

On semi-reduced quadratic forms, continued fractions and class number

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§1. Introduction

Let m be a square-free integer > 1 and let Δ be the discriminant of the real quadratic field $h(\mathbb{Q}(\sqrt{m}))$. Define

$$\Delta = \begin{cases} m, & \tilde{\omega} = \begin{cases} \frac{1}{2} + \frac{1}{2}\sqrt{m}, & \text{if } \begin{cases} m \equiv 1 \pmod{4}, \\ m \equiv 2, 3 \pmod{4}, \end{cases} \\ \sqrt{m}, & \end{cases} \end{cases}$$

$$\theta = \begin{cases} 0 & \text{if } m \equiv 1 \pmod{4}, \ell \text{ even and } 2 \nmid k_{\ell/2}, \\ 1 & \text{if } \begin{cases} m \equiv 1 \pmod{4}, \ell \text{ even and } 2 \mid k_{\ell/2}, \\ m \equiv 1 \pmod{4}, \ell \text{ odd}, \\ m \equiv 2, 3 \pmod{4}, \ell \text{ even and } 2 \nmid k_{\ell/2}, \end{cases} \\ 2 & \text{if } \begin{cases} m \equiv 2, 3 \pmod{4}, \ell \text{ even and } 2 \mid k_{\ell/2}, \\ m \equiv 2, 3 \pmod{4}, \ell \text{ odd}. \end{cases} \end{cases}$$

Write $\tilde{\omega}$ as a continued fraction:

$$\tilde{\omega} = [k_0, \overline{k_1, \dots, k_\ell}].$$

Let $\lambda_1(m)$ (resp. $\lambda_2(m)$) be the number of solutions of

$$x^2 + 4yz = \Delta \quad (\text{resp. } x^2 + 4y^2 = \Delta)$$

with $x, y, z \in \mathbb{N} = \{0, 1, 2, \dots\}$. Then H. Lu [Lu] proved the following result.

Theorem (Lu). *The class number h_Δ of $\mathbb{Q}(\sqrt{m})$ is equal to 1 if and only if*

$$\theta + \sum_{i=1}^{\ell} k_i = \lambda_1(m) + \lambda_2(m).$$

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Problem. Generalize Lu's result. More precisely, use continued fractions to list the elements of the set

$$\{aX^2 + bXY + cY^2 : a, b, c \in \mathbb{Z}, b^2 - 4ac = \Delta \text{ with } 0 \leq b < \sqrt{\Delta}\}$$

of *semi-reduced quadratic forms* of discriminant Δ and relate h_{Δ}^+ (resp. h_{Δ}) to the cardinality of this last set, i.e., to the cardinality of the set

$$\{(a, b, c) \in \mathbb{Z}^3 : \Delta = b^2 - 4ac \text{ with } 0 \leq b < \sqrt{\Delta}\}.$$

§2. Preliminaries

A quadratic form f is a homogeneous polynomial of the form

$$f = f(X, Y) = \langle a, b, c \rangle = aX^2 + bXY + cY^2, \text{ with } a, b, c \in \mathbb{Z},$$

which we may write in matrix form as

$$f(X, Y) = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

We say that the matrix of f is

$$M_f = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}.$$

By definition, the *discriminant* of f is

$$\Delta = \Delta_f = b^2 - 4ac;$$

moreover f is *primitive* if $\text{pgcd}(a, b, c) = 1$; f is *definite positive* if $\Delta < 0$, $a > 0$, $c > 0$; and f is *indefinite* if $\Delta > 0$.

Consider

$$\mathcal{F}_{\Delta} = \{\text{primitive quadratic forms of discriminant } \Delta\}.$$

We need

$$GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} r & s \\ t & u \end{pmatrix} : r, s, t, u \in \mathbb{Z}, ru - st = \pm 1 \right\},$$

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} r & s \\ t & u \end{pmatrix} : r, s, t, u \in \mathbb{Z}, ru - st = +1 \right\}.$$

We can define an action of $A \in GL_2(\mathbb{Z})$ on f by stating that

$$g = Af \text{ where } M_g = AM_f A^t.$$

Moreover we say:

$$f \sim g \text{ (} f \text{ is equivalent to } g \text{)} \iff g = Af \text{ for some } A \in GL_2(\mathbb{Z}),$$

$$f \approx g \text{ (} f \text{ is strictly equivalent to } g \text{)} \iff g = Af \text{ for some } A \in SL_2(\mathbb{Z}).$$

It turns out that the *class group* (resp. *strict class group*) of $\mathbb{Q}(\sqrt{\Delta})$ is $\mathcal{F}_\Delta / \sim$ (resp. $\mathcal{F}_\Delta / \approx$), its cardinality being denoted the *class number* h_Δ (resp. the *strict class number* h_Δ^+) of $\mathbb{Q}(\sqrt{\Delta})$.

