

A square principle in the context of $\mathcal{P}_\kappa\lambda$

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Abstract

We introduce a combinatorial principle for $\mathcal{P}_\kappa\lambda$ based upon $\square_\kappa$. Although we cannot transfer one of the clauses of $\square_\kappa$ to this context, we can replicate some of the desired consequences of that clause. We discuss this situation and its implications along with proving the relative consistency of some $\mathcal{P}_\kappa\lambda$ versions of $\square_\kappa$.

1 Introduction

In this paper, we discuss the problem of generalising the square principle to the context of $\mathcal{P}_\kappa\lambda$. The research presented below is discussed in the author's thesis, [9]. (In fact, the principles presented there are slight variations on those defined below.) This combinatorial research follows a well-established tradition and is guided by the idea of transferring interesting notions from the theory of the combinatorics of ordinal numbers. For example, Jensen's diamond principle (see [5]) has been usefully generalised to this context (originally by Jech in [4], but also by Matet in [8] and by Džamonja in [3]).

The square principle cannot be directly transferred to the context of $\mathcal{P}_\kappa\lambda$ for various reasons, as discussed below. The general approach that we follow is to establish a basic nontrivial square principle for $\mathcal{P}_\kappa\lambda$ then explicitly add

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further properties of square in more complex forcings. In this paper, following the basic principle, we define a second principle in which the square principle's non-reflection property is added.

Throughout this paper, κ is a regular infinite cardinal and λ is a cardinal with $\kappa \leq \lambda$. We now give some basic definitions and clarify the notation used in this paper.

Let $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$. $\mathcal{P}_\kappa\lambda$ is typically ordered by $\subseteq$ and will be throughout this paper. Combinatorial ideas such as clubs and stationarity can be defined in this context, as described in [6]. Note that κ and λ are arguments and may be replaced by specified cardinals or sets respectively. In this paper, we will frequently consider $\mathcal{P}_{|x|}(x)$ where x is a set. Note that $\mathcal{P}_\kappa\lambda$ is also commonly written as $[\lambda]^{<\kappa}$.

The notation used in this paper is mostly standard. By $x \subset y$ we mean ($x \subseteq y$ and $x \neq y$). We write $\lim(\alpha)$ as an abbreviation for “ α is a limit ordinal”; $\text{otp}(X)$ denotes the ordertype of a wellordered set X and for an ordinal α , $\text{cf}(\alpha)$ denotes the cofinality. For a function f , $\text{dom}(f)$ denotes the domain of f and $\text{im}(f)$ denotes image of f , while $f \restriction X$ denotes the restriction of f to X where $X \subseteq \text{dom}(f)$. In forcing proofs, we follow the convention that for two conditions p, q , $p \leq q$ means p is a *weaker* condition than q . Lastly, we use $\text{reg}(X)$ to denote the set of elements of X of regular cardinality.

Before we develop a version of the square principle in the context of $\mathcal{P}_\kappa\lambda$, we introduce the standard principle. This principle, denoted $\square_\kappa$, was developed by Jensen and has proved a useful tool in various areas of mathematical logic. It is defined as follows (although it should be noted that other equivalent formulations exist).

Definition 1.1 $\square_\kappa$ is the statement that there is a sequence $\langle C_\alpha : \alpha \in \kappa^+, \lim(\alpha) \rangle$ with the following properties:

- (i) C_α is a club subset of α
- (ii) if $\text{cf}(\alpha) < \kappa$ then $\text{otp}(C_\alpha) < \kappa$
- (iii) (Coherence:) if $\beta \in C_\alpha$ and $\lim(\beta)$ then $C_\beta = C_\alpha \cap \beta$.

Forcing can be used to produce a model of set theory in which $\square_\kappa$ holds. This approach uses a partial order whose elements are initial segments of potential square sequences. It is also known that $\square_\kappa$ holds in L , the universe

of constructible sets. The best-known proof uses fine structure theory and is due to Jensen; a good account of this proof is given in [2].

The square principle encapsulates various interesting properties. Coherence and anticoherence are discussed in the next section. Another property, non-reflection, is discussed further in the final section of this paper.

2 A square principle in the context of $\mathcal{P}_\kappa\lambda$

We will define a square-like principle that asserts the existence of a coherent set of subsets of $\mathcal{P}_\kappa\lambda$ indexed by the elements of $\mathcal{P}_\kappa\lambda$. Note that in considering $C_x \subseteq \mathcal{P}_{|x|}(x)$ for $x \in \mathcal{P}_\kappa\lambda$ we require $|x|$ to be regular and hence no club of $\mathcal{P}_{|x|}(x)$ will have cardinality $< |x|$. Thus, the cardinalities of the clubs cannot be limited as they are for those corresponding to singular ordinals in $\square_\kappa$. It is necessary, therefore, to introduce alternative non-triviality conditions that add some of the basic properties of $\square_\kappa$. Also, note that if κ is a successor cardinal, coherence is trivial for a club of $\mathcal{P}_\kappa\lambda$, that is for the elements of $[\lambda]^\kappa$. Thus, while $\square_\kappa$ actually asserts a property of κ^+ , the principle defined below does not “look ahead” at $\mathcal{P}_{\kappa+\lambda}$.

For the reasons mentioned above, we must assume that κ is a regular limit cardinal. In fact, since we require stationary-many regular cardinals below κ , for the remainder of this paper we assume that κ is a Mahlo cardinal.

Definition 2.1 *Suppose κ is a Mahlo cardinal and λ is an infinite cardinal with $\kappa \leq \lambda$. Suppose also that S is a stationary subset of $\mathcal{P}_\kappa\lambda$. Then $\square_{\mathcal{P}_\kappa\lambda}(S)$ is the statement that there is a family of sets $\{C_x : x \in S\}$ with the following properties:*

- (i) C_x is a club subset of $\mathcal{P}_{|x|}(x)$ for all $x \in S$
- (ii) (Coherence:) if $x \in S$ and $y \in C_x \cap S$ then $C_y = C_x \cap \mathcal{P}_{|y|}(y)$
- (iii) (Anticoherence:) the set $\{x \in S : \text{there is a cofinal set of } y \in S \cap \mathcal{P}_{|x|}(x) \text{ such that } C_y \neq C_x \cap \mathcal{P}_{|y|}(y)\}$ is stationary in $\mathcal{P}_\kappa\lambda$.

