

Crossed products of simple C^* -algebras by actions with the tracial Rokhlin property

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1 Introduction

In this note we will discuss "tracial" analogs of the Rokhlin property for actions of discrete groups, mainly, an integer group $\mathbf{Z}$ and a finite group G . This property is formally weaker than the various Rokhlin properties which have appeared in the literature, such as in [19], [26], and [21], at least for C^* -algebras which are tracially AF in the sense of [29] (those C^* -algebra are said to have tracial rank zero), in roughly the same way that being tracially AF is weaker than the local characterization of AF algebras.

Our main results are as follows.

- (1) Let A be a stably finite simple unital C^* -algebra, and let α be an automorphism of A which has the tracial Rokhlin property. Suppose A has real rank zero and stable rank one, and suppose that the order on projections over A is determined by traces (Blackadar's Second Fundamental Comparability Question, 1.3.1 of [2], for $M_\infty(A)$). Then a crossed product $C^*(\mathbf{Z}, A, \alpha)$ of A by $\mathbf{Z}$ also has these three properties.
- (2) Let A be a simple separable unital C^* -algebra with tracial rank zero, and suppose that A has a unique tracial state τ . Let $\pi_\tau: A \rightarrow L(H_\tau)$ be the Gelfand-Naimark-Segal representation associated with τ . Let $\alpha \in \text{Aut}(A)$. Then the following condition are equivalent: (i) α has the tracial Rokhlin property, (ii) The automorphism of $\pi_\tau(A)''$ induced by α^n is outer for every $n \neq 0$, that is, α^n is not weakly inner in π_τ for any $n \neq 0$, (iii) $C^*(\mathbf{Z}, A, \alpha)$ has a unique tracial state, (iv) $C^*(\mathbf{Z}, A, \alpha)$ has real rank zero.
- (3) Let A be a simple separable unital C^* -algebra which satisfies the Universal Coefficient Theorem, which has tracial rank zero, and which has a unique tracial state. If $\alpha \in \text{Aut}(A)$ has the Rokhlin property and if α^n is an approximately inner for some $n > 0$, then $C^*(\mathbf{Z}, A, \alpha)$ is a simple AH algebras with no dimensional growth and real rank zero.
- (4) We introduce a general class of automorphisms of rotation algebras, the noncommutative Furstenberg transformations. We prove that irrational noncommutative Furstenberg transfor-

mations have the tracial Rokhlin property.

- (5) The crossed product of an infinite dimensional simple unital C^* -algebra with tracial rank no more than one by an action of a cyclic group with the tracial Rokhlin property again has tracial rank no more than one.

Kishimoto proved ([26]) that if A is a simple unital AT algebra with real rank zero which has a unique tracial state, and $\alpha \in \text{Aut}(A)$ satisfies the approximate innerness, then α has Rokhlin property is equivalent to each of three conditions (ii), (iii), (iv) in (2). Moreover, if α is homotopic to an inner automorphism, then $C^*(\mathbb{Z}, A, \alpha)$ is again a simple unital AT algebra with real rank zero (Theorem 6.4 of [27]). (2) and (3) are generalization of Kishimoto's result. It seems reasonable to hope that whenever A is a simple tracially AF and α has the tracial Rokhlin property, then $C^*(\mathbb{Z}, A, \alpha)$ is again tracially AF. (This is still opened.) However, using the observation of (1) we have automorphisms of C^* -algebras which are not tracially AF, and for which the crossed products are also not tracially AF. (4) is our motivating example to consider the tracial Rokhlin property.

On the contrary, Phillips has proved recently ([44]) that the crossed product of an infinite dimensional simple unital C^* -algebra with tracial rank zero by an action of a finite group with the tracial Rokhlin property again has tracial rank zero. (5) is generalization of this result. We note that there is an action of period 2 on UHF-algebra with no tracial Rokhlin property whose crossed product is not tracially AF ([10]).

2 Classification of simple C^* -algebras of tracial rank zero

The following conventions will be used in this paper. Let A be a unital C^* -algebra.

- (i) We denote by $\text{Aut}(A)$ the set of all automorphisms on A and by $T(A)$ the tracial state space of A .
- (ii) Two projections $p, q \in A$ are said to be equivalent if they are Murray-von Neumann equivalent. That is, there exists a partial isometry $w \in A$ such that $w^*w = p$ and $ww^* = q$. Then we write $p \sim q$.
- (iii) Let $\mathcal{F}$ and S be subsets of A and $\varepsilon > 0$. We write $x \in_\varepsilon S$ if there exists $y \in S$ such that $\|x - y\| < \varepsilon$, and write $\mathcal{F} \subset_\varepsilon S$ if $x \in_\varepsilon S$ for all $x \in \mathcal{F}$.