We are extensively working with the matrices

$$E(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad A(u) = \begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix}.$$

We associate to f the quadratic numbers

$$\omega = \omega(f) = \frac{b + \sqrt{\Delta}}{2|c|} \text{ and } \Omega = \Omega(f) = \frac{b + \sqrt{\Delta}}{2|a|},$$

and define

$$\sigma(f) = \text{sign}(a).$$

Gauss defined the *right neighbour* $R(f)$ and the *left neighbour* $L(f)$ of f :

$$Rf = A(e)f = \langle c, -b - 2ce, a + be + ce^2 \rangle \quad \text{with } e = -\text{sign}(c)[\omega(f)],$$

$$Lf = A(E)^{-1}f = \langle c + bE + aE^2, -b - 2aE, a \rangle \quad \text{with } E = -\text{sign}(a)[\Omega(f)].$$

We say that $f = \langle a, b, c \rangle$ with $\Delta > 0$ is *reduced* if

$$\begin{cases} (i) & 0 < b < \sqrt{\Delta}, \\ (ii) & \sqrt{\Delta} - b < 2|c| < \sqrt{\Delta} + b, \end{cases}$$

which is equivalent to saying

$$\begin{cases} (i) & 0 < b < \sqrt{\Delta}, \\ (ii) & \sqrt{\Delta} - b < 2|a| < \sqrt{\Delta} + b. \end{cases}$$

Gauss showed how to calculate the class number h_Δ^+ . With the continued fraction algorithm, write the finite set of reduced quadratic forms of discriminant $\Delta > 0$ as a

union of h_Δ disjoint cycles, by picking up a reduced quadratic form $f = \langle a, b, c \rangle$ and taking the continued fraction of

$$\omega_0 = \omega(f) = \frac{b + \sqrt{\Delta}}{2|c|} = \frac{P_0 + \sqrt{\Delta}}{2Q_0},$$

i.e., for $i \geq 0$,

$$\begin{cases} \omega_i &= \frac{P_i + \sqrt{\Delta}}{2Q_i}, \\ P_{i+1} &= 2k_i Q_i - P_i \text{ with } k_i = [\omega_i], \\ Q_{i+1} &= \frac{\Delta - P_{i+1}^2}{4Q_i}, \end{cases}$$

where $[]$ is the greatest integer function. Hence

P_0	P_1	$\dots$	$P_{\ell-1}$	P_ℓ	$P_{\ell+1}$	$\dots$	$P_{2\ell-1}$	$\dots$
Q_0	Q_1	$\dots$	$Q_{\ell-1}$	Q_ℓ	$Q_{\ell+1}$	$\dots$	$Q_{2\ell-1}$	$\dots$
k_0	k_1	$\dots$	$k_{\ell-1}$	k_ℓ	$k_{\ell+1}$	$\dots$	$k_{2\ell-1}$	$\dots$

It is well known that f is reduced if and only if $\omega(f)$ is a reduced quadratic number (i.e., the continued fraction expansion of $\omega(f)$ is purely periodic).

Let

$$\tilde{\ell} = \begin{cases} \ell & \text{if } \ell \text{ is even,} \\ 2\ell & \text{if } \ell \text{ is odd.} \end{cases}$$

The cycle of f is

$$[f_1, f_2, \dots, f_{\tilde{\ell}}],$$

where

$$f = f_1 = \langle a, b, c \rangle = \langle \sigma(f)Q_0, P_1, -\sigma(f)Q_1 \rangle$$

and where for $i = 1, \dots, \tilde{\ell} - 1$,

$$\begin{aligned} f_{i+1} &= R(f_i) \\ &= \langle (-1)^i \sigma(f)Q_i, P_{i+1}, (-1)^{i+1} \sigma(f)Q_{i+1} \rangle \\ &= A_i f_i \end{aligned}$$

with

$$A_i = A(\sigma(f_i)k_i) = A((-1)^{i-1} \sigma(f)k_i).$$

In symbols,

$$\begin{cases} f_{i+1} = Rf_i & \text{for } 1 \leq i \leq \tilde{\ell} \text{ with } f_1 = Rf_{\tilde{\ell}}, \\ f_j = Lf_{j+1} & \text{for } 0 \leq j < \tilde{\ell} \text{ with } f_{\tilde{\ell}} = Lf_1, \end{cases}$$

and $k_i = [\omega(f_i)]$.

Fact: h_Δ^+ is the number of cycles. Moreover $h_\Delta^+ = h_\Delta$ (resp. $h_\Delta^+ = 2h_\Delta$) if the length of the continued fraction of $\omega(f)$ for any reduced quadratic form f is odd (resp. even).

See section 7 for examples of calculations of cycles.

§3. Semi-reduced forms

A quadratic form $f = \langle a, b, c \rangle$ of discriminant Δ is *semi-reduced* if

$$0 \leq b < \sqrt{\Delta},$$

and f is said to be *intermediate* if f est semi-reduced without being reduced.