We write $\square_{\mathcal{P}_\kappa\lambda}$ to mean that there is a stationary $S \subseteq \mathcal{P}_\kappa\lambda$ such that $\square_{\mathcal{P}_\kappa\lambda}(S)$ holds.

Note that the restriction to a stationary subset is not as serious a restriction

as it may appear since in this context we can at best have C_x defined for all $x \in \text{reg}(\mathcal{P}_\kappa\lambda)$, which is stationary if Mahlo is κ , but cannot be club. Note that stronger forms of stationarity could be substituted with appropriate adjustments to the forcing below.

The anticoherence property is implicit in the definition of $\square_\kappa$ but must be explicitly required for $\square_{\mathcal{P}_\kappa\lambda}$. This ensures that the principle cannot be satisfied trivially, e.g. by setting $C_x = \mathcal{P}_{|x|}(x)$ for all $x \in \text{reg}(\mathcal{P}_\kappa\lambda)$.

The $\square_{\mathcal{P}_\kappa\lambda}$ principle is consistent with ZFC+ “ κ is Mahlo”, as we assert in the following theorem. Important questions remain unanswered, however. Noteably, it is not known whether the principle holds in ZFC + “V=L” or even in ZFC.

Before we proceed with the consistency proof, note that we could also develop a principle based on clubs of $\mathcal{P}_{\kappa_x}(x)$ for each $x \in S$. Recall that $\kappa_x = x \cap \kappa$ if this is an ordinal and is undefined otherwise. Here, we would insist that S contains only elements x for which κ_x is a regular cardinal. Assuming that κ is Mahlo, the consistency of such a principle can be proved with a forcing analogous to the one for $\square_{\mathcal{P}_\kappa\lambda}$.

Theorem 2.2 *Suppose M is a countable model of a sufficiently rich fragment of ZFC in which κ is Mahlo and $\lambda \geq \kappa$. Then there is a generic extension of this model which preserves cofinalities and cardinalities and in which κ is Mahlo and $\square_{\mathcal{P}_\kappa\lambda}$ holds.*

This theorem is proved by forcing with the partial order defined below. Essentially, the partial order is composed of fragments of possible witnesses to $\square_{\mathcal{P}_\kappa\lambda}$.

Definition 2.3 *Let P be a set whose elements p are characterised as follows:*

- (i) p is a function with $\text{dom}(p) \in \mathcal{P}_\kappa(\text{reg}(\mathcal{P}_\kappa\lambda))$
- (ii) for all $x \in \text{dom}(p)$, $p(x)$ is either club in $\mathcal{P}_{|x|}(x)$ or the empty set
- (iii) if $x \in \text{dom}(p)$ and $y \in p(x) \cap \text{reg}(\mathcal{P}_\kappa\lambda)$ then $y \in \text{dom}(p)$ and either $p(y) = p(x) \cap \mathcal{P}_{|y|}(y)$ or $p(y) = \emptyset$
- (iv) if $\text{dom}(p) \cap \mathcal{P}_{|y|}(y)$ is stationary in $\mathcal{P}_{|y|}(y)$ then $y \in \text{dom}(p)$.

For $p, q \in P$, $p \leq q$ (meaning q is stronger than p) iff $p \subseteq q$. We will also use the symbols $<, \geq$ and $>$ in the natural way.

Note that if we let $\emptyset$ be the function with empty domain then $\emptyset$ is the unique minimal element of P . Clearly, P is non-empty. We must now establish various properties of $(P, \leq)$ to show that a suitable generic object exists and that the resulting forcing preserves cofinalities and cardinalities.

Lemma 2.4 $(P, \leq)$ is separative.

Proof. Let $p \in P$ and let $x \in \text{reg}(\mathcal{P}_{\kappa\lambda})$ be such that $\bigcup \text{dom}(p) \in \mathcal{P}_{|x|}(x)$, which is possible because $|\bigcup \text{dom}(p)| < \kappa$. We now define $q \in P$ such that $p \leq q$ and $x \in \text{dom}(q)$ and $q(x) \neq \emptyset$. For $y \in \text{dom}(p)$, let $q(y) = p(y)$. Let $q(x)$ be any club of $\mathcal{P}_{|x|}(x)$ that does not intersect $\text{dom}(p)$. Such a club exists because p satisfies (iv) of Definition 2.3. It is straightforward to check that $q \in P$, by checking against conditions (i)-(iv).

Now let $r \geq p$ be defined as follows. Let $r(y) = q(y)$ if $y \neq x$, let $r(x) = \emptyset$ and let $r(y)$ be undefined otherwise. Then $r \in P$ and q, r are clearly incompatible extensions of p .

□

Since P is separative, there is a generic object G in $M[G]$ that is not in the ground model, M . We will see that this generic provides an example of a $\square_{\mathcal{P}_{\kappa\lambda}}$ set. First, however, we must show that the forcing preserves cofinalities and cardinalities.

Lemma 2.5 P satisfies the κ^+ -chain condition.

Proof. Suppose $X \subseteq P$ and $|X| = \kappa^+$. We show that X is not an antichain. Let $\mathcal{A} = \{\text{dom}(p) : p \in X\}$. By a Δ -system argument, using the fact that κ is strongly inaccessible, we can find $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| = \kappa^+$ and $\mathcal{B}$ is a Δ -system with root R . That is, for all $X, Y \in \mathcal{B}$, $X \cap Y = R$.

Consider the numbers of functions with domain R such that for each function f and each $x \in R$, $f(x) \subseteq \mathcal{P}_{|x|}(x)$. Clearly, if we impose no further conditions on the value of $f(x)$, the number of distinct functions is equal to $\prod_{x \in R} |\mathcal{P}(\mathcal{P}_{|x|}(x))|$. Now for all $x \in R$, $|\mathcal{P}_{|x|}(x)| < \kappa$ and since $\kappa^{<\kappa} = \kappa$, it follows that $|\mathcal{P}(\mathcal{P}_{|x|}(x))| \leq \kappa$. Furthermore, since $|R| < \kappa$, it follows that $\prod_{x \in R} |\mathcal{P}(\mathcal{P}_{|x|}(x))| \leq \kappa$. In other words there are only κ -many suitable functions defined on R . But $\mathcal{B} = \kappa^+$ so by the pigeonhole principle there must

be some function f defined on R such that $p \restriction R = f$ for κ^+ many $p \in X$ with $\text{dom}(p) \in \mathcal{B}$.