We begin by introducing the tracial rank zero for a simple unital C^* -algebra.

Definition 2.1. ([30]) Let A be a simple unital C^* -algebra. Then A has *tracial rank no more than one* (write $\text{TR}(A) \leq 1$) if the following holds: For any $\varepsilon > 0$ and any finite set $\mathcal{F} \subset A$ containing a nonzero positive element $a \in A^+$, there is a sub C^* -algebra C in A where $C = \oplus_{i=1}^k M_{n_i}(C(X_i))$ and X_i is a finite CW complex with dimension no more than one such that $1_C = p$ satisfying the following:

- (i) $\|px - xp\| < \varepsilon$ for $x \in \mathcal{F}$,
- (ii) $pxp \in_\varepsilon C$ for any $x \in \mathcal{F}$ and
- (iii) $1 - p$ is equivalent to a projection in pAp .

When each X_i is a point, A is said to have *tracial rank zero* and write $\text{TR}(A) = 0$.

If each X_i is a point and $p = 1$, the above definition gives AF-algebras. The definition says that in a simple unital C^* -algebra A with $\text{TR}(A) = 0$, the part that may not be approximated by finite dimensional C^* -algebras must have small "measure". This observation comes from the following:

Theorem 2.2. ([30, Corollary 6.15]) Let A be a simple unital C^* -algebra with stable rank one which satisfies the Fundamental Comparison Property. Then $\text{TR}(A) \leq 1$ if and only if for any finite set $\mathcal{F} \subset A$, $\varepsilon > 0$, and any non-zero positive element $a \in A$, there exists a sub C^* -algebra $C \subset A$, where $C = \oplus_{i=1}^k M_{n_i}(C(X_i))$, and X_i is a finite CW complex with dimension no more than one such that $1_C = p$ satisfying the following:

- (1) $\| [x, p] \| < \varepsilon$ for all $x \in \mathcal{F}$,
- (2) $pxp \in_\varepsilon C$ for all $x \in \mathcal{F}$,
- (3) $\tau(1 - p) < \varepsilon$ for all $\tau \in T(A)$.

For a unital separable simple unital C^* -algebra with tracial rank no more than one we have

Theorem 2.3. ([30]) Let A be a unital separable simple unital C^* -algebra with $\text{TR}(A) \leq 1$. Then

- A is quasidiagonal (i.e. there exists a faithful representation $\pi: A \rightarrow B(H)$ and an increasing sequence of finite rank projections $p_1 \leq p_2 \leq \dots$ such that $\|p_n \pi(a) - \pi(a)p_n\| \rightarrow 0$ ($\forall a \in A$) and $p_n \rightarrow 1_H$ (strongly operator topology) ($n \rightarrow \infty$)) ;
- A has real rank zero (i.e. any self-adjoint element in A can be approximated by an invertible self-adjoint element in A) or one (i.e. any self-adjoint elements $x_1, x_2 \in A$ can be approximated by self-adjoint elements $y_1, y_2 \in A$ such that $y_1^2 + y_2^2$ is invertible.) ([7]);
- A has stable rank one (i.e. any element in A can be approximated by an invertible element in A) ([49]) ;
- $K_0(A)$ is weakly unperforated (i.e. $nx > 0$ for some $n > 0$ implies $x > 0$) and with Riesz interpolation property (i.e. $x_1, x_2, y_1, y_2 \in K_0(A)$ with $x_1, x_2 \leq y_1, y_2$, then there is a $z \in K_0(A)$ with $x_1, x_2 \leq z \leq y_1, y_2$);
- A has the fundamental comparison property (Blackadar's Fundamental Comparability Question): if $p, q \in A$ are two projections and $\tau(p) < \tau(q)$ for all $T(A)$, then $p \sim q'$ with $q' \leq q$.

Remark 2.4. If A is a simple separable unital C^* -algebra with $\text{TR}(A) \leq 1$ and real rank zero, then $\text{TR}(A) = 0$.

Theorem 2.3 and Remark 2.4 suggest that the class of separable nuclear simple unital C^* -algebras with $\text{TR}(A) = 0$ is a reasonable replacement for the class of separable nuclear simple unital quasidaigonal C^* -algebras with real rank zero, stable rank one and with weakly unperforated K_0 -groups. But there exists an exact, quasidaigonal simple C^* -algebra with real rank zero, stable rank one, the Universal Coefficient Theorem, unperforated K_0 -group, Riesz interpolation property, and the fundamental comparison property which has not tracial rank zero ([8, Corollary 7.2]).

