

Multiplicity and invariants in birational geometry

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Definition 1

Let X be a variety and let x be a point of X . The multiplicity of x on X , denoted $\text{mult}_x X$, means $\text{mult}_{O_{X,x}}$.

For a normal variety X , a prime divisor E over X means that a prime divisor E appears on a resolution $f: Y \rightarrow X$. Let E be a prime divisor over X and appear on a log resolution $f: Y \rightarrow X$ of X . The log discrepancy of X with respect to E is

$$a_E(X) := \text{ord}_E(K_Y - f^*K_X) + 1.$$

For a closed subset Z of X , the minimal log discrepancy $\text{mld}_Z(X)$ over Z is the infimum of $a_E(X)$ for all prime divisor E over X with center in Z .

Let $I \subset O_X$ be a nonzero ideal. The log-canonical threshold of I in X is defined as follows:

$$\text{lct}(I) = \sup\{c \in \mathbb{Q} \mid (X, I^c) \text{ is log canonical}\}.$$

Conjecture 1

Let X be an n -dimensional locally a complete intersection variety. Then $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$ for a closed point x of X and equality holds if and only if $\text{emb}(X, x) = 2n - \text{mld}_x X$.

Conjecture 2

Let X be an n -dimensional locally a complete intersection variety with log canonical singularities. Then $\text{mult}_x X \leq 2^{n-\lceil \text{lct}(m_x) \rceil}$ for a closed point x of X and equality holds if and only if $\text{emb}(X, x) = 2n - \lceil \text{lct}(m_x) \rceil$.

Watanabe's Conjecture

Let X be an n -dimensional locally a complete intersection variety with canonical singularities. Then $\text{mult}_x X \leq 2^{n-1}$ for a closed point x of X and equality holds if and only if $\text{emb}(X, x) = 2n - 1$.

Remark

- $\lceil \text{lct}(m_x) \rceil \leq \text{mld}_x X$
- Conjecture 1 $\Rightarrow$ Conjecture 2 $\Rightarrow$ Watanabe's Conjecture
- Watanabe's Conjecture is a generalization of Proposition 3.1 in [W].

Proposition 1

Let X be an n -dimensional Gorenstein variety with canonical singularities and let x in X be a closed point. Let $\text{emb}(X, x) = v$ and $\lceil \text{lct}(m_x) \rceil = c$.

(1) If $n + 1 - c = 2r + 1$, then

$$\text{mult}_x X \leq \binom{v-n+r}{r} + \binom{v-n+r-1}{r-1}.$$

(2) If $n + 1 - c = 2r$, then

$$\text{mult}_x X \leq 2 \binom{v-n+r-1}{r-1}.$$

Remark

This Proposition 1 is a generalization of Theorem 3.1 in [HW] of characteristic zero. But our proof is just an imitation of the proof of Theorem 3.1 in [HW].

Proposition 2

Let X be an n -dimensional normal locally complete intersection variety with canonical singularities.

If $n \leq 5$, $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$ for a closed point x of X .

If $n \leq 4$, $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$ for a closed point x of X and equality holds if and only if $\text{emb}(X, x) = 2n - \text{mld}_x X$.

Theorem 1

Let X be an n -dimensional locally a complete intersection variety with canonical singularities. Let $\lceil \text{lct}(m_x) \rceil = c$. Let $e = \text{emb}(X, x) - n$. Let $c + 2e - n > 0$.

(1) If $n - c + 1 = 2r$, then

$$\text{mult}_x X \leq 2 \sum_{j=0}^{c+2e-n} \binom{c+2e-n}{j} \binom{n-c-e+r-j-1}{r-j-1}.$$

(2) If $n - c + 1 = 2r + 1$, then

$$\text{mult}_x X \leq \sum_{j=0}^{c+2e-n} \binom{c+2e-n}{j} \left\{ \binom{n-c-e+r-j-1}{r-j-1} + \binom{n-c-e+r-j}{r-j} \right\}.$$

Key idea

By the formula (13) in [IR], we obtain

$$\text{mld}_x X = \inf_m \{ (m+1)n - \dim \pi_m^{-1}(x) \}, \text{ where}$$

$\pi_m: X_m \rightarrow X$ is a truncation map. By this equality, we get a regular sequence consisting the initial terms of some defining polynomials of X . Using this fact and Proposition 1, I proved Proposition 2 and Theorem 1.

Corollary 1

Let X be an n -dimensional locally a complete intersection variety with canonical singularities. If $n \leq 32$, then

$\text{mult}_x X \leq 2^{n-\lceil \text{lct}(m_x) \rceil}$ for a closed point x of X and equality holds if and only if $\text{emb}(X, x) = 2n - \lceil \text{lct}(m_x) \rceil$.

Corollary 2

Watanabe's Conjecture is true if $n \leq 32$.

Theorem 2

Let X be an n -dimensional locally a complete intersection simplicial toric variety. Then $\text{mld}_x X = \lceil \text{lct}(m_x) \rceil$ for a fixed closed point x of X .

Theorem 3

Let X be an n -dimensional locally a complete intersection simplicial toric variety. Then $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$ for a fixed closed point x of X and equality holds if and only if $\text{emb}(X, x) = 2n - \text{mld}_x X$.

Key idea

Nakajima classified cones corresponding to locally a complete intersection toric varieties in [N]. Using this classification, I proved Theorem 2 and Theorem 3.

Reference

- [HW] C. Huneke and K. Watanabe, Upper bound of multiplicity of F-rational rings and F-pure rings, <http://arxiv.org/abs/1310.0584>
- [IR] S. Ishii and A.J. Reguera, Singularities with the highest Mather minimal log discrepancy. Math. Z. 275 (2013), no. 3-4, 1255–1274.
- [N] H. Nakajima, Affine torus embeddings which are complete intersections, Tôhoku Math. Journal 38, 1986, 85–98.
- [W] K. Watanabe, Invariant subrings which are complete intersections, I. (Invariant subrings of finite Abelian groups), Nagoya Math. J. 77(1980), 89–98.