Now let $Y = \{p \in X : \text{dom}(p) \in \mathcal{B} \text{ and } p \restriction R = f\}$. For $p, q \in Y$, if $p(x) = q(x)$ for all $x \in \text{dom}(p) \cap \text{dom}(q)$, it is easily proved that $p \cup q$ is a common extension of p, q and hence that p, q are compatible. Thus, the elements of Y are pairwise compatible because they agree on R , which is the intersection of their domains, by the definition of $\mathcal{B}$. Hence, X is not an antichain.

⊥.

We can now conclude that the forcing preserves cofinalities and cardinalities $> \kappa$. We now prove that P is $< \kappa$ -directed closed. It will then follow that the forcing preserves cofinalities and cardinalities $\leq \kappa$.

Lemma 2.6 *P is $< \kappa$ -directed closed.*

Proof. Suppose $\mu < \kappa$ and $\{p_\alpha : \alpha < \mu\}$ is a set of pairwise compatible conditions from P . We define $p_\mu^* = \bigcup_{\alpha < \mu} p_\alpha$. This is a function since the conditions are pairwise compatible. It is easily checked that p_μ^* satisfies (i)-(iii) of Definition 2.3. However, there may be $x \in \text{reg}(\mathcal{P}_\kappa \lambda) \setminus \text{dom}(p_\mu^*)$ such that $\text{dom}(p_\mu^*)$ is stationary in $\mathcal{P}_{|x|}(x)$ so condition (iv) may not hold. We now make a small adjustment to p_μ^* to obtain p_μ still satisfying (i)-(iii) but also satisfying (iv). Let $p_\mu(x) = p_\mu^*(x)$ for all $x \in \text{dom}(p_\mu^*)$. For $x \in (\mathcal{P}(\bigcup \text{dom}(p_\mu^*)) \setminus \text{dom}(p_\mu^*))$, let $p_\mu(x) = \emptyset$. Then p_μ is as required since (i)-(iv) hold and for all $\alpha < \mu$, $p_\alpha < p_\mu$.

⊥.

Note that since the forcing is $< \kappa$ -closed, no new sets of ordinals of size $< \kappa$ are introduced. Hence, $(\mathcal{P}_\kappa \lambda)^{M[G]} = (\mathcal{P}_\kappa \lambda)^M$ and we can write $\mathcal{P}_\kappa \lambda$ for the name $\mathcal{P}_\kappa \lambda$.

We must now ensure that for any generic G of P , the set $\{x \in \mathcal{P}_\kappa \lambda : (\exists p \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset)\}$ is stationary in $\mathcal{P}_\kappa \lambda$ in the generic extension. Before we do this we give a lemma that will be needed several times in the proof.

Lemma 2.7 *Suppose $p \in P$ and $p \Vdash \neg(C \text{ is a club of } \mathcal{P}_\kappa \lambda)$ and suppose $y \in \mathcal{P}_\kappa \lambda$. Then there is $x \in \mathcal{P}_\kappa \lambda$ and $q \in P$ such that $q \geq p$ and $q \Vdash \neg(y \subset x \text{ and } x \in C)$.*

Proof. Let $\underline{x}$ be a name such that $p \Vdash (y \subseteq \underline{x} \text{ and } \underline{x} \in \dot{C})$. This is possible because $p \Vdash (\dot{C} \text{ is club})$. Also, $p \Vdash ((\exists \mu < \kappa)(|\underline{x}| < \mu))$ because κ is a limit cardinal.

Let $p_0 \geq p$ and ν a cardinal such that $p_0 \Vdash (\mu = \nu)$. So $p_0 \Vdash (|\underline{x}| < \nu \text{ and } \nu < \kappa)$. Thus we can find a condition $p_1 \geq p_0$ and a name for an enumeration of $\underline{x}$ in an ordertype $< \alpha$ so that: $p_1 \Vdash (i^* < \nu \text{ and } \underline{x} = \{\gamma_i : i < i^*\})$ and we can extend again to obtain β and p_2 such that $p_2 \Vdash (\underline{x} = \{\gamma_i : i < \beta\})$.

Now let $q_0 \geq p_2$ be such that $q_0 \Vdash (\gamma_0 = \delta_0)$. That is, q_0 identifies the value of the name γ_i . By induction on $i < \beta$ we construct an increasing sequence $\langle q_\alpha : \alpha < \beta \rangle$ and a sequence $\langle \delta_\alpha : \alpha < \beta \rangle$ such that $q_\alpha \Vdash (\forall \xi < \alpha)(\gamma_\xi = \delta_\xi)$. This is possible because P is $< \kappa$ -closed.

Again, by the $< \kappa$ -closure of P , it is possible to find $q \in P$ that identifies all the elements of $\underline{x}$. That is, there is $z \in \mathcal{P}_\kappa \lambda$ and $q \geq p$ such that $q \Vdash (y \subseteq z \text{ and } z \in \dot{C})$, as required.

⊥.

Lemma 2.8 *Let G be a generic of P . Then $M[G] \models \{x \in \mathcal{P}_\kappa \lambda : (\exists p \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset)\}$ is stationary in $\mathcal{P}_\kappa \lambda$.*

Proof. Let $\underline{S}$ be a name of the set $\{x \in \mathcal{P}_\kappa \lambda : (\exists p \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset)\}$.

Suppose $p_0 \in G$ is such that $p_0 \Vdash (\dot{C} \text{ is club in } \mathcal{P}_\kappa \lambda \text{ and } \dot{C} \cap \underline{S} = \emptyset \text{ and } x_0 \in \dot{C} \cap \mathcal{P}_\kappa \lambda)$. Note that we use the previous lemma to obtain $p_0 \Vdash (x_0 \in \dot{C} \cap \mathcal{P}_\kappa \lambda)$. We derive a contradiction by finding $p \geq p_0$ such that $p \Vdash (\dot{C} \cap \underline{S} \neq \emptyset)$. The strategy is to fix a chain of elements of $\dot{C}$ and $\underline{S}$ up to a regular limit where the two chains intersect.

