x \in B(L^2(\mathbb{G})).$$

Theorem 2.1. *The following hold:*

- $R(\mathbb{G})$ and $L(\mathbb{G})$ are von Neumann algebras;
- The restrictions $\Delta := \beta|_{R(\mathbb{G})}$ and $\Delta^\ell := \beta^\ell|_{L(\mathbb{G})}$ define coproducts, i.e.

$$\Delta(R(\mathbb{G})) \subset R(\mathbb{G}) \otimes R(\mathbb{G}), \quad (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$$

and

$$\Delta(L(\mathbb{G})) \subset L(\mathbb{G}) \otimes L(\mathbb{G}), \quad (\Delta^\ell \otimes \text{id}) \circ \Delta^\ell = (\text{id} \otimes \Delta^\ell) \circ \Delta^\ell.$$

Proof. We prove the second statement. Take $(\text{id} \otimes \omega)(V) \in R(\mathbb{G})$. Then using the pentagon equation, we have

$$\begin{aligned} \Delta((\text{id} \otimes \omega)(V)) &= V^*(1 \otimes (\text{id} \otimes \omega)(V))V \\ &= (\text{id} \otimes \text{id} \otimes \omega)(V_{12}^* V_{23} V_{12}) = (\text{id} \otimes \text{id} \otimes \omega)(V_{13} V_{23}) \in R(\mathbb{G}) \otimes R(\mathbb{G}). \end{aligned}$$

□

This theorem implies that the pair $(R(\mathbb{G}), \Delta)$ is a bialgebra. In fact, it is known that there exist weights φ and ψ on $R(\mathbb{G})$ such that

- $\varphi((\omega \otimes \text{id})(\Delta(x))) = \omega(1)\psi(x)$ for all $\omega \in R(\mathbb{G})_*^+, x \in R(\mathbb{G})_+$.
- $\psi((\text{id} \otimes \omega)(\Delta(x))) = \omega(1)\varphi(x)$ for all $\omega \in R(\mathbb{G})_*^+, x \in R(\mathbb{G})_+$;

Therefore, $\widehat{\mathbb{G}} := (R(\mathbb{G}), \Delta)$ is a quantum group in the sense of [19].

Using Theorem 1.15, we obtain the isomorphism,

$$R(\mathbb{G}) \rightarrow \bigoplus_{\pi \in \text{Irr}(\mathbb{G})} B(H_\pi).$$

Hence $\widehat{\mathbb{G}}$ is also called a *discrete quantum group*. Similarly $L(\mathbb{G})$ is a discrete quantum group, too. They are acting on $L^2(\mathbb{G})$ standardly, and

$$R(\mathbb{G})' = L(\mathbb{G}).$$

A discrete quantum group is also characterized by the existence of a normal counit. Indeed in our case, the normal counit $\hat{\varepsilon}$ is given by the evaluation of $x \in R(\mathbb{G})$ at $\pi = 1$, i.e. $\hat{\varepsilon}(x) = x_1 \in \mathbb{C}$.

When we study actions of $\widehat{\mathbb{G}}$, the following description of the coproduct Δ is quite useful:

$$\Delta(x_\pi) = \sum_{\rho, \sigma \in \text{Irr}(\mathbb{G})} \sum_{S \in \text{ONB}(\text{Mor}(\pi, \rho \otimes \sigma))} S x_\pi S^* \quad \text{for } x_\pi \in B(H_\pi) \subset R(\mathbb{G}),$$

where $\text{ONB}(\text{Mor}(\pi, \rho \otimes \sigma))$ is a set of orthonormal bases of $\text{Mor}(\rho \otimes \sigma, \pi)$ with the inner product $(S, T) := T^* S$.

2.2. Actions of quantum groups

Definition 2.2. Let $\mathbb{G} = (L^\infty(\mathbb{G}), \delta)$ be a locally compact quantum group and M a von Neumann algebra. A map $\alpha: M \rightarrow M \otimes L^\infty(\mathbb{G})$ is called a (*right*) *action* when it satisfies the following:

- α is a unital faithful normal $*$ -homomorphism;
- $(\text{id} \otimes \delta) \circ \alpha = (\alpha \otimes \text{id}) \circ \alpha$.

A left action is similarly defined.

