

The Godbillon-Vey class of codimension one foliations  
without holonomy

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In this note we prove the following result.

THEOREM. Let  $F$  be a codimension one  $C^2$ -foliation on a compact smooth manifold  $M$  and assume that  $F$  is without holonomy, namely the holonomy group of each leaf is trivial. Then the Godbillon-Vey characteristic class of  $F$  defined in  $H^3(M; \mathbb{R})$  ([3]) vanishes.

For the proof of the above result, the argument of Herman used in [4] to prove the triviality of the Godbillon-Vey invariant of foliations by planes of  $T^3$  and also the work of Novikov [7] and Imanishi [5] on codimension one foliations without holonomy play very important roles.

1. Codimension one foliations without holonomy.

Let  $M$  be a compact connected smooth manifold and let  $F$  be a codimension one  $C^2$ -foliation without holonomy on  $M$ . We fix a base point  $x_0$ , a flow  $\Phi: M \times \mathbb{R} \rightarrow M$  whose orbits are transverse to leaves of  $F$  and we denote  $\varphi(t)$  for  $\Phi(x_0, t)$  ( $t \in \mathbb{R}$ ). Following Novikov [7] (also see Imanishi [5]), we define a homomorphism

$$\chi: \pi_1(M, x_0) \longrightarrow \text{Diff}_+^2(\mathbb{R})$$

as follows, where  $\text{Diff}_+^2(\mathbb{R})$  is the group of orientation preserving diffeomorphisms of class  $C^2$  of  $\mathbb{R}$ . Let  $\omega$  be an element of  $\pi_1(M, x_0)$  represented by a closed curve  $p: (I, \dot{I}) \rightarrow (M, x_0)$

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and let  $t$  be a point of  $\mathbb{R}$ . Then  $\chi(\omega)(t)$  is defined to be a point  $t_1$  of  $\mathbb{R}$  such that there is a leaf curve  $\ell$ :  $(I, 0, 1) \rightarrow (L, \varphi(t_1), \varphi(t))$  ( $L$  is the leaf passing through  $\varphi(t)$ ) satisfying the condition: two curves  $p_+$  and  $\ell_-$  are homotopic, where  $p_+$  is the product of two curves  $p$  and  $\varphi([0, t])$  (if  $t \geq 0$ ) or  $\varphi([t, 0])$  (if  $t < 0$ ), while  $\ell_-$  is the product of two curves  $\varphi([0, t_1])$  (or  $\varphi([t_1, 0])$ ) and  $\ell$ .

$\chi$  is a well defined homomorphism (we define the product of two elements  $f$  and  $g$  of  $\text{Diff}_+^2(\mathbb{R})$  to be  $f \circ g$ ) and it is known that  $\text{Image}(\chi)$  is abelian (see [5] [7]). Now using the homomorphism  $\chi$ , we can construct a locally trivial foliated  $\mathbb{R}$ -bundle (or the suspension foliation)  $E$  over  $M$  as follows. Let  $\tilde{M}$  be the universal covering space of  $M$ . Then  $\pi_1(M, x_0)$  acts on  $\tilde{M} \times \mathbb{R}$  by the deck transformation on the first factor and through the homomorphism  $\chi$  on the second. This action is free and preserves the trivial foliation on  $\tilde{M} \times \mathbb{R}$  defined by  $\{t = \text{constant}\}$ . Therefore the quotient manifold  $E = \tilde{M} \times \mathbb{R} / \pi_1(M, x_0)$  has the structure of a locally trivial foliated  $\mathbb{R}$ -bundle over  $M$ .

Now our first important step is the following.

PROPOSITION 1. Let  $E$  be the locally trivial foliated  $\mathbb{R}$ -bundle over  $M$  defined by the homomorphism  $\chi$ . Then there is a cross-section  $\sigma: M \rightarrow E$  such that  $\text{Image}(\sigma)$  is transverse to the codimension one foliation on  $E$  and the induced foliation on  $M$  is the same as the original one  $F$ .