Let $y_0 \in \text{reg}(\mathcal{P}_\kappa \lambda)$ be such that $(\bigcup \text{dom}(p_0) \cup x_0) \in \mathcal{P}_{|y_0|}(y_0)$. We now identify $p_0^* \geq p_0$ such that $y_0 \in \text{dom}(p_0^*)$.

Let D_0 be a linearly ordered club of $\mathcal{P}_{|y_0|}(y_0)$ that does not intersect $\text{dom}(p_0)$. Such a club exists by definition of P (in particular, clause (iv) of Definition 2.3). Note that having D_0 linearly ordered is convenient but not strictly necessary; it is possible because $|y_0|$ is regular.

$$\text{Let } p_0^*(u) = \begin{cases} p_0(u) & \text{if } u \in \text{dom}(p_0) \\ D_0 & \text{if } u = y_0 \\ \emptyset & \text{if } u \in \text{reg}(\mathcal{P}(y_0) \setminus (\text{dom}(p_0) \cup \{y_0\})) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Then by checking against Definition 2.3, it is apparent that $p_0^* \in P$. Note also that $p_0^* \geq p_0$.

Now using the preceding lemma, let $p_1 \geq p_0^*$ be such that for some $x_1 \in \mathcal{P}_\kappa \lambda$, $p_1 \Vdash (x_1 \in \mathcal{C} \cap \mathcal{P}_\kappa \lambda \text{ and } y_0 \subseteq x_1)$.

We now proceed inductively to define $p_\alpha, x_\alpha, y_\alpha, p_\alpha^*$ so that for all $\beta < \alpha$, $y_\beta \in p_\alpha(y_\alpha)$ and $p_\beta \leq p_\alpha \leq p_\alpha^*$. In the case when α is a limit ordinal, we describe the condition under which the induction will stop. We will then observe that this condition will be met at some stage $\alpha < \kappa$.

Case 1: $\alpha = \beta + 1$

By the inductive definition, p_α and x_α are already defined. We now define p_α^* and y_α then also define $p_{\alpha+1}$ and $x_{\alpha+1}$. Let $y_\alpha \in \text{reg}(\mathcal{P}_\kappa \lambda)$ be such that $\bigcup \text{dom}(p_\beta) \cup x_\alpha \in \mathcal{P}_{|y_\alpha|}(y_\alpha)$. We now identify $p_\alpha^* \geq p_\alpha$ such that $y_\alpha \in \text{dom}(p_\alpha^*)$. Unlike in the case $\alpha = 0$, we will define $p_\alpha^*(y_\alpha)$ so that it has non-trivial coherence. In particular, for all $\beta < \alpha$, we will have $y_\beta \in p_\alpha^*(y_\alpha)$.

The inductive hypothesis implies that $y_\beta \in \text{dom}(p_\alpha)$ so we can find a linearly ordered club D_α of $\mathcal{P}_{|y_\alpha|}(y_\alpha)$ that does not intersect $\text{dom}(p_\alpha)$ and satisfies $u \in D_\alpha \Rightarrow y_\beta \subseteq u$. Such a club exists by (iv) of Definition 2.3 and by intersecting with the club $\{u \in \mathcal{P}_\kappa \lambda : y_\beta \subseteq u\}$. Now let $D_\alpha^* = p_\alpha(y_\beta) \cup \{y_\beta\} \cup D_\alpha$.

$$\text{Let } p_\alpha^*(u) = \begin{cases} p_\alpha(u) & \text{if } u \in \text{dom}(p_\alpha) \\ D_\alpha^* & \text{if } u = y_\alpha \\ \emptyset & \text{if } u \in \text{reg}(\mathcal{P}(y_\alpha) \setminus (\text{dom}(p_\alpha) \cup \{y_\alpha\})) \\ \text{undefined} & \text{otherwise} \end{cases}$$

It is easily checked that p_α^* satisfies (i) to (iv) of Definition 2.3 and that $p_\alpha^* \geq p_\alpha$. Note also that $y_\beta \in p_\alpha^*(y_\alpha)$.

Now using the previous lemma, let $p_{\alpha+1} \geq p_\alpha^*$ be such that for some $x_{\alpha+1} \in \mathcal{P}_\kappa \lambda$, $p_{\alpha+1} \Vdash (x_{\alpha+1} \in \mathcal{C} \cap \mathcal{P}_\kappa \lambda \text{ and } y_\alpha \subseteq x_{\alpha+1})$.

Case 2: α is a limit ordinal $< \kappa$

Note that x_α and p_α are not yet defined. Let $p_\alpha \in P$ be such that $p_\alpha \geq p_\beta$ for all $\beta < \alpha$. This is possible because P is $< \kappa$ -closed. Let $s_\alpha = \bigcup \{y_\beta : \beta < \alpha\}$.

If $|s_\alpha|$ is regular then this will be the final stage of the induction. We then proceed to define y and p as described below. So suppose now that $|s_\alpha|$ is singular. Note in particular that $s_\alpha \notin \text{reg}(\mathcal{P}_\kappa\lambda)$ so $s_\alpha \notin \text{dom}(p_\alpha^*)$.

By the inductive definitions of y_β , $s_\alpha = \bigcup \{\text{dom}(p_\beta) : \beta < \alpha\}$, that is s_α is the set of ordinals that are in at least one element of the domain of at least one p_β . Let $y_\alpha \in \text{reg}(\mathcal{P}_\kappa\lambda)$ be such that $s_\alpha \in \mathcal{P}_{|y_\alpha|}(y_\alpha)$. Thus, for any $\beta < \alpha$, if $u \in \text{dom}(p_\beta)$ then $u \in \mathcal{P}_{|y_\alpha|}(y_\alpha)$.