Example 2.3. The map $\beta: B(L^2(\mathbb{G})) \rightarrow R(\mathbb{G}) \otimes B(L^2(\mathbb{G}))$ is a left action of $\widehat{\mathbb{G}}$. Similarly $\alpha: B(L^2(\mathbb{G})) \rightarrow B(L^2(\mathbb{G})) \otimes L^\infty(\mathbb{G})$ defined by $\alpha(x) = V(x \otimes 1)V^*$ is a right action of $\mathbb{G}$.

2.3. Quantum subgroups and left (right) coideals

There are several ways to define a quantum subgroup of a c.q.g. Here, we adopt the following definition.

Definition 2.4. Let $\mathbb{G}$ and $\mathbb{H}$ be compact quantum groups. We say that $\mathbb{H}$ is a *quantum subgroup* of $\mathbb{G}$ if there exists a unital $*$ -homomorphism $r_{\mathbb{H}}: A(\mathbb{G}) \rightarrow A(\mathbb{H})$ such that

- $r_{\mathbb{H}}$ is surjective;
- $\delta_{\mathbb{H}} \circ r_{\mathbb{H}} = (r_{\mathbb{H}} \otimes r_{\mathbb{H}}) \circ \delta_{\mathbb{G}}$.

This definition is weaker than the usual C^* -version, which requires $r_{\mathbb{H}}$ is a C^* -homomorphism from $C(\mathbb{G})$ onto $C(\mathbb{H})$.

Note that $\mathbb{H}$ acts on $A(\mathbb{G})$ from the both sides as $\mathbb{H} \curvearrowright^{\gamma^\ell} A(\mathbb{G}) \curvearrowright^{\gamma^r} \mathbb{H}$ defined by

$$\gamma^\ell := (r_{\mathbb{H}} \otimes \text{id}) \circ \delta_{\mathbb{G}}, \quad \gamma^r := (\text{id} \otimes r_{\mathbb{H}}) \circ \delta_{\mathbb{G}}.$$

Then we define the *non-commutative quotient spaces* by the following fixed point algebras:

$$\begin{aligned} A(\mathbb{H} \backslash \mathbb{G}) &:= \{a \in A(\mathbb{G}) \mid \gamma^\ell(a) = 1 \otimes a\}, \\ A(\mathbb{G} / \mathbb{H}) &:= \{a \in A(\mathbb{G}) \mid \gamma^r(a) = a \otimes 1\}. \end{aligned}$$

The weak closures in $B(L^2(\mathbb{G}))$ are denoted by $L^\infty(\mathbb{H} \backslash \mathbb{G})$ and $L^\infty(\mathbb{G} / \mathbb{H})$.

Note that the left $\mathbb{H}$ -action γ^ℓ and the right $\mathbb{G}$ -action $\delta_{\mathbb{G}}$ are commuting, i.e. $(\text{id} \otimes \delta_{\mathbb{G}}) \circ \gamma^\ell = (\gamma^\ell \otimes \text{id}) \circ \delta_{\mathbb{G}}$. Hence $\mathbb{G}$ is also acting $A(\mathbb{H} \backslash \mathbb{G})$ by δ . Similarly the coproduct δ defines a left action on $A(\mathbb{G} / \mathbb{H})$. Since these actions are preserving the Haar state, they extend to the quotient spaces $L^\infty(\mathbb{H} \backslash \mathbb{G})$ or $L^\infty(\mathbb{G} / \mathbb{H})$, respectively. They are typical examples of right or left coideals.

Definition 2.5. Let $B \subset L^\infty(\mathbb{G})$ be a von Neumann subalgebra. Then we say that

- B is a left coideal if $\delta(B) \subset L^\infty(\mathbb{G}) \otimes B$;
- B is a right coideal if $\delta(B) \subset B \otimes L^\infty(\mathbb{G})$;
- a left (right) coideal B is of *quotient type* if $B = L^\infty(\mathbb{G}/\mathbb{H})$ (resp. $L^\infty(\mathbb{H} \setminus \mathbb{G})$) for some quantum subgroup $\mathbb{H}$.

Thanks to Gelfand theorem, every left coideal is of quotient type when $\mathbb{G}$ is a compact group [1]. However, this is not true in general [26, 27, 30]. Indeed, we have the following characterization [31].

Theorem 2.6 (Tomatsu). *Let $B \subset L^\infty(\mathbb{G})$ be a right coideal. Then the following are equivalent:*

- B is of quotient type;
- There exists an expectation $E_B: L^\infty(\mathbb{G}) \rightarrow B$ preserving the Haar state, and moreover $\widehat{\mathbb{G}}$ acts on B , i.e. $\beta(B) \subset R(\mathbb{G}) \otimes B$.