Proof. We define a mapping  $\psi: \tilde{M} \rightarrow \mathbb{R}$  as follows. Let  $\tilde{q}$  be a point of  $\tilde{M}$  represented by a path  $q: (I, 0) \rightarrow (M, x_0)$ . Then  $\psi(\tilde{q})$  is defined to be a point of  $\mathbb{R}$  such that there is a leaf curve  $\ell: (I, 0, 1) \rightarrow (M, \varphi \circ \psi(\tilde{q}), q(1))$ , so that two curves  $q$  and  $\ell_-$  are homotopic where  $\ell_-$  is the product

of two curves  $\varphi([0, \psi(\tilde{q})])$  (or  $\varphi([\psi(\tilde{q}), 0])$ ) and  $\ell$ . Now we define an imbedding  $\tilde{\sigma} : \tilde{M} \rightarrow \tilde{M} \times \mathbb{R}$  by  $\tilde{\sigma}(\tilde{q}) = (\tilde{q}, \psi(\tilde{q}))$ . Then it can be checked that  $\tilde{\sigma}$  is equivariant with respect to the  $\pi_1(M, x_0)$ -actions. Moreover  $\tilde{\sigma}$  is transverse to the trivial foliation on  $\tilde{M} \times \mathbb{R}$  defined by  $\{t = \text{constant}\}$  and the induced codimension one foliation on  $\tilde{M}$  coincides with the lift to  $\tilde{M}$  of the original foliation  $F$ . Therefore the induced mapping  $\sigma : M \rightarrow E$  satisfies the required conditions.

q.e.d.

REMARK 2. In the construction above, suppose that the orbit  $\text{Image}(\varphi)$  is periodic, namely for some  $k$  the equality  $\varphi(t+k) = \varphi(t)$  holds for every  $t \in \mathbb{R}$ . Then for any element  $\omega$  of  $\pi_1(M, x_0)$ ,  $\chi(\omega)$  is a periodic diffeomorphism of  $\mathbb{R}$ ;  $\chi(\omega)(t+k) = \chi(\omega)(t)$ . Thus  $\chi$  induces a homomorphism  $\chi' : \pi_1(M, x_0) \rightarrow \text{Diff}_+^2(S^1)$  where we identify  $\mathbb{R} \bmod k\mathbb{Z}$  with  $S^1$ . Imanishi [5] has proved, among other things, that  $\text{Image}(\chi')$  is topologically conjugate to rotations. Now the same proof as that of Proposition 1 gives the following.

PROPOSITION 1'. Let  $E'$  be the foliated  $S^1$ -bundle over  $M$  defined by the homomorphism  $\chi'$ . Then there is a cross-section  $\sigma' : M \rightarrow E'$  such that  $\text{Image}(\sigma')$  is transverse to the codimension one foliation on  $E'$  and the induced foliation on  $M$  is the same as the original one  $F$ .

## 2. The Godbillon-Vey class of foliated $S^1$ and $\mathbb{R}$ -bundles.

Let  $E$  be a foliated  $S^1$ -bundle of class  $C^2$  over a smooth manifold  $M$  defined by a homomorphism  $\pi_1(M) \rightarrow \text{Diff}_+^2(S^1)$ . For such object, the Godbillon-Vey class (integrated over the fibres)

is defined as an element of  $H^2(\text{Diff}_+^2(S^1); \mathbb{R})$  (the 2-dimensional cohomology group with trivial coefficients  $\mathbb{R}$  of  $\text{Diff}_+^2(S^1)$  considered as an abstract group). According to Thurston (cf. [1] [4]), this element is represented by the following cocycle  $\alpha \in C^2(\text{Diff}_+^2(S^1); \mathbb{R})$ .

DEFINITION 3. Let  $u, v$  be elements of  $\text{Diff}_+^2(S^1)$ .

Then

$$\alpha(u, v) = \int_{S^1} \log Dv(t) D \log D(u)(v(t)) dt.$$

Now let  $E$  be a locally trivial foliated  $\mathbb{R}$ -bundle over a smooth manifold  $M$  defined by a homomorphism  $\pi_1(M) \rightarrow \text{Diff}_+^2(\mathbb{R})$ . Then similarly as above, the Godbillon-Vey class for such objects is defined as an element of  $H^3(\text{Diff}_+^2(\mathbb{R}); \mathbb{R})$  as follows.

Let  $f, g, h$  be elements of  $\text{Diff}_+^2(\mathbb{R})$  and we set

$$A = \log Df^{-1}(t)$$

$$B = \log Dg^{-1}(f^{-1}(t))$$

$$C = \log Dh^{-1}(g^{-1}f^{-1}(t)).$$

Let  $\Delta^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1, x_2, x_3 \geq 0, x_1 + x_2 + x_3 \leq 1\}$  be the 3-simplex and let  $s : \Delta^3 \rightarrow \mathbb{R}$  be a function defined by

$$s(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2 + x_3)^f \left( \frac{x_2 + x_3}{x_1 + x_2 + x_3} g \left( \frac{x_3}{x_2 + x_3} h(0) \right) \right), & x_2 + x_3 \neq 0 \\ x_1 f(0), & x_2 + x_3 = 0. \end{cases}$$

$s$  is  $C^\infty$  on the interior of  $\Delta^3$ ,  $\overset{\circ}{\Delta^3}$ , and continuous on  $\Delta^3$ .