Let D_α be a linearly ordered club of $\mathcal{P}_{|y_\alpha|}(y_\alpha)$ that does not intersect $\text{dom}(p_\alpha)$ and such that if $u \in D_\alpha$ then $s_\alpha \subseteq u$. Let $D_\alpha^* = \bigcup \{p_\beta(y_\beta) : \beta < \alpha\} \cup \{s_\alpha\} \cup D_\alpha$.

$$\text{Let } p_\alpha^*(u) = \begin{cases} p_\alpha(u) & \text{if } u \in \text{dom}(p_\alpha) \\ D_\alpha^* & \text{if } u = y_\alpha \\ \emptyset & \text{if } u \in \text{reg}(\mathcal{P}(y_\alpha) \setminus (\text{dom}(p_\alpha) \cup \{y_\alpha\})) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Then $p_\alpha^* \in P$ and $(\forall \beta < \alpha)(p_\alpha^* \geq p_\beta)$.

As before, let $p_{\alpha+1} \geq p_\alpha^*$ be such that for some $x_{\alpha+1} \in \mathcal{P}_\kappa\lambda$, $p_\alpha \Vdash (x_\alpha \in \mathcal{C} \cap \mathcal{P}_\kappa\lambda \text{ and } y_\alpha \subseteq x_\alpha)$.

We repeat this procedure until we reach a limit ordinal $\alpha = \mu < \kappa$ such that s_α (as defined in Case 2) has inaccessible cardinality. There must be such a μ because κ is Mahlo. Otherwise the set $\{s_\alpha : \alpha < \kappa \text{ and } \text{lim}(\alpha)\}$ would be a club subset of κ that does not intersect the set of regular cardinals, contradicting the fact that κ is Mahlo. So suppose $|s_\alpha|$ is regular. Then $|s_\alpha|$ is inaccessible because the sequence $\langle |y_\beta| : \beta < \alpha \rangle$ is strictly increasing by the inductive definitions of y_β for $\beta < \alpha$.

Let $y = s_\alpha$ and let $E = \bigcup \{\text{dom}(p_\beta) : \beta < \alpha\}$. Now define p as follows.

$$\text{Let } p(u) = \begin{cases} p_\beta(u) & \text{if } (\exists \beta < \mu)(u \in \text{dom}(p_\beta)) \\ \bigcup \{p_\beta(y_\beta) : \beta < \mu\} & \text{if } u = y \\ \emptyset & \text{if } u \in \text{reg}(\mathcal{P}(y) \setminus (E \cup \{y\})) \\ \text{undefined} & \text{otherwise} \end{cases}$$

As before, by checking against (i)-(iv) of Definition 2.3, we see that $p \in P$. We now show that $p \Vdash \mathcal{C} \cap \mathcal{S} \neq \emptyset$.

Note that $\bigcup_{\beta < \mu} y_\beta = y = \bigcup_{\beta < \mu} x_\beta$ because for any $\beta < \mu$, $x_\beta \subset y_\beta \subseteq x_{\beta+1} \subset y_{\beta+1}$. By the definition of p , it is clear that $p(y) \neq \emptyset$ and hence that $p \Vdash y \in \mathcal{S}$. Also, since $p \Vdash (\mathcal{C} \text{ is club in } \mathcal{P}_\kappa\lambda \text{ and } (\forall \beta < \mu)(x_\beta \in \mathcal{C}))$ it

follows that $p \Vdash -y \in \mathcal{C}$. Hence, $p \Vdash -y \in \mathcal{C} \cap \mathcal{S}$, which is a contradiction because $p \geq p_0$ and $p_0 \Vdash \mathcal{C} \cap \mathcal{S} = \emptyset$.

⊥.

We now establish that the proposed witness to $\square_{\mathcal{P}_\kappa \lambda}$ satisfies the anticohereence condition.

Lemma 2.9 *Let G be a generic of P . Then let $\mathcal{S} = \{x \in \mathcal{P}_\kappa \lambda : (\exists p \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset)\}$ and let $\mathcal{T} = \{x \in \mathcal{S} : \text{there is a cofinal set of } y \in \mathcal{S} \cap \mathcal{P}_{|x|}(x) \text{ such that } (\exists p \in G)(\{x, y\} \subseteq \text{dom}(p) \text{ and } p(y) \neq p(x) \cap \mathcal{P}_{|y|}(y))\}$.*

Then $M[G] \models \mathcal{T}$ is stationary in $\mathcal{P}_\kappa \lambda$.

Proof (outline). We proceed as in Lemma 2.8, forming the sequence of forcing conditions as before but at each stage, we interrupt the induction after setting p_α^* but before setting $x_{\alpha+1}$. We set $z_\alpha \supset y_\alpha$ and define $q \geq p_\alpha^*$ such that $z_\alpha \in \text{dom}(q)$ but $q(z_\alpha) \cap q(y_\alpha) = \emptyset$. Now continue as before but defining $x_{\alpha+1}$ so that $z_\alpha \subset x_{\alpha+1}$ and with $q \leq p$.

⊥.

Finally, we need to verify that κ is Mahlo in the generic extension $M[G]$.

Lemma 2.10 *If G is a generic of P then $M[G] \models \kappa$ is Mahlo.*

Proof. Working in $M[G]$, suppose C is a club in κ . Then if $C^* = \{x \in \mathcal{P}_\kappa \lambda : |x| \in C\}$, it follows that C^* is club in $\mathcal{P}_\kappa \lambda$. By Lemma 2.8, we can find y in $C^* \cap \{x \in \mathcal{P}_\kappa \lambda : (\exists p \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset)\}$. Then $|y|$ is a regular cardinal in both M and $M[G]$, by the preservation of cofinalities and cardinalities. Furthermore, $|y| \in C$. Hence the set of regular cardinals is stationary in κ . To see that κ remains a strong limit, note that for all $\mu < \kappa$, $(2^\mu)^{M[G]} = (2^\mu)^M$ by $< \kappa$ -closure so κ remains a strong limit in the generic extension. Hence κ is Mahlo in $M[G]$ as required.

⊥.

Given generic G of P , let $\mathcal{S} = \{x \in \text{reg}(\mathcal{P}_\kappa \lambda) : (\exists p \in G)(p(x) \neq \emptyset)\}$ and for $x \in \mathcal{S}$, let $C_x = p(x)$ where p is an element of G with $x \in \text{dom}(p)$. The

preceding series of lemmas together prove that this S and $\{C_x : x \in S\}$ provides a witness to $\square_{\mathcal{P}_\kappa\lambda}$ in $M[G]$. Thus, Theorem 2.2 is proved.