This theorem has been proved for co-amenable quantum groups [31], but the same proof works because we have changed the definition of quantum subgroups.

2.4. Amenability and co-amenable

For details of the theory of amenability for quantum groups, readers are referred to [5, 6, 7, 29] and references therein.

Definition 2.7. We say that $\widehat{\mathbb{G}}$ is *amenable* when there exists an *invariant mean* m on $R(\mathbb{G})$, that is, $m \in R(\mathbb{G})^*$ is a state such that

$$m((\text{id} \otimes \omega)(\Delta(x))) = \omega(1)m(x) = m((\omega \otimes \text{id})(\Delta(x))).$$

In this case, $\mathbb{G}$ is said to be *co-amenable*.

Theorem 2.8 (Bedos-Murphy-Tuset, Tomatsu). *The following are equivalent:*

- $\mathbb{G}$ is co-amenable;
- $C(\mathbb{G}_{\text{red}})$ has a bounded counit ε , which is a $*$ -homomorphism $\varepsilon: C(\mathbb{G}_{\text{red}}) \rightarrow \mathbb{C}$ such that $(\varepsilon \otimes \text{id}) \circ \delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \delta$.

If $\mathbb{G}$ is co-amenable, then $C(\mathbb{G}_{\text{red}})$ is the universal C^* -algebra of $A(\mathbb{G})$, which is proved by using Fell absorption technique.

It is easy to see that any quantum subgroup of a co-amenable c.q.g. is also co-amenable. Since $A_o(F)$ ($n \geq 3$) or $A_u(F)$ has non-co-amenable quantum subgroups, they are not co-amenable. It is known that the q -deformation of a classical compact Lie group is co-amenable. In particular, $SU_q(2)$ is co-amenable.

3. PROBLEMS

In this section, some interesting open problems are listed.

3.1. Minimal actions

One of the most difficult problem in a quantum group theory is the existence of minimal actions on amenable factors. The definition of minimality is the following:

Definition 3.1. An action $M \curvearrowright^\alpha \mathbb{G}$ is said to be *minimal* if it satisfies

- (trivial relative commutant) $(M^\alpha)' \cap M = \mathbb{C}$;
- (full spectrum) $L^\infty(\mathbb{G}) = \overline{\text{span}}^w \{(\omega \otimes \text{id})(\alpha(M)) \mid \omega \in M_*\}$.

The first condition is concerned with the high non-commutativity of the fixed point algebra in the ambient algebra. In particular, M must be a factor. The second one is also called *faithfulness* of α . Indeed if $\mathbb{G}$ is a compact group, then that is equivalent to the injectivity of the group homomorphism $\alpha: \mathbb{G} \rightarrow \text{Aut}(M)$, and the trivial action is excluded.

Example 3.2. Let $\mathbb{G} \subset U(n)$ be a closed subgroup. We present a typical construction of a minimal action of $\mathbb{G}$ on the amenable type II₁ factor $\mathcal{R}_0$.

Set the unitary representation $\pi: \mathbb{G} \rightarrow U(n+1)$,

$$\pi(g) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\mathbb{G}$ acts on $M_{n+1}(\mathbb{C})^{\otimes k}$ by adjoint action $\text{Ad } \pi^{\otimes k}(g)$ for each $k \geq 0$. The action extends to the UHF $M_{n+1}(\mathbb{C})^{\otimes \infty}$, which preserves the unique trace. Hence we obtain an action of $\mathbb{G}$ on its weak closure, i.e. the amenable type II₁ factor $\mathcal{R}_0$. Indeed it is minimal [44]. It is known that any minimal action of a compact group on $\mathcal{R}_0$ is unique up to conjugacy [25, 22].

We have seen an example of a minimal action of a compact Lie group. The first example of a minimal action of a c.q.g. was constructed by Y. Ueda [33].

Let $\mathbb{G}$ be a c.q.g. Take a von Neumann algebra N such that $N' \cap (N \star L^\infty(\mathbb{G})) = \mathbb{C}$. Then consider the *Ueda action* $\alpha := \text{id} \star \delta$ on a factor $M := N \star L^\infty(\mathbb{G})$. By definition, α is minimal. The point is that M is not amenable. So, our problem is the following:

Problem 3.3. Let $\mathbb{G}$ be a compact quantum group of non-Kac type. Does there exist a minimal action on an amenable factor?