In each of the following examples, we give the list of semi-reduced quadratic forms, the reduced forms being written in boldface, the intermediate forms appearing in roman style.

Example 3.1. Let $m = 14$. There are 24 semi-reduced quadratic forms of discriminant $\Delta = 56$:

$$\begin{array}{llllll} \langle 1, 0, -14 \rangle & \langle -1, 0, 14 \rangle & \langle 2, 0, -7 \rangle & \langle -2, 0, 7 \rangle & \langle 7, 0, -2 \rangle & \langle -7, 0, 2 \rangle \\ \langle 1, 2, -13 \rangle & \langle -1, 2, 13 \rangle & \langle 14, 0, -1 \rangle & \langle -14, 0, 1 \rangle & \langle 13, 2, -1 \rangle & \langle -13, 2, 1 \rangle \\ \langle 1, 4, -10 \rangle & \langle -1, 4, 10 \rangle & \langle 2, 4, -5 \rangle & \langle -2, 4, 5 \rangle & \langle 5, 4, -2 \rangle & \langle -5, 4, 2 \rangle \\ \langle 10, 4, -1 \rangle & \langle -10, 4, 1 \rangle & \langle 1, 6, -5 \rangle & \langle -1, 6, 5 \rangle & \langle 5, 6, -1 \rangle & \langle -5, 6, 1 \rangle \end{array}$$

Example 3.2. Let $m = 19$. There are 36 semi-reduced quadratic forms of discriminant $\Delta = 76$:

$$\begin{array}{llllll} \langle 1, 0, -19 \rangle & \langle -1, 0, 19 \rangle & \langle 19, 0, -1 \rangle & \langle -19, 0, 1 \rangle & \langle 1, 2, -18 \rangle & \langle -1, 2, 18 \rangle \\ \langle 2, 2, -9 \rangle & \langle -2, 2, 9 \rangle & \langle 3, 2, -6 \rangle & \langle -3, 2, 6 \rangle & \langle 6, 2, -3 \rangle & \langle -6, 2, 3 \rangle \\ \langle 9, 2, -2 \rangle & \langle -9, 2, 2 \rangle & \langle 18, 2, -1 \rangle & \langle -18, 2, 1 \rangle & \langle 1, 4, -15 \rangle & \langle -1, 4, 15 \rangle \\ \langle 3, 4, -5 \rangle & \langle -3, 4, 5 \rangle & \langle 5, 4, -3 \rangle & \langle -5, 4, 3 \rangle & \langle 15, 4, -1 \rangle & \langle -15, 4, 1 \rangle \\ \langle 1, 6, -10 \rangle & \langle -1, 6, 10 \rangle & \langle 2, 6, -5 \rangle & \langle -2, 6, 5 \rangle & \langle 5, 6, -2 \rangle & \langle -5, 6, 2 \rangle \\ \langle 10, 6, -1 \rangle & \langle -10, 6, 1 \rangle & \langle 1, 8, -3 \rangle & \langle -1, 8, 3 \rangle & \langle 3, 8, -1 \rangle & \langle -3, 8, 1 \rangle \end{array}$$

Example 3.3. Let $m = 26$. There are 36 semi-reduced quadratic forms of discriminant $\Delta = 104$:

$$\begin{array}{llllll} \langle 1, 0, -26 \rangle & \langle -1, 0, 26 \rangle & \langle 2, 0, -13 \rangle & \langle -2, 0, 13 \rangle & \langle 13, 0, -2 \rangle & \langle -13, 0, 2 \rangle \\ \langle 26, 0, -1 \rangle & \langle -26, 0, 1 \rangle & \langle 1, 2, -25 \rangle & \langle -1, 2, 25 \rangle & \langle 5, 2, -5 \rangle & \langle -5, 2, 5 \rangle \\ \langle 25, 2, -1 \rangle & \langle -25, 2, 1 \rangle & \langle 1, 4, -22 \rangle & \langle -1, 4, 22 \rangle & \langle 2, 4, -11 \rangle & \langle -2, 4, 11 \rangle \\ \langle 11, 4, -2 \rangle & \langle -11, 4, 2 \rangle & \langle 22, 4, -1 \rangle & \langle -22, 4, 1 \rangle & \langle 1, 6, -17 \rangle & \langle -1, 6, 17 \rangle \\ \langle 17, 6, -1 \rangle & \langle -17, 6, 1 \rangle & \langle 1, 8, -10 \rangle & \langle -1, 8, 10 \rangle & \langle 2, 8, -5 \rangle & \langle -2, 8, 5 \rangle \\ \langle 5, 8, -2 \rangle & \langle -5, 8, 2 \rangle & \langle 10, 8, -1 \rangle & \langle -10, 8, 1 \rangle & \langle 1, 10, -1 \rangle & \langle -1, 10, 1 \rangle \end{array}$$