We proved in Lemma 2.10 that this forcing preserves the fact that κ is Mahlo. In fact, we can do more than this and preserve supercompactness. Since forcing with P is κ -directed closed, if κ is supercompact in the ground model and we first force with a Laver preparation, then the supercompactness of κ is preserved when we force with P .

Theorem 2.11 *Suppose M is a countable model of a sufficiently rich fragment of ZFC in which κ is supercompact and $\lambda \geq \kappa$. Then there is a generic extension of this model which preserves cofinalities and cardinalities and in which κ is supercompact and $\square_{\mathcal{P}_\kappa\lambda}$ holds.*

Proof. This follows by forcing with a Laver preparation followed by forcing with P . We use the fact that P is κ -directed closed.

⊥.

3 A $\mathcal{P}_\kappa\lambda$ version of square with a non-reflection property

One of the useful properties encapsulated by the square sequence is that of stationary non-reflection. This is demonstrated in the theorem presented below, which makes use of Fodor's Lemma, which we present here without proof.

Lemma 3.1 (Fodor's Lemma) *Suppose that S is a stationary subset of a regular cardinal μ . Suppose also that $f : S \rightarrow \mu$ is such that $f(\alpha) < \alpha$ for all $\alpha \in S$. Then there is a stationary subset $T \subseteq S$ such that f is constant on T .*

The following well-known theorem is presented here with proof to motivate the work towards a $\mathcal{P}_\kappa\lambda$ version of the theorem discussed in the remainder of this section.

Theorem 3.2 *If $\square_\kappa$ holds then κ^+ has a non-reflecting stationary subset.*

Proof. Suppose $\langle C_\alpha : \alpha < \kappa^+ \text{ and } \lim(\alpha) \rangle$ is as specified in the definition of $\square_\kappa$. Let $T = \{\alpha < \kappa^+ : \text{cf}(\alpha) < \kappa < \alpha\}$. To see that this is stationary, let C be an arbitrary club of κ^+ and let $C^* = C \setminus \kappa$. Then the ω th element of C^* is an element of T .

Now define $F : T \rightarrow \kappa$ by $F(\alpha) = \text{otp}(C_\alpha)$. By part (ii) of Definition 1.1 and the definition of T , $F(\alpha) < \kappa < \text{otp}(\alpha)$ for all $\alpha \in T$. Hence, by Fodor's Lemma, we can select a stationary subset $R \subseteq T$ such that F is constant on R .

Now suppose R reflects in α for some $\alpha \in R$. Let $\beta, \gamma \in R \cap C_\alpha$ with $\beta < \gamma$. Then $C_\beta \cup \{\beta\} \subseteq C_\gamma$ as $\beta = \sup(C_\beta)$. Thus $F(\gamma) = \text{otp}(C_\gamma) \geq \text{otp}(C_\beta) + 1 > F(\beta)$. But this is a contradiction because F is constant on R .

⊥.

We now extend $\square_{\mathcal{P}_{\kappa\lambda}}$ to produce a square principle that has a non-reflection property explicitly built into the definition. We then give a non-reflection theorem using this new principle.

Definition 3.3 $\square_{\mathcal{P}_{\kappa\lambda}}(S, f)$ holds if $f : S \rightarrow \kappa$ and S is stationary and there is a witness $\{C_x : x \in S\}$ to $\square_{\mathcal{P}_{\kappa\lambda}}(S)$ such that in addition to (i)-(iii) from Definition 2.1 we have:

(iv) $f(x) \in x$

(v) if $y \in C_x$ then $f(x) \neq f(y)$.

We now prove the relative consistency of this principle by extending the partial order P used in the proof of Theorem 2.2.

Theorem 3.4 Suppose M is a countable model of a sufficiently rich fragment of ZFC in which κ is Mahlo and $\lambda \geq \kappa$. Then there is a generic extension of this model which preserves cofinalities and cardinalities and in which κ is Mahlo and for some f, S , $\square_{\mathcal{P}_{\kappa\lambda}}(S, f)$ holds.

We force with the poset Q defined below.

Definition 3.5 $p, q \in Q$ iff $p \in P$ and q is as follows:

(i) q is a function with domain $\{x \in \text{dom}(p) : p(x) \neq \emptyset\}$

(ii) $q(x) \in x$ for all $x \in \text{dom}(q)$

(iii) if $x \in \text{dom}(p)$ and $y \in p(x) \cap \text{dom}(p)$ and $p(y) \neq \emptyset$ then $q(y) \neq q(x)$.

If $(p, q), (p', q') \in Q$ then $(p, q) \leq (p', q')$ iff $p \subseteq p'$ and $q \subseteq q'$.

We do not present all of the details of the forcing proof. Instead we describe how to upgrade the proof of Theorem 2.2 to include the new property.

Note that $(\emptyset, \emptyset) \in Q$ so Q is non-empty and has a minimal element. We must now establish various properties of $(Q, \leq)$ to show that a suitable generic object exists and that the resulting forcing preserves cofinalities and cardinalities.

Lemma 3.6 $(Q, \leq)$ is separative.

Proof. Let $(p, q) \in Q$ and let $x \in \text{reg}(\mathcal{P}_\kappa \lambda) \setminus \text{dom}(p)$ such that there is $\gamma \in x \setminus \text{im}(q)$. Let $(p_0, q_0) \geq (p, q)$ be such that $p_0(x)$ is a club in $\mathcal{P}_{|x|}(x)$ that does not intersect $\text{dom}(p)$ and let $q_0(x) = \gamma$. Such a p_0 can be found by Definition 2.3 (iv) and because $|\text{dom}(p)| < \kappa \leq |\text{reg}(\mathcal{P}_\kappa \lambda)|$ so there must be some $x \in \text{reg}(\mathcal{P}_\kappa \lambda) \setminus \text{dom}(p)$. Now let $(p_1, q_1) \geq (p, q)$ be such that $x \in \text{dom}(p_1)$ and $p_1(x) = \emptyset$ and hence $x \notin \text{dom}(q_1)$. Clearly (p_0, q_0) and (p_1, q_1) are incompatible extensions of (p, q) . Hence, Q is separative.

⊥.