If there would exist such an action, $\mathbb{G}$ would have to be co-amenable [36]. Although we have not known if it is true or not even for $SU_q(2)$, Izumi's observation seems to be relevant.

Let $v \in B(H) \otimes L^\infty(\mathbb{G})$ be a finite dimensional representation. Then as in the previous example, we can define the infinite tensor product action α on an amenable factor $\mathcal{R}$, which is a Powers factor. According to Izumi theory, we have the isomorphism between $\mathbb{G}$ -algebras,

$$(\mathcal{R}^\alpha)' \cap \mathcal{R} \cong H^\infty(\widehat{\mathbb{G}}, P_\mu),$$

where $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is the *Poisson boundary* associated with the random walk on $\widehat{\mathbb{G}}$ with a probability measure μ [13]. Therefore, α is never minimal when $\mathbb{G}$ is of non-Kac type.

Many operator algebraists have interest in this problem because it has an application to Jones' problem, that is, to determine the index values for irreducible subfactors of $\mathcal{R}_0$ [16]. If Proposition 3.3 would be affirmatively solved for $SU_q(2)$, then the set of possible values greater than 4 would become $(4, \infty)$ [34]. They are attained for Wassermann subfactors:

$$M^\alpha \subset (B(\mathbb{C}^2) \otimes M)^{\tilde{\alpha}},$$

where $\tilde{\alpha} = \text{Ad}(v_{\pi_{1/2}})_{13} \circ (\text{id} \otimes \alpha)$.

However, we might solve Jones' problem without proving the existence. In fact, if we had found an example of an action α on an amenable factor M such that

- $M = M^\alpha \oplus M_{\pi_{1/2}} \oplus 0 \oplus M_{\pi_{3/2}} \oplus \dots$ (spectral decomposition);
- $\{M, \alpha\}$ is conjugate to $\{B(\mathbb{C}^2) \otimes M, \tilde{\alpha}\}$,

then the Wassermann inclusion would give an index $\dim_q(\pi_{1/2})^2 = (q^{-1} + q)^2$.

3.2. Centers of Poisson boundaries

We briefly recall the notion of the Poisson boundary for a discrete quantum group. We refer to [13] for definitions of terminology.

Let $\phi_\pi \in B(H_\pi)_*$ be the right $\mathbb{G}$ -invariant state. Define a transition operator P_π on $R(\mathbb{G})$ by $P_\pi(x) = (\text{id} \otimes \phi_\pi)(\Delta_R(x))$ for $x \in R(\mathbb{G})$. When $\widehat{\mathbb{G}}$ is a discrete group, P_g , $g \in \widehat{\mathbb{G}}$, is nothing but the right translation of functions by $g \in \widehat{\mathbb{G}}$, which is an automorphism. However, the map P_π is not an automorphism but a faithful normal u.c.p. map in general.

For a probability measure μ on $\text{Irr}(\mathbb{G})$, we set a non-commutative Markov operator,

$$P_\mu := \sum_{\pi \in \text{Irr}(\mathbb{G})} \mu(\pi) P_\pi.$$

We assume μ is generating, that is, $\text{supp}(\mu)$ generates $\text{Irr}(\mathbb{G})$ as a “semigroup”, i.e. for any $\pi \in \text{Irr}(\mathbb{G})$, there exist $\rho_1, \dots, \rho_n \in \text{supp}(\mu)$ such that the representation π is contained in the tensor product representation $\rho_1 \otimes \dots \otimes \rho_n$.

Then we define an operator system,

$$H^\infty(\widehat{\mathbb{G}}, P_\mu) := \{x \in R(\mathbb{G}) \mid P_\mu(x) = x\}.$$

We often regard $\text{id} - P_\mu$ as a Laplace operator on $\widehat{\mathbb{G}}$, and we say that each element of $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is P_μ -harmonic. That operator system has the von Neumann algebra structure defined by

$$x \cdot y = \lim_{n \rightarrow \infty} P_\mu^n(xy) \quad \text{for } x, y \in H^\infty(\widehat{\mathbb{G}}, P_\mu), \quad (3.1)$$

where the limit is taken in the strong topology [13, Theorem 3.6]. The von Neumann algebra $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is called the (non-commutative) *Poisson boundary* of $\{R(\mathbb{G}), P_\mu\}$.