Example 3.4. Let $m = 33$. There are 20 semi-reduced quadratic forms of discriminant $\Delta = 33$:

$$\begin{array}{ccccc} \langle 1, 1, -8 \rangle & \langle -1, 1, 8 \rangle & \langle 2, 1, -4 \rangle & \langle -2, 1, 4 \rangle & \langle 4, 1, -2 \rangle \\ \langle -4, 1, 2 \rangle & \langle 8, 1, -1 \rangle & \langle -8, 1, 1 \rangle & \langle 1, 3, -6 \rangle & \langle -1, 3, 6 \rangle \\ \langle 2, 3, -3 \rangle & \langle -2, 3, 3 \rangle & \langle 3, 3, -2 \rangle & \langle -3, 3, 2 \rangle & \langle 6, 3, -1 \rangle \\ \langle -6, 3, 1 \rangle & \langle 1, 5, -2 \rangle & \langle -1, 5, 2 \rangle & \langle 2, 5, -1 \rangle & \langle -2, 5, 1 \rangle \end{array}$$

Example 3.5. Let $m = 35$. There are 40 semi-reduced quadratic forms of discriminant $\Delta = 140$:

$$\begin{array}{ccccc} \langle 19, 8, -1 \rangle & \langle -19, 8, 1 \rangle & \langle 1, 10, -10 \rangle & \langle -1, 10, 10 \rangle & \langle 1, 8, -19 \rangle \\ \langle -1, 8, 19 \rangle & \langle 1, 0, -35 \rangle & \langle -1, 0, 35 \rangle & \langle 5, 0, -7 \rangle & \langle -5, 0, 7 \rangle \\ \langle 7, 0, -5 \rangle & \langle -7, 0, 5 \rangle & \langle 35, 0, -1 \rangle & \langle -35, 0, 1 \rangle & \langle 1, 2, -34 \rangle \\ \langle -1, 2, 34 \rangle & \langle 2, 2, -17 \rangle & \langle -2, 2, 17 \rangle & \langle 1, 4, -31 \rangle & \langle -1, 4, 31 \rangle \\ \langle 17, 2, -2 \rangle & \langle -17, 2, 2 \rangle & \langle 34, 2, -1 \rangle & \langle -34, 2, 1 \rangle & \langle 31, 4, -1 \rangle \\ \langle -31, 4, 1 \rangle & \langle 1, 6, -26 \rangle & \langle -1, 6, 26 \rangle & \langle 2, 6, -13 \rangle & \langle -2, 6, 13 \rangle \\ \langle 13, 6, -2 \rangle & \langle -13, 6, 2 \rangle & \langle 26, 6, -1 \rangle & \langle -26, 6, 1 \rangle & \langle 2, 10, -5 \rangle \\ \langle -2, 10, 5 \rangle & \langle 5, 10, -2 \rangle & \langle -5, 10, 2 \rangle & \langle 10, 10, -1 \rangle & \langle -10, 10, 1 \rangle \end{array}$$

Let $f = \langle a, b, c \rangle$ be a reduced form of discriminant Δ such that $e = \sigma(f)[\omega(f)]$. Consider the quadratic forms associated to f and given by the following two cases (the first may be empty):

Case (i)	Case (ii)
$E(u)f = \langle a + bu + cu^2, b + 2cu, c \rangle$ with $1 \leq \sigma(f)u \leq \frac{b}{2 c }$	$A(u)f = \langle c, -b - 2cu, a + bu + cu^2 \rangle$ with $\frac{b}{2 c } \leq \sigma(f)u \leq \sigma(f)e = [\omega(f)]$

We can show that the quadratic forms of Cases (i) and (ii) are all semi-reduced (i.e., are either reduced or intermediate) and that the intermediate forms $\langle a', b', c' \rangle$ of Cases (i) and (ii) respectively verify the following properties:

$$\text{Case (i)} \begin{cases} a'c' < 0, \\ |a'| > |c'|, \\ R\langle a', b', c' \rangle = Rf, \end{cases} \quad \text{Case (ii)} \begin{cases} a'c' < 0, \\ |a'| < |c'|, \\ L\langle a', b', c' \rangle = f. \end{cases}$$

These last properties characterize intermediate quadratic forms. We can show that

intermediate forms come from reduced forms. More precisely, let $g = \langle a', b', c' \rangle$ be an intermediate quadratic form. Then the following properties hold true:

(i) If $|a'| > |c'|$, then the forms $f_1 = Rg$ and $f = Lf_1 = \langle a, b, c \rangle$ are both reduced and

$$g = E(u)f = \langle a + bu + cu^2, b + 2cu, c \rangle$$

with

$$1 \leq \sigma(f)u \leq \frac{b}{2|c|}.$$

(ii) If $|a'| < |c'|$, then the form $f = Lg = \langle a, b, c \rangle$ is reduced and

$$g = A(u)f = \langle c, -b - 2cu, a + bu + cu^2 \rangle$$

with

$$\frac{b}{2|c|} \leq \sigma(f)u \leq \sigma(f)e = [\omega(f)].$$

We can now state our first result.