We now prove that forcing with Q preserves cofinalities and cardinalities by showing that Q has the κ^+ -chain condition and is $< \kappa$ -directed closed.

We now use the Δ -System Lemma to show that Q has the κ^+ -chain condition.

Lemma 3.7 Q satisfies the κ^+ -chain condition.

Proof. Let A be a subset of Q of size κ^+ . Now let $\mathcal{A} = \{\text{dom}(p) : \exists q(p, q) \in A\}$. By the Δ -System Lemma, using the fact that κ is a strong limit, we can find $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| = \kappa^+$ and $\mathcal{B}$ is a Δ -system with root R .

Consider the number of pairs of functions (p, q) definable on R such that for each function (p, q) and each $x \in R$, $p(x) \in \mathcal{P}(\mathcal{P}_{|x|}(x))$ and $q(x) \in x$. By the

argument in the proof of Lemma 2.5, the number of possible values that $p(x)$ can take is $< \kappa$. The number of possible values that $q(x)$ can take is clearly $|x|$. Since $|x| < \kappa$, the number of possible pairs $(p(x), q(x))$ is $< \kappa$. But $|\mathcal{B}| = \kappa^+$ so by the pigeonhole principle there must be some pair of functions (g, h) defined on R such that $p \restriction R = g$ and $q \restriction R = h$ for κ^+ many $(p, q) \in X$ with $\text{dom}(p) \in \mathcal{B}$.

Now let $Y = \{(p, q) \in X : p \restriction R = g \text{ and } q \restriction R = h\}$. For any $(p_0, q_0), (p_1, q_1) \in Y$, using the fact that p_0, p_1 and q_0, q_1 agree R , it is straightforward to verify that $(p_0 \cup p_1, q_0 \cup q_1) \in Q$. Thus, $(p_0, q_0), (p_1, q_1)$ have a common extension in Q and hence are compatible. Hence, A is not an antichain.

⊥.

Lemma 3.8 *Q is $< \kappa$ -directed closed.*

Proof. Suppose $\mu < \kappa$ and $\{(p_\alpha, q_\alpha) : \alpha < \mu\}$ is a set of pairwise compatible conditions from Q . We define $p_\mu^* = \bigcup_{\alpha < \mu} p_\alpha$ and $q_\mu^* = \bigcup_{\alpha < \mu} q_\alpha$. Now extend p_μ^* to p_μ as in the proof of the $< \kappa$ -directed closure of P . Note that we need not add new elements to the domain of q_μ^* since $x \in \text{dom}(p_\mu) \setminus \text{dom}(p_\mu^*) \Rightarrow p_\mu(x) = \emptyset$. That is, we may set $q_\mu = q_\mu^*$. Now for any $x, y \in \text{dom}(q_\mu)$, there is some $\alpha < \mu$ such that $x, y \in \text{dom}(q_\alpha)$. Since $(p_\gamma, q_\gamma) \in Q$ it follows that $x \in p_\mu(x) \Rightarrow q_\mu(x) \neq q_\mu(y)$ and vice versa as required. It follows that $(p_\alpha, q_\alpha) \in Q$ and for all $\beta < \mu$, $(p_\alpha, q_\alpha) \leq (p_\mu, q_\mu)$.

⊥.

It follows from the preceding lemmas that forcing with Q preserves cofinalities and cardinalities. As with P , this forcing is $< \kappa$ -closed so for a generic G of Q , $(\mathcal{P}_\kappa \lambda)^{M[G]} = (\mathcal{P}_\kappa \lambda)^M$ and we can write $\mathcal{P}_\kappa \lambda$ for the name $\mathcal{P}_\kappa \lambda$ in the following. We must now ensure that for any generic G of Q , the set $\{x \in \mathcal{P}_\kappa \lambda : (\exists (p, q) \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset)\}$ is stationary in $\mathcal{P}_\kappa \lambda$. Note that the following variation on Lemma 2.7 holds. The proof is almost identical to the proof of Lemma 2.7.

Lemma 3.9 *Suppose $(p, q) \in Q$ and $(p, q) \Vdash (C \text{ is a club of } \mathcal{P}_\kappa \lambda)$. Then there is $x \in \mathcal{P}_\kappa \lambda$ and $(p', q') \in Q$ such that $(p', q') \geq (p, q)$ and $(p', q') \Vdash x \in C$.*

Lemma 3.10 *Let G be a generic of Q . Then $M[G] \models \{x \in \mathcal{P}_\kappa \lambda : (\exists p \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset)\}$ is stationary in $\mathcal{P}_\kappa \lambda$.*

Proof. We proceed as in the proof of Lemma 2.8 but define (p_α, q_α) and (p_α^*, q_α^*) at each stage. We now describe how to set q_α . Let $\gamma \in y_0 \setminus \{q(y_0)\}$. We insist, without loss of generality, that for all α , γ is not in the image of q_α or q_α^* . For all $\alpha < \mu$ we set $q(y_\alpha) = \gamma_\alpha \in y_\alpha \setminus \bigcup_{\beta < \alpha} y_\beta$. By definition of y_α , such a γ_α will always exist. At the final stage, when defining (p, q) , we define p as before and set $q(y) = \gamma$.

⊥.

The last two lemmas that we need follow by arguments exactly analogous to the corresponding lemmas for P .

Lemma 3.11 *Let G be a generic of Q . Then let $\mathcal{S} = \{x \in \mathcal{P}_\kappa \lambda : (\exists(p, q) \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset)\}$ and let $\mathcal{T} = \{x \in \mathcal{S} : \text{there is a cofinal set of } y \in \mathcal{S} \cap \mathcal{P}_{|x|}(x) \text{ such that } (\exists(p, q) \in G)(\{x, y\} \subseteq \text{dom}(p) \text{ and } p(y) \neq p(x) \cap \mathcal{P}_{|y|}(y))\}$.*

Then $M[G] \models \mathcal{T}$ is stationary in $\mathcal{P}_\kappa \lambda$.