We know that $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is isomorphic to $(\mathcal{R}^\mathbb{G})' \cap \mathcal{R}$, where $\mathcal{R} \curvearrowright \mathbb{G}$ is an ITP action [13]. Thanks to Takesaki theorem on an expectation [28], we see that $(\mathcal{R}^\mathbb{G})' \cap \mathcal{R}$ is amenable, and so is $H^\infty(\widehat{\mathbb{G}}, P_\mu)$.

Now we recall the actions $\widehat{\mathbb{G}} \overset{\beta}{\curvearrowright} B(L^2(\mathbb{G})) \overset{\alpha}{\curvearrowright} \mathbb{G}$ defined by

$$\beta(x) := V^*(1 \otimes x)V, \quad \alpha(x) = V(x \otimes 1)V^*.$$

Since we can prove P_μ and α or β are commuting on $R(\mathbb{G})$, the Poisson boundary is a $\widehat{\mathbb{G}}$ - $\mathbb{G}$ -von Neumann algebra. We should note that if $\widehat{\mathbb{G}}$ is a discrete group, then α is trivial. Hence a non-triviality of α on $R(\mathbb{G})$ is a purely quantum phenomenon.

In Poisson boundary theory, one of the most important problem is the following:

Problem 3.4 (Identification problem). *Realize $\widehat{\mathbb{G}}$ - $\mathbb{G}$ -von Neumann algebra $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ more concretely.*

For example, if $\mathbb{G}$ is the q -deformation of a classical compact Lie group, then the Poisson boundary is isomorphic to a quantum flag manifold [13, 15, 31]. Also for $A_o(F)$ or $A_u(F)$, the computation has been done [38, 39, 40].

Here, we propose a new problem on a Poisson boundary. Recall that P_μ and α are commuting, and P_μ acts on the fixed point algebra $R(\mathbb{G})^\alpha$, which is nothing but the center $Z(R(\mathbb{G})) = \ell_\infty(\text{Irr}(\mathbb{G}))$. Hence we introduce the classical part of a Poisson boundary,

$$H^\infty(\widehat{\mathbb{G}}, P_\mu)_{\text{class}} := H^\infty(\widehat{\mathbb{G}}, P_\mu) \cap Z(R(\mathbb{G})).$$

Let us denote the center of $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ by $Z(H^\infty(\widehat{\mathbb{G}}, P_\mu))$. It is trivial by (3.1) that the classical part $H^\infty(\widehat{\mathbb{G}}, P_\mu)_{\text{class}}$ is contained in $Z(H^\infty(\widehat{\mathbb{G}}, P_\mu))$. Now we present following our problem:

Conjecture 3.5. *Let $\mathbb{G}$ be a compact quantum group and μ a generating probability measure. Then the following equality holds:*

$$H^\infty(\widehat{\mathbb{G}}, P_\mu)_{\text{class}} = Z(H^\infty(\widehat{\mathbb{G}}, P_\mu)).$$

When $\mathbb{G}$ has a commutative fusion, then the classical part is trivial [12, 13]. So, the conjecture means the factoriality of $H^\infty(\widehat{\mathbb{G}}, P_\mu)$.

There are some positive observations about this conjecture. For $SU_q(2)$ case, it is true because the quantum flag manifold (or a Podleś sphere) $L^\infty(\mathbb{T} \backslash SU_q(2))$ is a type I_∞ factor. However, that is unknown for other q -deformations. The conjecture holds even for non-amenable examples such as $A_o(F)$ and $A_u(F)$ [37, 39]. It has seemed to be affirmative so far.

3.3. 2-cocycles

The next problem is about 2-cocycle deformations of a locally compact quantum group. Let $\mathbb{G}$ be a locally compact quantum group with the coproduct δ .

Definition 3.6. An element $\Omega \in L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$ is called a *2-cocycle* if it satisfies the following:

- Ω is a unitary;
- $(\Omega \otimes 1)(\delta \otimes \text{id})(\Omega) = (1 \otimes \Omega)(\text{id} \otimes \delta)(\Omega)$.

Let us define the map $\delta_\Omega := \text{Ad } \Omega \circ \delta$, which is a new coproduct on $L^\infty(\mathbb{G})$. Hence $(L^\infty(\mathbb{G}), \delta_\Omega)$ is a bi-algebra. In fact, K. De Commer recently proved that there exist the left and right invariant weights on it, that is, $\mathbb{G}_\Omega := (L^\infty(\mathbb{G}), \delta_\Omega)$ is a locally compact quantum group [10].