Theorem. (1) For each reduced form $f = \langle a, b, c \rangle$ of discriminant Δ , the quadratic forms associated to f and defined by

$$\begin{cases} E(u)f & \text{with } 1 \leq \sigma(f)u \leq \frac{b}{2|c|}, \\ A(u)f & \text{with } \frac{b}{2|c|} \leq \sigma(f)u \leq [\omega(f)], \end{cases}$$

are all different from one another and form a set $I(f)$ of cardinality

$$\#I(f) = \begin{cases} [\omega(f)] & \text{if } (2c) \nmid b, \\ 1 + [\omega(f)] & \text{if } (2c) \mid b. \end{cases}$$

(2) If f and g are two different reduced forms, then $I(f) \cap I(g) = \phi$.

(3) Moreover the (disjoint) union of the $I(f)$'s when f runs through the set $\text{Red}(\Delta)$ of reduced forms is a set equal to the set of semi-reduced forms and is of cardinality

$$2 \left(\sum_{\substack{0 \leq t < \sqrt{\Delta} \\ t \equiv \Delta \pmod{2}}} \tau \left(\frac{\Delta - t^2}{4} \right) \right),$$

where τ is the number of divisors function.

§5. The palindrome level

Suppose that the continued fraction of a quadratic number α is purely periodic (i.e. α is reduced):

$$\alpha = [\overline{k_1, k_2, \dots, k_\ell}]$$

with ℓ minimal. We associate to α the word

$$w(\alpha) = k_1 k_2 \dots k_\ell$$

and say that it is defined up to a cyclic permutation (of order ℓ).

Recall that a *palindrome* ($KA I BU N$) is a word which once read from left to right or from right to left is the same.

Examples. In french, english, german and japanese:

(i) ELU PAR CETTE CRAPULE

(ii) NAME NO ONE MAN

(iii) EIN NEGER MIT GAZELLE ZAGT IM REGEN NIE

(iv) さくはなの のもきよきかに かおるよる
おかにかきよき もののなはくさ

(v) てきをのし いさましいし まさいしのおきて

(vi) たけやぶ やけた

When ℓ is odd, we say that α has 1 (resp. 0) *central element* if $w(\alpha)$ is (resp. is not) a palindrome, the central element being the center of $w(\alpha)$. When ℓ is even, we say α has 2 *central elements* (resp. 0 *central element*) if after an eventual cyclic permutation, $w(\alpha)$ is (resp. is not) the concatenation of a palindrome of length $\ell - 1$ and of a palindrome of length 1, the two central elements being the two centers of the concatenated palindromes.

Definition. The *palindrome level* $s = s(\alpha) \in \{0, 1, 2\}$ of a reduced quadratic number is equal to the number of even central elements.

In examples 7.1 to 7.5, s is respectively equal to 2, 1, 1, 0, 0.

§6. The main results

We can now relate the class number h_Δ of $\mathbb{Q}(\sqrt{\Delta})$ to the cardinality of the set of semi-reduced forms by writing this cardinality in terms of the partial quotients of non equivalent reduced quadratic numbers.

Theorem. Let Δ be the discriminant of a real quadratic field $\mathbb{Q}(\sqrt{\Delta})$. Put

$$g = \begin{cases} 0 & \text{if } \Delta \equiv 1 \pmod{4}, \\ 2^{r+1} & \text{if } \Delta \equiv 4 \text{ ou } 12 \pmod{16}, \\ 2^{r+2} & \text{if } \Delta \equiv 8 \pmod{16}, \end{cases}$$

where r is the number of odd primes dividing Δ . Then the class number h_Δ of $\mathbb{Q}(\sqrt{\Delta})$ is equal to h_0 if and only if there exist h_0 reduced quadratic numbers $\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(h_0)}$ associated to h_0 quadratic forms non equivalent to one another such that the number of all semi-reduced forms of discriminant Δ is

$$2 \left(s_\Delta + \sum_{j=1}^{h_0} \sum_{t=1}^{\ell_j} k_t^{(j)} \right),$$

where for $j = 1, \dots, h_0$,

$$\alpha^{(j)} = \left[k_1^{(j)}, k_2^{(j)}, \dots, k_{\ell_j}^{(j)} \right]$$

with ℓ_j minimal, and where the palindrome level s_Δ of Δ , defined as the sum of all the palindrome levels of the $\alpha^{(j)}$'s, is equal to