Recall that a C^* -algebra A is AH if

$$A = \lim_{n \rightarrow \infty} A_n,$$

where $A_n = \bigoplus_{i=1}^{k(n)} P_{(i,n)} M_{(i,n)}(C(X_{(i,n)})) P_{(i,n)}$, $P_{(i,n)} \in C(X_{(i,n)})$ is a projection and $X_{(i,n)}$ is a connected CW-complex. If A is simple, we say A has slow dimension growth if

$$\lim_{n \rightarrow \infty} \max_i \frac{\dim X_{(i,n)}}{1 + \text{rank} P_{(i,n)}} = 0.$$

A is said to have no dimension growth, if there is integer $m > 0$ such that

$$\dim X_{(i,n)} \leq m$$

for all i and n . When each $X_{(i,n)}$ is an interval I (resp. a circle S^1), then A is said to be an AI-algebra (resp. an AT-algebra).

Note that simple AH algebras with the slow dimension growth and with real rank zero have no dimension growth ([9], [14], [15]).

Elliott and Gong ([12]) showed that every simple AH-algebra with no dimension growth and with real rank zero has tracial rank zero.

Theorem 2.5. ([12]) Let A and B be two simple unital AH-algebras with slow dimension growth and with real rank zero. Then $A \cong B$ if and only if

$$(K_0(A), K_0(A)_+, [1_A]_0, K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B]_0, K_1(B)).$$

For simple separable unital C^* -algebras with tracial rank zero Lin showed

Theorem 2.6. ([31]) Let A and B be two simple, separable, nuclear unital C^* -algebras with $\text{TR}(A) = \text{TR}(B) = 0$ which satisfy the Universal Coefficient Theorem. Then $A \cong B$ if and only if

$$(K_0(A), K_0(A)_+, [1_A]_0, K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B]_0, K_1(B)).$$

Note that the above two classes of simple C^* -algebras in Theorems 2.5 and 2.6 coincide ([31]).

3 Tracial Rokhlin property: Integer group $\mathbb{Z}$ case

3.1 Definition and basic facts

We start by defining the tracial Rokhlin property for single automorphisms (actions of $\mathbb{Z}$). It is closely related to, but slightly weaker than, the approximate Rokhlin property of Definition 4.2 of [24]. To our knowledge, the idea was first introduced in [5]. It is closely related to the tracial Rokhlin property for actions of finite cyclic groups, as in [44].

Definition 3.1. ([39]) Let A be a stably finite simple unital C^* -algebra and let $\alpha \in \text{Aut}(A)$. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every nonzero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \dots, e_n \in A$ such that:

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$.
- (2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$.
- (3) With $e = \sum_{j=0}^n e_j$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x .

We do not say anything about $\alpha(e_n)$.

In all applications so far, in addition to the conditions in Definition 3.1, the algebra A has real rank zero, and the order on projections over A is determined by traces. In this case, we can replace the third condition by one involving traces:

Lemma 3.2. ([39]) Let A be a stably finite simple unital C^* -algebra such that $\text{RR}(A) = 0$ and the order on projections over A is determined by traces. Let $\alpha \in \text{Aut}(A)$. Then α has the tracial Rokhlin property if and only if for every finite set $F \subset A$, every $\varepsilon > 0$, and every $n \in \mathbb{N}$, there are mutually orthogonal projections $e_0, e_1, \dots, e_n \in A$ such that:

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$.
- (2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$.
- (3) With $e = \sum_{j=0}^n e_j$, we have $\tau(1 - e) < \varepsilon$ for all $\tau \in T(A)$.

We now want to relate the tracial Rokhlin property to forms of the Rokhlin property which have appeared in the literature. The most important of these is as follows. (See, for example, Definition 2.5 of [21], and Condition (3) in Proposition 1.1 of [26].)

Definition 3.3. Let A be a simple unital C^* -algebra and let $\alpha \in \text{Aut}(A)$. We say that α has the *Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, there are mutually orthogonal projections

$$e_0, e_1, \dots, e_{n-1}, f_0, f_1, \dots, f_n \in A$$

such that:

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-2$ and $\|\alpha(f_j) - f_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$.
- (2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and all $a \in F$, and $\|f_j a - a f_j\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$.
- (3) $\sum_{j=0}^{n-1} e_j + \sum_{j=0}^n f_j = 1$.