If $\mathbb{G}$ is discrete, $\mathbb{G}_\Omega$ is again discrete (use the counit). If $\mathbb{G}$ is a compact Kac, then $\mathbb{G}_\Omega$ is also compact. However, the compactness is not invariant in general [10]. His example is involving free products of $SU_q(2)$. So, we propose the following problem.

Problem 3.7. Find a co-amenable compact quantum group $\mathbb{G}$ which has a 2-cocycle Ω so that $\mathbb{G}_\Omega$ is not compact.

3.4. Monoidal equivalences

Definition 3.8. Let $\mathbb{G}$ and $\mathbb{G}_1$ be compact quantum groups. We say that they are *monoidally equivalent* when there exist the following maps both denoted by φ :

- a bijection $\varphi: \text{Irr}(\mathbb{G}) \rightarrow \text{Irr}(\mathbb{G}_1)$;
- bijective linear maps $\varphi: \text{Mor}(x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_n) \rightarrow \text{Mor}(\varphi(x_1) \otimes \cdots \otimes \varphi(x_m), \varphi(y_1) \otimes \cdots \otimes \varphi(y_n))$ for $x_1, \dots, x_m, y_1, \dots, y_n \in \text{Irr}(\mathbb{G})$,

such that

- $\varphi(1) = 1$;
- $\varphi(S^*) = \varphi(S)^*$;
- $\varphi(ST) = \varphi(S)\varphi(T)$;
- $\varphi(S \otimes T) = \varphi(S) \otimes \varphi(T)$.

This means the tensor category of $\mathbb{G}$ and $\mathbb{G}_1$ are isomorphic, and so are their fusion algebras, in particular. The dimensions of their representation spaces may not be equal, but their quantum dimensions are invariant, i.e. $\dim_q(x) = \dim_q(\varphi(x))$.

The following problem is proposed by T. Banica.

Problem 3.9. Assume that $\mathbb{G}$ has commutative fusion rules. Then does there exist a co-amenable compact quantum group $\mathbb{G}_1$ such that $\mathbb{G} \sim \mathbb{G}_1$?

The fusion algebra of $A_o(F)$ is exactly same as the one of $SU_q(2)$ [3]. So, the problem is true for $A_o(F)$.

3.5. Universal quantum groups

Denote $A_u(1_n)$ by $A_u(n)$. It is known that $L^\infty(A_u(n))$ is a type II_1 factor (the trace is the Haar state) [4, 40].

Problem 3.10 (Banica). *Is the factor $L^\infty(A_u(n))$ isomorphic to a free group factor $L(\mathbb{F}_r)$ for some $r \in \mathbb{N}$?*

If $n = 2$, it is proved by Banica [4] with $r = 2$.

Even if the statement of the previous problem does not hold, it is expectable that the factor $L^\infty(A_o(n))$ or $L^\infty(A_u(n))$ have similar properties to free group factors. Indeed, $L^\infty(A_o(n))$ has Akemann-Ostrand property, it is solid [23, 40].

Problem 3.11. *Show that the factor $L^\infty(A_o(n))$ has no Cartan subalgebras.*

3.6. Ergodic actions of $SU(n)$

We have considered several problems related with compact quantum groups, but there still remain some problems for compact groups.

Problem 3.12 (Jones). *Can $SU(n)$ ergodically act on $\mathcal{R}_0$?*

The first breakthrough was made by A. Wassermann [42, 43]. He classified all ergodic actions of $SU(2)$, and indeed there does not exist an ergodic action on $\mathcal{R}_0$. Hence probably $SU(n)$ can not act on $\mathcal{R}_0$ ergodically. One of the key point of this problem is to require the factoriality. If $n \geq 3$, then $\mathbb{T}^2$ embeds into $SU(n)$. Inducing an ergodic action from $\mathbb{T}^2 \curvearrowright \mathcal{R}_0$ (consider the irrational rotation algebra), we see the induced algebra, which is of type II_1 , has a non-trivial center.

Even if we focus on very restricted cases, the answer has not been given. Let $\mathbb{G} = SU(n)$ and $\omega \in R(\mathbb{G}) \otimes R(\mathbb{G})$ a 2-cocycle. Then $\mathbb{G}$ ergodically acts on the twisted group algebra $R_\omega(\mathbb{G})$, whose type has not been computed yet.