Let  $S : \Delta^3 \rightarrow \Delta^3 \times \mathbb{R}$  be defined by  $S(x_1, x_2, x_3) =$

$(x_1, x_2, x_3, s(x_1, x_2, x_3))$ . Now we define a cochain  $\beta \in$

$C^3(\text{Diff}_+^2(\mathbb{R}); \mathbb{R})$  by the formula

DEFINITION 4.

$$\beta(f, g, h) = \int_{\Delta^3} S^* \{ A dx_1 + (A+B) dx_2 + (A+B+C) dx_3 \} \{ A' dtdx_1 + (A'+B') dtdx_2 + (A'+B'+C') dtdx_3 \}.$$

Since the derivatives  $\frac{\partial s}{\partial x_1}$ ,  $\frac{\partial s}{\partial x_2}$ ,  $\frac{\partial s}{\partial x_3}$  are bounded over  $\Delta^3$ , the integral exists. We can show

PROPOSITION 5. The cochain  $\beta$  is a cocycle.

Thus  $\beta$  defines an element  $[\beta] \in H^3(\text{Diff}_+^2(\mathbb{R}); \mathbb{R})$ .

A proof of Proposition 5 together with related topics will be given in [6]. This is because, for a proof of our THEOREM, the form of the cocycle  $\beta$  is not essential. We need only the fact that the Godbillon-Vey class of a locally trivial foliated  $\mathbb{R}$ -bundle can be calculated by group cohomology argument. More precisely, let  $\rho : \pi_1(T^3) = \mathbb{Z}^3 \rightarrow \text{Diff}_+^2(\mathbb{R})$  be a homomorphism defined by three mutually commuting diffeomorphisms  $f, g, h$  of  $\mathbb{R}$  and let  $E$  be the locally trivial foliated  $\mathbb{R}$ -bundle over  $T^3$  defined by  $\rho$ . Then the Godbillon-Vey class of this foliation on  $E$  is an element of  $H^3(E; \mathbb{R}) \cong H^3(T^3; \mathbb{R}) \cong \mathbb{R}$ . Let us denote  $GV(f, g, h)$  for the corresponding real number. Under these situation, we have

PROPOSITION 6. Let  $f, g, h$  be mutually commuting elements of  $\text{Diff}_+^2(\mathbb{R})$ . Then  $z = (f, g, h) - (f, h, g) + (g, h, f) - (g, f, h) + (h, f, g) - (h, g, f)$  is a cycle (of the group  $\text{Diff}_+^2(\mathbb{R})$ ) and the equality

$$GV(f, g, h) = \beta(z)$$

holds.

A proof of this Proposition will also be given in [6].

### 3. Foliated $S^1$ and $\mathbb{R}$ -bundles over tori.

In [4], Herman has proved the following

**THEOREM 7.** Let  $E$  be a foliated  $S^1$ -bundle of class  $C^2$  over  $T^2$ . Then the Godbillon-Vey invariant of the codimension one foliation on  $E$  is zero.

In this section, we prove the following results which can be considered as generalizations of Theorem 7.

**THEOREM 8.** Let  $E$  be a foliated  $S^1$ -bundle of class  $C^2$  over a torus  $T^k$  ( $k \geq 2$ ). Then the Godbillon-Vey class of the codimension one foliation on  $E$  vanishes.

**THEOREM 9.** Let  $E$  be a locally trivial foliated  $\mathbb{R}$ -bundle over a torus  $T^k$  ( $k \geq 3$ ). Then the Godbillon-Vey class of the codimension one foliation on  $E$  vanishes.

Before proving the above Theorems, let us recall the argument of Herman [4] briefly. Let  $E$  be a foliated  $S^1$ -bundle over  $T^2$  defined by commuting diffeomorphisms  $u, v \in \text{Diff}_+^2(S^1)$ . Then  $c = (u, v) - (v, u)$  is a cycle of the group  $\text{Diff}_+^2(S^1)$  and by Thurston (cf. [1] [4]), the Godbillon-Vey invariant of  $E$ , denoted by  $Gv(u, v)$ , is given by

$$Gv(u, v) = \alpha(c).$$

Herman has proved  $\alpha(c) = 0$  by an elegant argument using known properties of elements of  $\text{Diff}_+^2(S^1)$ . Now we prove Theorems 8 and 9.