Lemma 3.12 *If G is a generic of Q then $M[G] \models \kappa$ is Mahlo.*

By forcing with the partial order $(Q, \leq)$, Theorem 3.4 is proved. We set $\mathcal{S} = \{x \in \mathcal{P}_\kappa \lambda : (\exists(p, q) \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset)\}$ and set $f = \bigcup \{q : \exists p((p, q) \in G)\}$. Then f and $\{C_x : (\exists(p, q) \in G)(C_x = p(x) \neq \emptyset)\}$, together witness that $\square_{\mathcal{P}_\kappa \lambda}(\mathcal{S}, f)$ holds, as required.

We now show how $\square_{\mathcal{P}_\kappa \lambda}(\mathcal{S}, f)$ gives non-reflection in $\mathcal{P}_{|x|}(x)$ for stationary many $x \in \mathcal{P}_\kappa \lambda$. We then state without proof some related results proved by Abe in [1] and by Koszmider in [7].

The following is proved by forcing and draws on Gitik's method of shooting clubs in $\mathcal{P}_\kappa \lambda$.

Theorem 3.13 (Abe) *Let $V \subset W$ be two models of ZFC with the same ordinals, $(\kappa^+)^V = (\kappa^+)^W$; let C be a club subset of κ of V -inaccessibles; let κ be an inaccessible cardinal in W and let $T = \{x \in \mathcal{P}_\kappa \kappa^+ : V \models |x| \text{ is not inaccessible}\}$. Then there is a forcing notion in W that preserves cofinalities and cardinalities and such that there is a stationary $S \subset \mathcal{P}_\kappa \kappa^+$ such that $S \cap \mathcal{P}_{\kappa_x}(x)$ is non-stationary for any $x \in T$.*

Koszmider in [7] gives a different kind of non-reflection result, considering reflection in $\mathcal{P}_\kappa(X)$ where $X \subset \lambda$.

Theorem 3.14 (Koszmider) *It is consistent that there is a stationary set $S \subset \mathcal{P}_\kappa \lambda$ such that $S \cap \mathcal{P}_\kappa X$ is non-stationary in $\mathcal{P}_\kappa X$ for any $X \subset \lambda$ with $|X| \geq \kappa$ in the generic extension.*

Finally we consider the following theorem of Abe which gives a form of non-reflection when κ is supercompact.

Theorem 3.15 (Abe) *If it is consistent that there is a supercompact cardinal then it is consistent that there is a supercompact κ , a cardinal $\lambda \geq \kappa$ and a stationary set $X \subset \mathcal{P}_\kappa \lambda$ such that $X \cap \mathcal{P}_\kappa \alpha$ is non-stationary in $\mathcal{P}_\kappa \alpha$ for any $\alpha < \lambda$.*

The following definition presents the form of non-reflection that we examine with $\square_{\mathcal{P}_\kappa \lambda}(S, f)$.

Definition 3.16 *A stationary set $S \subseteq \mathcal{P}_\kappa \lambda$ reflects in $\mathcal{P}_{|x|}(x)$ if $S \cap \mathcal{P}_{|x|}(x)$ is stationary in $\mathcal{P}_{|x|}(x)$.*

The non-reflection theorem follows easily from the $\square_{\mathcal{P}_\kappa \lambda}(S, f)$ principle. Note that the proof is closely analogous to the proof of non-reflection from $\square_\kappa$ in the theory of cardinals. This theorem draws on the variation on Fodor's Lemma presented below. Lacking a suitable reference, we present a proof.

Lemma 3.17 *Suppose that S is a stationary subset of $\mathcal{P}_\kappa \lambda$. Suppose also that $f : S \rightarrow \lambda$ is such that $f(x) \in x$ for all $x \in S$. Then there is a stationary subset $T \subseteq S$ such that f is constant on T .*

Proof. Suppose $f : S \rightarrow \lambda$ is a counterexample. For each $\alpha < \lambda$ choose C_α club in $\mathcal{P}_\kappa \lambda$ with $(f^{-1}(\alpha)) \cap C_\alpha = \emptyset$. Now let D be the diagonal intersection of the C_α , $D = \Delta \langle C_\alpha : \alpha < \lambda \rangle$ and take $y \in S \cap D$, guaranteed to exist because D is club. Then $f(y) \in y$ so since $y \in D$ we have $y \in C_{f(y)}$. Hence, $y \in f^{-1}(f(y)) \cap C_{f(y)}$, contradicting the choice of $C_{f(y)}$.

⊥.

Theorem 3.18 *Suppose κ is Mahlo and $\lambda \geq \kappa$. Then if $\square_{\mathcal{P}_\kappa\lambda}(S, f)$ holds then there is a stationary set $T \subseteq S$ such that T does not reflect in $\mathcal{P}_{|x|}(x)$ for any $x \in S$.*

Proof. Let $\{C_x : x \in S\}$ witness $\square_{\mathcal{P}_\kappa\lambda}(S, f)$. Note that since $f(x) \in x$, by the preceding lemma it follows that there is a stationary set $T \subseteq S$ such that $f(x)$ is constant on T . Now suppose T reflects in $\mathcal{P}_{|x|}(x)$ for some $x \in S$. Let $y \in T \cap C_x$. The set $\{u \in \mathcal{P}_{|x|}(x) : y \subseteq u \text{ and } |y| < |u|\}$ is club in $\mathcal{P}_{|x|}(x)$ so we can find $z \in T \cap C_x$ such that $y \in \mathcal{P}_{|z|}(z)$. By the definition of $\square_{\mathcal{P}_\kappa\lambda}(S, f)$, we have that $C_z = C_x \cap \mathcal{P}_{|z|}(z)$ so $y \in C_z$. But then $f(y) \neq f(z)$, contradicting the definition of T . Thus T cannot reflect in $\mathcal{P}_{|x|}(x)$.

⊥.

It should be noted that for some κ , for example the first Mahlo cardinal, the conclusion of this theorem holds in ZFC. (Simply let $S = T = \text{reg}(\mathcal{P}_\kappa\lambda)$.) The theorem becomes more relevant for cardinals higher in the Mahlo hierarchy (i.e. those that are α -Mahlo for $\alpha > 0$).

As with $\square_{\mathcal{P}_\kappa\lambda}(S)$ we may use a Laver preparation to prove that $\square_{\mathcal{P}_\kappa\lambda}(S, f)$ is consistent even for supercompact κ . Thus, supercompactness of κ does not prevent this principle or the corresponding non-reflection theorem.

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