There is a relating problem [41]. Let us introduce the 2-cohomology set $H^2(\widehat{\mathbb{G}})$ which consists of equivalence classes of 2-cocycles.

Conjecture 3.13 (A. Wassermann). *Each element of $H^2(\widehat{\mathbb{G}})$ is coming from $H^2(\widehat{\mathbb{T}_{\max}})$ via the embedding $R(\mathbb{T}_{\max}) \subset R(\mathbb{G})$, where $\mathbb{T}_{\max}$ is the maximal torus of $\mathbb{G} = SU(n)$. In particular, $H^2(\widehat{\mathbb{G}})$ is a finite set.*

Of course, the following problem is still open.

Problem 3.14. *Classify all ergodic actions of $SU_q(2)$.*

Readers are referred to [8, 24, 30] for examples or partial classification results.

3.7. Galois correspondence

In [14, 32], a Galois correspondence for a minimal action of a c.q.g. is proved. The next step is to generalize this result to locally compact quantum group actions.

Problem 3.15. *Does the Galois correspondence hold for any minimal actions of locally compact quantum groups?*

Readers are referred to [11, 14, 32].

REFERENCES

- [1] H. Araki, D. Kastler, M. Takesaki and R. Haag, *Extension of KMS states and chemical potential*, Comm. Math. Phys. **53** (1977), no. 2, 97–134.
- [2] S. Baaj and G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de C^* -algèbres*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 4, 425–488.
- [3] T. Banica, *Théorie des représentations du groupe quantique compact libre $O(n)$* , C. R. Acad. Sci. Paris Sér. I Math. **322** (1996), no. 3, 241–244.
- [4] T. Banica, *Le groupe quantique compact libre $U(n)$* , Comm. Math. Phys. **190** (1997), no. 1, 143–172.
- [5] E. Bédos, R. Conti and L. Tuset, *On amenability and co-amenable of algebraic quantum groups and their corepresentations*, Canad. J. Math. **57** (2005), no. 1, 17–60.
- [6] E. Bédos, G. Murphy and L. Tuset, *Co-amenable of compact quantum groups*, J. Geom. Phys. **40** (2001), no. 2, 130–153.
- [7] E. Bédos, G. Murphy and L. Tuset, *Amenability and coamenability of algebraic quantum groups*, Int. J. Math. Math. Sci. **31** (2002), no. 10, 577–601.
- [8] J. Bichon, A. De Rijdt and S. Vaes, *Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups*, Comm. Math. Phys. **262** (2006), no. 3, 703–728.
- [9] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994. xvi+651 pp.
- [10] K. De Commer, *Monoidal equivalence for locally compact quantum groups*, arXiv.math.OA/0804.2405.
- [11] M. Enock, *Sous-facteurs intermédiaires et groupes quantiques mesurés*, J. Operator Theory **42** (1999), no. 2, 305–330.
- [12] T. Hayashi, *Harmonic function spaces of probability measures on fusion algebras*, Publ. Res. Inst. Math. Sci. **36** (2000), no. 2, 231–252.
- [13] M. Izumi, *Non-commutative Poisson boundaries and compact quantum group actions*, Adv. Math. **169** (2002), no. 1, 1–57.
- [14] M. Izumi, R. Longo and S. Popa, *A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras*, J. Funct. Anal. **155** (1998), no. 1, 25–63.
- [15] M. Izumi, S. Neshveyev and L. Tuset, *Poisson boundary of the dual of $SU_q(n)$* , Comm. Math. Phys. **262** (2006), no. 2, 505–531.
- [16] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), no. 1, 1–25.
- [17] G. I. Kac and V. G. Paljutkin, *Finite ring groups*, Trudy Moskov. Mat. Obšč. **15** 1966 224–261.
- [18] L. I. Korogodski and Y. S. Soibelman, *Algebras of functions on quantum groups. Part I*, Mathematical Surveys and Monographs **56**, American Mathematical Society, Providence, RI, 1998. x+150 pp.
- [19] J. Kustermans and S. Vaes, *Locally compact quantum groups in the von Neumann algebraic setting*, Math. Scand. **92** (2003), no. 1, 68–92.
- [20] Tetsuya Masuda, K. Mimachi, Y. Nakagami, M. Noumi and K. Ueno, *Representations of the quantum group $SU_q(2)$ and the little q -Jacobi polynomials*, J. Funct. Anal. **99** (1991), no. 2, 357–386.
- [21] Toshihiko Masuda and R. Tomatsu, *Classification of minimal actions of a compact Kac algebra with amenable dual*, Comm. Math. Phys. **274** (2007), no. 2, 487–551.
- [22] Toshihiko Masuda and R. Tomatsu, *Classification of minimal actions of a compact Kac algebra with amenable dual on injective factors of type III*, arXiv.math.OA/0806.4259.
- [23] N. Ozawa, *Solid von Neumann algebras*, Acta Math. **192** (2004), no. 1, 111–117.
- [24] C. Pinzari and J. E. Roberts, *A duality theorem for ergodic actions of compact quantum groups on C^* -algebras*, Comm. Math. Phys. **277** (2008), no. 2, 385–421.