**Proof of Theorem 9.** Since the cohomology group  $H^3(T^k; \mathbb{R})$  ( $k \geq 3$ ) is generated by 3-dimensional cohomologies of various 3-dimensional subtori of  $T^k$ , we have only to prove the case  $k = 3$ . Thus let  $f, g, h \in \text{Diff}_+^2(\mathbb{R})$  be mutually commuting diffeomorphisms and let  $E$  be the locally trivial foliated

$\mathbb{R}$ -bundle over  $T^3$  defined by them. We have to prove  $GV(f, g, h) = 0$ . We consider two cases.

Case 1. All of  $f, g, h$  have fixed points.

In this case it can be proved that  $f, g, h$  have a common fixed point. In fact this follows from the following general statement.

PROPOSITION 10. Let  $f_1, \dots, f_r$  be mutually commuting homeomorphisms of  $\mathbb{R}$  and assume that all of  $f_i$  have fixed points. Then there is a common fixed point of  $f_1, \dots, f_r$ .

Proof. If  $f$  is an orientation reversing homeomorphism of  $\mathbb{R}$ , then  $f$  has a unique fixed point  $p$  and for any homeomorphism  $g$  of  $\mathbb{R}$  such that  $f \circ g = g \circ f$ , clearly  $g(p) = p$  holds. Therefore if at least one of  $f_1, \dots, f_r$  reverses the orientation, then the assertion is clear. Hence we assume that all of  $f_1, \dots, f_r$  preserve the orientation. Now first assume that at least one of  $f_1, \dots, f_r$ , say  $f_1$ , has a maximum (or minimum) fixed point  $p$ . Then since any  $f_j$  ( $j = 1, \dots, r$ ) leaves the fixed point set of  $f_1$ ,  $F(f_1)$ , invariant, we have  $f_j(p) = p$ . So  $p$  is a common fixed point. Next assume the contrary and let  $(a, b)$  be a maximal open interval contained in  $\mathbb{R} - F(f_1)$ , thus  $a, b \in F(f_1)$ . Let  $(a_1, b_1)$  be the maximal open interval containing  $(a, b)$  such that  $(a_1, b_1)$  is contained in  $\mathbb{R} - F(f_i)$  for some  $i$ . We claim that  $a_1$  and  $b_1$  are common fixed points of  $f_1, \dots, f_r$ . For from the definition, either  $(a_1, b_1) \subset \mathbb{R} - F(f_j)$  or  $f_j$  has a fixed point on  $(a_1, b_1)$ . But in either case we should have  $f_j(a_1) = a_1$  and  $f_j(b_1) = b_1$ . This completes the proof of Proposition 10.

REMARK 11. In Proposition 10, if we assume that  $f_1, \dots, f_r$  are orientation preserving diffeomorphisms of class  $C^2$ , then

we can obtain a stronger statement that if  $(a, b)$  is a maximal open interval contained in  $\mathbb{R} - F(f_1)$ , then  $a$  and  $b$  are common fixed points of  $f_1, \dots, f_r$  (cf. [4] Lemma 1).

Now we go back to the proof of Theorem 9, Case 1.

We have just proved that  $f, g, h$  have a common fixed point  $p$ . Then this fixed point defines a cross-section  $\sigma: T^3 \rightarrow E$  such that  $\text{Image}(\sigma)$  is a compact leaf of the foliation on  $E$ . Since the restriction of the Godbillon-Vey class to any leaf is trivial and since  $\text{Image}(\sigma)$  generates the 3-dimensional homology group of  $E$ , we conclude that  $GV(f, g, h) = 0$ .

Case 2. At least one of  $f, g, h$  has no fixed point.