Generally, the tracial Rokhlin property is weaker than the above Rokhlin property as follows:

Theorem 3.4. ([39]) Let A be a stably finite simple unital C^* -algebra and let $\alpha \in \text{Aut}(A)$. Assume either that A has tracial rank zero, or that A is approximately divisible ([3]), every quasitrace on A is a trace, and that projections in A distinguish the tracial states of A . Suppose that α has the Rokhlin property in the sense of Definition 3.3. Then α has the tracial Rokhlin property.

We have no example of an automorphism on a simple C^* -algebra with tracial rank zero which has the tracial Rokhlin property, but does not have the Rokhlin property.

Question 3.5. Let A be a simple separable unital C^* -algebra with tracial rank zero and let $\alpha \in \text{Aut}(A)$. Suppose that α has the tracial Rokhlin property.

Is this true that α has the Rokhlin property in the sense of Definition 3.3 ?

Theorem 3.6. ([39]) Let A be a stably finite simple unital C^* -algebra, and let α be an automorphism of A which has the tracial Rokhlin property. Suppose A has real rank zero and stable rank one, and suppose that the order on projections over A is determined by traces. Then $C^*(\mathbb{Z}, A, \alpha)$ also has these three properties.

Question 3.7. Let A be a simple unital C^* -algebra with tracial rank zero, and let α be an automorphism of A which has the tracial Rokhlin property.

Is it true that $C^*(\mathbb{Z}, A, \alpha)$ also has the tracial rank zero ?

There is an example of a simple C^* -algebra A which has three conditions in Theorem 3.6, but does not have tracial rank zero, and an automorphism α on A such that α has the tracial Rokhlin property, but the crossed product algebra $C^*(\mathbb{Z}, A, \alpha)$ does not have tracial rank zero.

Example 3.8. Let $n \in \{2, 3, \dots, \infty\}$, let F_n be the free group on n generators, and let α be any automorphism of $C_r^*(F_n)$. (An example which is particularly interesting in this context is to take $n = \infty$ and to take α to be induced by an infinite order permutation of the free generators of F_n . Another possibility is to have α multiply the k -th generating unitary by an irrational number λ_k .) Let B be the 2^∞ UHF algebra and let $\beta \in \text{Aut}(B)$ have the Rokhlin property in [6]. Then $\alpha \otimes \beta$

generates an action with the Rokhlin property from the elementary observation. Since $C_r^*(F_n)$ has a unique tracial state, it follows from Corollary 6.6 of [50] that $C_r^*(F_n) \otimes B$ has stable rank one. Moreover, $C_r^*(F_n) \otimes B$ is exact, so every quasitrace is a trace ([17][18]), whence Theorem 7.2 of [51] implies that $C_r^*(F_n) \otimes B$ has real rank zero and Theorem 5.2(b) of [51] implies that the order on projections over $C_r^*(F_n) \otimes B$ is determined by traces. (In fact, $K_0(C_r^*(F_n) \otimes B)$ is $\mathbf{Z}[\frac{1}{2}]$ with its usual order.) We can now use Theorem 3.4 to conclude that $\alpha \otimes \beta$ generates an action with the tracial Rokhlin property. On the other hand, the corollary to Theorem A1 of [52] shows that $C_r^*(F_n)$ is not quasidiagonal, so $C_r^*(F_n) \otimes B$ is not quasidiagonal either. Theorem 3.4 of [29] (or Theorem 2.3) therefore shows that $C_r^*(F_n) \otimes B$ does not have tracial rank zero. Theorem 3.6 shows that the crossed product $C^*(\mathbf{Z}, C_r^*(F_n) \otimes B, \alpha \otimes \beta)$ has real rank zero and stable rank one, and that the order on projections over this algebra is determined by traces. However, it does not have tracial rank zero by Theorem 2.3 because it contains the nonquasidiagonal C^* -algebra $C_r^*(F_n)$.

We can now give a version of Kishimoto's result, Theorem 2.1 of [26], giving conditions for the Rokhlin property on a simple unital AT-algebra with real rank zero and a unique tracial state.

Theorem 3.9. ([40]) Let A be a simple separable unital C^* -algebra with tracial rank zero, and suppose that A has a unique tracial state τ . Let $\pi_\tau : A \rightarrow L(H_\tau)$ be the Gelfand-Naimark-Segal representation associated with τ . Let $\alpha \in \text{Aut}(A)$. Then the following conditions are equivalent:

- (1) α has the tracial Rokhlin property.
- (2) The automorphism of $\pi_\tau(A)''$ induced by α^n is outer for every $n \neq 0$, that is, α^n is not weakly inner in π_τ for any $n \neq 0$.
- (3) $C^*(\mathbf{Z}, A, \alpha)$ has a unique tracial state.
- (4) $C^*(\mathbf{Z}, A, \alpha)$ has real rank zero.