- [25] S. Popa and A. Wassermann, *Actions of compact Lie groups on von Neumann algebras* (English. English, French summary), C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), no. 4, 421–426.
- [26] P. Podleś, *Quantum spheres*, Lett. Math. Phys. **14** (1987), no. 3, 193–202.
- [27] P. Podleś, *Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups*, Comm. Math. Phys. **170** (1995), no. 1, 1–20.
- [28] M. Takesaki, *Conditional expectations in von Neumann algebras*, J. Funct. Anal. **9** (1972), 306–321.
- [29] R. Tomatsu, *Amenable discrete quantum groups*, J. Math. Soc. Japan **58** (2006), no. 4, 949–964.
- [30] R. Tomatsu, *Compact quantum ergodic systems*, J. Funct. Anal. **254** (2008), no. 1, 1–83.
- [31] R. Tomatsu, *A characterization of right coideals of quotient type and its application to classification of Poisson boundaries*, Comm. Math. Phys. **275** (2007), 271–296.
- [32] R. Tomatsu, *A Galois correspondence for compact quantum group actions*, To appear in Journal für die reine und angewandte Mathematik (Crelle's journal).
- [33] Y. Ueda, *A minimal action of the compact quantum group $SU_q(n)$ on a full factor*, J. Math. Soc. Japan **51** (1999), no. 2, 449–461.
- [34] Y. Ueda, *Free product actions and their applications*, Non-commutativity, infinite-dimensionality and probability at the crossroads, 388–411, QP-PQ: Quantum Probab. White Noise Anal., 16, World Sci. Publ., River Edge, NJ, 2002.
- [35] A. van Daele and S.-Z. Wang, *Universal quantum groups*, Internat. J. Math. **7** (1996), no. 2, 255–263.
- [36] S. Vaes, *Strictly outer actions of groups and quantum groups*, J. Reine Angew. Math. **578** (2005), 147–184.
- [37] R. Tomatsu and S. Vaes, Work in progress.
- [38] S. Vaes and N. Vander Vennet, *Identification of the Poisson and Martin boundaries of orthogonal discrete quantum groups*, J. Inst. Math. Jussieu **7** (2008), no. 2, 391–412.
- [39] S. Vaes and N. Vander Vennet, To appear.
- [40] S. Vaes and R. Vergnioux, *The boundary of universal discrete quantum groups, exactness and factoriality*, ArXiv:math.OA/0509706.
- [41] A. Wassermann, *Coactions and Yang-Baxter equations for ergodic actions and subfactors. Operator algebras and applications*, Vol. 2, 203–236, London Math. Soc. Lecture Note Ser. **136**, Cambridge Univ. Press, Cambridge (1988).
- [42] A. Wassermann, *Ergodic actions of compact groups on operator algebras. I. General theory*, Ann. of Math. (2) **130** (1989), no. 2, 273–319.
- [43] A. Wassermann, *Ergodic actions of compact groups on operator algebras. III. Classification for $SU(2)$* , Invent. Math. **93** (1988), no. 2, 309–354.
- [44] S. L. Woronowicz, *Twisted $SU(2)$ group. An example of a noncommutative differential calculus*, Publ. Res. Inst. Math. Sci. **23** (1987), no. 1, 117–181.
- [45] S. L. Woronowicz, *Compact quantum groups*, Symétries quantiques (Les Houches, 1995), 845–884, North-Holland, Amsterdam, (1998).

E-mail address: tomatsu@ms.u-tokyo.ac.jp