First we claim that

$$GV(f, g, h) = GV(g, h, f) = GV(h, f, g).$$

This follows from the definition of  $GV$ . It also follows from Proposition 6. Therefore to prove our assertion  $GV(f, g, h) = 0$ , we may assume that  $h$  has no fixed points. Now let us define a  $\mathbb{Z}$ -action on  $\mathbb{R}$  by  $n(t) = h^n(t)$  ( $n \in \mathbb{Z}, t \in \mathbb{R}$ ). Then since  $h$  has no fixed points, this action is free and the quotient manifold can be identified with  $S^1$  by an orientation preserving diffeomorphism  $k: \mathbb{R}/\{h^n\} \cong S^1$ . Let  $\tilde{k}: \mathbb{R} \rightarrow \mathbb{R}$  be the lift of  $k$  such that  $\tilde{k}(0) = 0$ . It is a diffeomorphism of class  $C^2$ . Now we set  $f_1 = \tilde{k}^{-1}f\tilde{k}$ ,  $g_1 = \tilde{k}^{-1}g\tilde{k}$ ,  $h_1 = \tilde{k}^{-1}h\tilde{k}$ . Then  $f_1, g_1, h_1$  are mutually commuting diffeomorphisms of class  $C^2$  of  $\mathbb{R}$ . Let  $\mathcal{Z} = (f, g, h) - (f, h, g) + (g, h, f) - (g, f, h) + (h, f, g) - (h, g, f)$  and  $\mathcal{Z}_1 = (f_1, g_1, h_1) - (f_1, h_1, g_1) + (g_1, h_1, f_1) - (g_1, f_1, h_1) + (h_1, f_1, g_1) - (h_1, g_1, f_1)$ . Then the cycle  $\mathcal{Z}_1$  is conjugate to  $\mathcal{Z}$ :  $\mathcal{Z}_1 = \tilde{k}^{-1}\mathcal{Z}\tilde{k}$ . Since inner automorphisms of a group induce the

identity on the homology groups ([2]), we have

$$\beta(z_1) = \beta(z).$$

Therefore from Proposition 6, we obtain

$$GV(f, g, h) = GV(f_1, g_1, h_1).$$

Now from the construction,  $h_1$  is the translation of  $\mathbb{R}$  by 1 (denoted by  $T$ ) or by  $-1$  according as  $h(0) > 0$  or  $h(0) < 0$  respectively. By the definition of  $GV$ , clearly we have

$$GV(f_1, g_1, h_1) = -GV(f_1, g_1, h_1^{-1}).$$

Therefore we may assume that  $h_1 = T$ . Since  $f_1$  and  $g_1$  commute with  $h_1 = T$ ,  $f_1$  and  $g_1$  are lifts of some diffeomorphisms  $f'_1$  and  $g'_1$  of  $S^1$ . Now we claim

PROPOSITION 12. Let  $u, v$  be mutually commuting elements of  $\text{Diff}_+^2(S^1)$  and let  $\tilde{u}, \tilde{v}$  be their arbitrary lifts to  $\mathbb{R}$ . Then we have

$$GV(\tilde{u}, \tilde{v}, T) = Gv(u, v).$$

Proof. We consider  $\mathbb{R}^2 \times \mathbb{R} = \{(x_1, x_2, t); x_i, t \in \mathbb{R}\}$ ,  
 $\mathbb{R}^3 \times \mathbb{R} = \{(x_1, x_2, x_3, t); x_i, t \in \mathbb{R}\}$  and let

$$\lambda(x_1, x_2, t) = (x_1+1, x_2, \tilde{u}(t)), \quad \lambda_1(x_1, x_2, x_3, t) = (x_1+1, x_2, x_3, \tilde{u}(t))$$

$$\mu(x_1, x_2, t) = (x_1, x_2+1, \tilde{v}(t)), \quad \mu_1(x_1, x_2, x_3, t) = (x_1, x_2+1, x_3, \tilde{v}(t))$$

$$\nu(x_1, x_2, t) = (x_1, x_2, t+1), \quad \nu_1(x_1, x_2, x_3, t) = (x_1, x_2, x_3, t+1).$$

Then  $\lambda, \mu, \nu$  and  $\lambda_1, \mu_1, \nu_1$  generate free  $\mathbb{Z}^3$ -actions on  $\mathbb{R}^2 \times \mathbb{R}$  and  $\mathbb{R}^3 \times \mathbb{R}$  respectively. These actions preserve the trivial foliations defined by  $\{t = \text{constant}\}$ . The quotient manifolds  $E$  and  $E_1$  carry the structures of foliated  $S^1$ -bundle over  $T^2$  defined by  $u$  and  $v$  and locally trivial foliated