$$s_\Delta = \sum_{j=1}^{h_0} s(\alpha^{(j)}) = \frac{g}{2}.$$

Theorem. Let Δ be the discriminant of a real quadratic field $\mathbb{Q}(\sqrt{\Delta})$. Put

$$g_1 = \begin{cases} 0 & \text{if } \Delta \equiv 1 \pmod{4}, \\ 2^r & \text{if } \Delta \equiv 4 \text{ ou } 12 \pmod{16}, \\ 2^{r+1} & \text{if } \Delta \equiv 8 \pmod{16}, \end{cases}$$

where r is the number of odd primes dividing Δ . Then the class number h_Δ of $\mathbb{Q}(\sqrt{\Delta})$ is equal to h_1 if and only if there exist h_1 reduced quadratic numbers $\beta^{(1)}, \beta^{(2)}, \dots, \beta^{(h_1)}$ associated to h_1 quadratic forms non equivalent to one another such that

$$\#\{(A, B, C) \in \mathbb{N}^3 : \Delta = B^2 + 4AC\} = s_\Delta + \sum_{j=1}^{h_1} \sum_{t=1}^{\ell_j} k_t^{(j)},$$

where for $j = 1, \dots, h_1$,

$$\beta^{(j)} = \left[k_1^{(j)}, k_2^{(j)}, \dots, k_{\ell_j}^{(j)} \right]$$

with ℓ_j minimal, and where the palindrome level s_Δ of Δ , defined as the sum of all the palindrome levels of the $\beta^{(j)}$'s, is

$$s_\Delta = \sum_{j=1}^{h_1} s(\beta^{(j)}) = g_1.$$

§7. Five examples

One has a chance of verifying the last theorems with the following examples.

Example 7.1. Let $m = 14$, so $\Delta = 56$. Here $h_{\Delta}^+ = 2$ and $h_{\Delta} = 1$. The continued fraction expansion of $\omega = \frac{4+\sqrt{56}}{2 \cdot 2}$ is $[2, 1, 6, 1]$:

$$\omega = \frac{2 + 2\sqrt{10}}{2 \cdot 3} \longleftrightarrow \begin{array}{|c|c|c|c|c|c|} \hline 4 & 4 & 6 & 6 & 4 & \dots \\ \hline 2 & 5 & 1 & 5 & 2 & \dots \\ \hline 2 & 1 & 6 & 1 & 2 & \dots \\ \hline \end{array}$$

Since $w(\omega) = 1, \underline{6}, 1, \underline{2}$, we have $s(\omega) = 2$. There are two cycles of reduced forms (in boldface) to which are attached the intermediate forms:

$$\begin{array}{lcl} \langle -7, 0, 2 \rangle & \leftarrow & \langle -5, 4, 2 \rangle \Rightarrow \langle 2, 4, -5 \rangle \\ \langle 2, 0, -7 \rangle & \leftarrow & \uparrow \downarrow \\ & & \langle 1, 6, -5 \rangle \Leftarrow \langle -5, 6, 1 \rangle \end{array} \begin{array}{l} \rightarrow \langle -10, 4, 1 \rangle \\ \rightarrow \langle -13, 2, 1 \rangle \\ \rightarrow \langle -14, 0, 1 \rangle \\ \rightarrow \langle 1, 0, -14 \rangle \\ \rightarrow \langle 1, 2, -13 \rangle \\ \rightarrow \langle 1, 4, -10 \rangle \end{array}$$

$$\begin{array}{lcl} \langle 7, 0, -2 \rangle & \leftarrow & \langle 5, 4, -2 \rangle \Rightarrow \langle -2, 4, 5 \rangle \\ \langle -2, 0, 7 \rangle & \leftarrow & \uparrow \downarrow \\ & & \langle -1, 6, 5 \rangle \Leftarrow \langle 5, 6, -1 \rangle \end{array} \begin{array}{l} \rightarrow \langle 10, 4, -1 \rangle \\ \rightarrow \langle 13, 2, -1 \rangle \\ \rightarrow \langle 14, 0, -1 \rangle \\ \rightarrow \langle -1, 0, 14 \rangle \\ \rightarrow \langle -1, 2, 13 \rangle \\ \rightarrow \langle -1, 4, 10 \rangle \end{array}$$

Example 7.3. Let $m = 26$, so $\Delta = 104$. Here $h_{\Delta}^+ = h_{\Delta} = 2$. The continued fraction expansion of $\omega_1 = \frac{8+\sqrt{104}}{2 \cdot 2}$ is $[4, 1, 1]$:

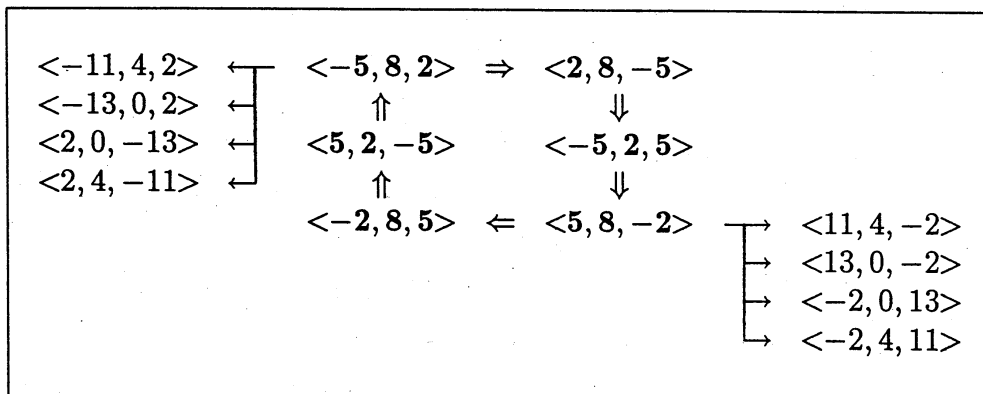
$$\omega_1 = \frac{8 + \sqrt{104}}{2 \cdot 2} \longleftrightarrow \begin{array}{|c|c|c|c|c|} \hline 8 & 8 & 2 & 8 & \dots \\ \hline 2 & 5 & 5 & 2 & \dots \\ \hline 4 & 1 & 1 & 4 & \dots \\ \hline \end{array}$$

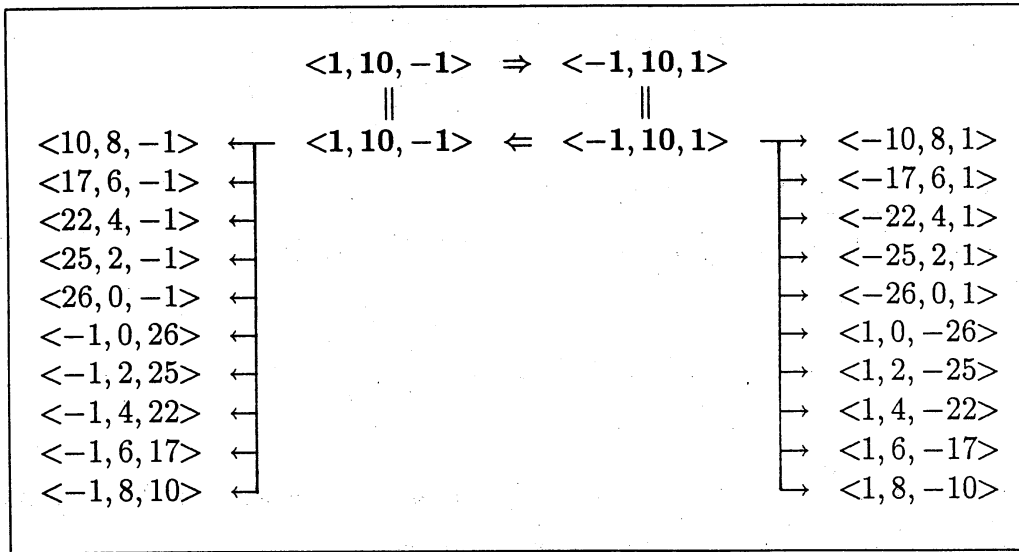
Since $w(\omega_1) = 1, \underline{4}, 1$, we have $s(\omega_1) = 1$.

The continued fraction expansion of $\omega_2 = \frac{10+\sqrt{104}}{2 \cdot 1}$ is $[\underline{10}]$:

$$\omega_2 = \frac{10 + \sqrt{104}}{2 \cdot 1} \longleftrightarrow \begin{array}{|c|c|c|} \hline 10 & 10 & \dots \\ \hline 1 & 1 & \dots \\ \hline 10 & 10 & \dots \\ \hline \end{array}$$

Since $w(\omega_2) = \underline{10}$, we have $s(\omega_2) = 1$. There are two cycles of reduced forms (in boldface) to which are attached the intermediate forms:

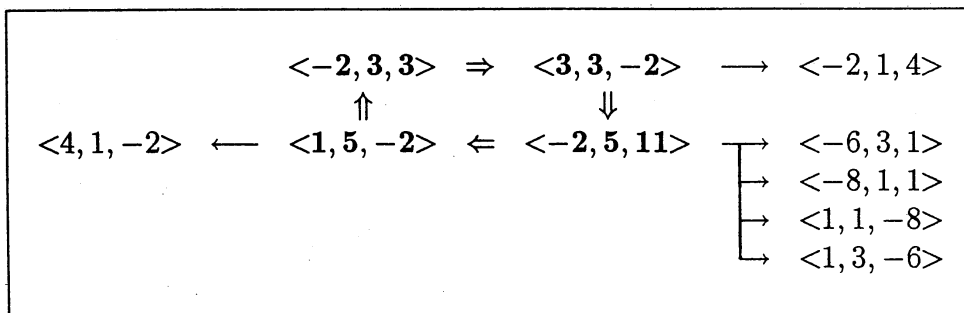


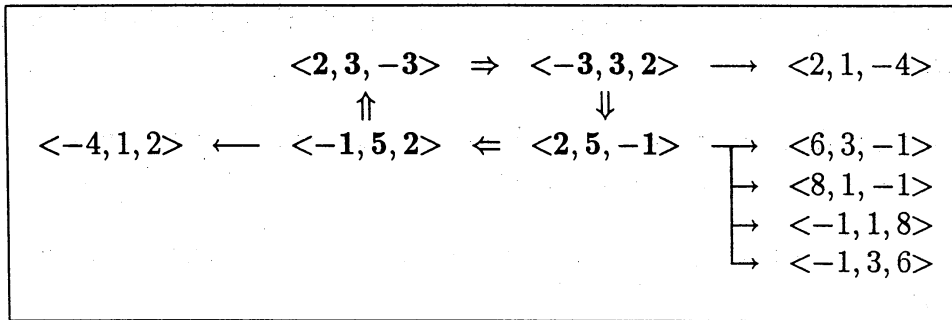


Example 7.4. Let $m = 33$, so $\Delta = 33$. Here $h_{\Delta}^{+} = 2$ and $h_{\Delta} = 1$. The continued fraction expansion of $\omega = \frac{5+\sqrt{33}}{2 \cdot 1}$ is $[5, 2, 1, 2]$:

$$\omega = \frac{5 + \sqrt{33}}{2 \cdot 1} \longleftrightarrow \begin{array}{|c|c|c|c|c|c|} \hline 5 & 5 & 3 & 3 & 5 & \dots \\ \hline 1 & 2 & 3 & 2 & 1 & \dots \\ \hline 5 & 2 & 1 & 2 & 5 & \dots \\ \hline \end{array}$$

Since $w(\omega) = 2, \underline{1}, 2, \underline{5}$, we have $s(\omega) = 0$. There are two cycles of reduced forms (in boldface) to which are attached the intermediate forms:

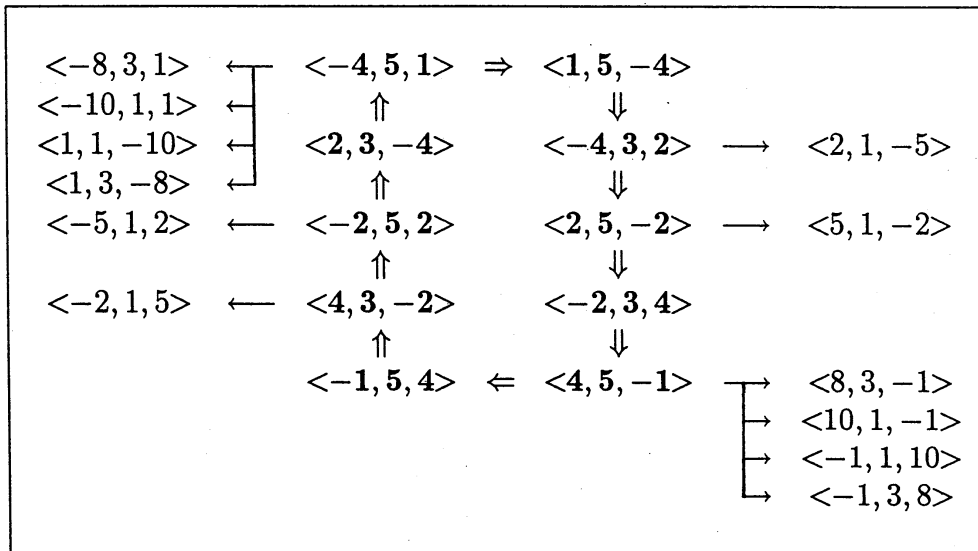




Example 7.5. Let $m = 41$, so $\Delta = 41$. We will see $h_{\Delta}^{+} = h_{\Delta} = 1$. The continued fraction expansion of $\omega = \frac{5+\sqrt{41}}{2 \cdot 1}$ is $[5, 1, 2, 2, 1]$:

$$\omega = \frac{5 + \sqrt{41}}{2 \cdot 1} \longleftrightarrow \begin{array}{|c|c|c|c|c|c|c|} \hline 5 & 5 & 3 & 5 & 3 & 5 & \dots \\ \hline 1 & 4 & 2 & 2 & 4 & 1 & \dots \\ \hline 5 & 1 & 2 & 2 & 1 & 5 & \dots \\ \hline \end{array}$$

Since $w(\omega) = 2, 1, \mathbf{5}, 1, 2$, we have $s(\omega) = 0$. There is only one cycle of reduced forms (in boldface) to which are attached the intermediate forms.



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