$\mathbb{R}$ -bundle over  $T^3$  defined by  $\tilde{u}, \tilde{v}, T$  respectively. Now define a mapping  $\pi: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^2 \times \mathbb{R}$  by  $\pi(x_1, x_2, x_3, t) = (x_1, x_2, t)$ . Then  $\pi$  is equivariant with respect to the  $\mathbb{Z}^3$ -actions. Therefore it induces a mapping  $\pi': E_1 \rightarrow E$ . Moreover it is easy to see that the pull back of the foliation on  $E$  by the submersion  $\pi'$  coincides with the given foliation on  $E_1$ . Therefore from the naturality of the Godbillon-Vey class, we obtain

$$(\pi')^*(gv(E)) = gv(E_1),$$

where  $gv(E)$  (resp.  $gv(E_1)$ ) is the Godbillon-Vey class of the foliation on  $E$  (resp.  $E_1$ ). Now since  $(\pi')^*$  gives an isomorphism  $H^3(E; \mathbb{R}) \cong H^3(E_1; \mathbb{R}) \cong \mathbb{R}$ , we obtain

$$GV(\tilde{u}, \tilde{v}, T_1) = Gv(u, v).$$

This completes the proof of Proposition 11.

Now by the above Proposition and the argument before it, we have

$$GV(f, g, h) = Gv(f'_1, g'_1).$$

But Herman's result (Theorem 7) implies

$$Gv(f'_1, g'_1) = 0.$$

Hence  $GV(f, g, h) = 0$ . This completes the proof of Case 2 and hence Theorem 9. q.e.d.

Next we prove Theorem 8.

Proof of Theorem 8. Since the case  $k = 2$  is just Theorem 7, we assume that  $k \geq 3$  and let  $E$  be a foliated  $S^1$ -bundle of class  $C^2$  over  $T^k$  defined by mutually commuting diffeomorphisms  $u_1, \dots, u_k \in \text{Diff}_+^2(S^1)$ . Since  $E$  is a trivial bundle as a differentiable  $S^1$ -bundle, there is a cross-section  $\sigma: T^k \rightarrow E$ .

$\sigma$  defines an isomorphism  $E \cong T^k \times S^1$ . Now the Godbillon-Vey class of the foliation on  $E$ ,  $gv(E)$ , lies in  $H^3(E; \mathbb{R}) \cong H^3(T^k; \mathbb{R}) \oplus H^2(T^k; \mathbb{R}) \otimes H^1(S^1; \mathbb{R})$ . However Herman's result (Theorem 7) implies that the second component of  $gv(E)$  is zero. Now let  $\tilde{E} = T^k \times \mathbb{R}$  be the covering space of  $E = T^k \times S^1$  corresponding to the subgroup  $\pi_1(T^k) \subset \pi_1(E)$ . Then the projection  $\pi: \tilde{E} \rightarrow E$  induces a codimension one foliation on  $\tilde{E}$ . In fact  $\tilde{E}$  has the structure of locally trivial foliated  $\mathbb{R}$ -bundle over  $T^k$  defined by mutually commuting diffeomorphisms  $\tilde{u}_1, \dots, \tilde{u}_k \in \text{Diff}_+^2(\mathbb{R})$ , where  $\tilde{u}_1$  is a suitable lift of  $u_1$  to  $\mathbb{R}$  defined by the cross-section  $\sigma$ . Hence  $gv(\tilde{E}) = 0$  by Theorem 9. Therefore we obtain  $\pi^*(gv(E)) = gv(\tilde{E}) = 0$ . Now since  $gv(E)$  lies in  $H^3(T^k; \mathbb{R}) \subset H^3(E; \mathbb{R})$  as remarked before, we conclude  $gv(E) = 0$ . q.e.d.

## 5. Proof of THEOREM.

Let  $M$  be a compact smooth manifold,  $F$  a codimension one foliation of class  $C^2$  over  $M$  and assume that  $F$  is without holonomy. Then by Proposition 1, there is a locally trivial foliated  $\mathbb{R}$ -bundle  $E$  over  $M$  defined by a homomorphism  $\chi: \pi_1(M) \rightarrow \text{Diff}_+^2(\mathbb{R})$  and an imbedding of  $M$  in  $E$  transverse to the codimension one foliation on  $E$  such that the induced foliation on  $M$  coincides with the original one  $F$ . Moreover  $\text{Image}(\chi)$  is abelian. Therefore by Theorem 9, we conclude that  $gv(E) = 0$ . Then by the naturality of the Godbillon-Vey class, we obtain  $gv(F) = 0$ . This completes the proof of THEOREM. We could also use Proposition 1' and Theorem 8 instead of Proposition 1 and Theorem 9. q.e.d.

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