When α satisfies the approximate innerness, we have the following:

Theorem 3.10. ([32]) Let A be a simple separable unital C^* -algebra which satisfies the Universal Coefficient Theorem, which has tracial rank zero, and which has a unique tracial state. If $\alpha \in \text{Aut}(A)$ has the Rokhlin property and if α^n is an approximately inner for some $n > 0$, then $C^*(\mathbf{Z}, A, \alpha)$ is a simple AH-algebras with no dimensional growth and real rank zero.

3.2 Noncommutative Furstenberg transformations

Furstenberg introduced in [13] a family of homeomorphisms of $S^1 \times S^1$, now called Furstenberg transformations. They have the form

$$h_{\gamma, d, f}(\zeta_1, \zeta_2) = (e^{2\pi i \gamma} \zeta_1, \exp(2\pi i f(\zeta_1)) \zeta_1^d \zeta_2),$$

with fixed $\gamma \in \mathbf{R}$, $d \in \mathbf{Z}$, and $f: S^1 \rightarrow \mathbf{R}$ continuous. For $\gamma \notin \mathbf{Q}$ and $d \neq 0$, Furstenberg proved that $h_{\gamma,d,f}$ is minimal. These homeomorphisms, and higher dimensional analogs (which also appear in [13]), have attracted significant interest in operator algebras (see, for example, [42], [22], [28], and [48]) and in dynamics (see, for example, [20] and [53]).

For any $\theta \in \mathbf{R}$, the formula for the automorphism $g \mapsto g \circ h_{\gamma,d,f}$ of $C(S^1 \times S^1)$ also defines an automorphism of the rotation algebra A_θ . Taking the generators of A_θ to be unitaries u and v satisfying $vu = e^{2\pi i \theta} uv$, we obtain an automorphism $\alpha_{\theta,\gamma,d,f}$ of A_θ as follows:

Definition 3.11. ([40]) Let $\theta, \gamma \in \mathbf{R}$, let $d \in \mathbf{Z}$, and let $f: S^1 \rightarrow \mathbf{R}$ be a continuous function. The *Furstenberg transformation* on A_θ determined by (θ, γ, d, f) is the automorphism $\alpha_{\theta,\gamma,d,f}$ of A_θ such that

$$\alpha_{\theta,\gamma,d,f}(u) = e^{2\pi i \gamma} u \quad \text{and} \quad \alpha_{\theta,\gamma,d,f}(v) = \exp(2\pi i f(u)) u^d v.$$

When $\theta \notin \mathbf{Q}$, it is the most general automorphism α of A_θ for which $\alpha(u)$ is a scalar multiple of u . That is,

Proposition 3.12. ([40]) Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$ and let $\gamma \in \mathbf{R}$. Let $\alpha \in \text{Aut}(A_\theta)$ be an automorphism such that $\alpha(u) = e^{2\pi i \gamma} u$. Then there exist $d \in \mathbf{Z}$ and a continuous function $f: S^1 \rightarrow \mathbf{R}$ such that $\alpha = \alpha_{\theta,\gamma,d,f}$.

The noncommutative Furstenberg transformations are our motivating example to define the tracial Rokhlin property. In fact we can conclude the following.

Theorem 3.13. ([40]) Let $\theta, \gamma \in \mathbf{R}$ and suppose that $1, \theta, \gamma$ are linearly independent over $\mathbf{Q}$. Let $d \in \mathbf{Z}$. Then the automorphism $\alpha = \alpha_{\theta,\gamma,d,f} \in \text{Aut}(A_\theta)$, of Definition 3.11, has the tracial Rokhlin property.

It follows from Theorem 4 and Remark 6 of [11] and Proposition 2.6 of [29] that when $\theta \notin \mathbf{R} \setminus \mathbf{Q}$, A_θ has tracial rank zero. Also, A_θ has a unique tracial state τ . The statement comes from Theorem 3.9(2).

Corollary 3.14. Let $\theta, \gamma \in \mathbf{R}$ and suppose that $1, \theta, \gamma$ are linearly independent over $\mathbf{Q}$. Let $d \in \mathbf{Z}$. Let $\alpha_{\theta,\gamma,d,f} \in \text{Aut}(A_\theta)$ be as in Definition 3.11. Then:

- (1) $C^*(\mathbf{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ is simple.
- (2) $C^*(\mathbf{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ has a unique tracial state.
- (3) $C^*(\mathbf{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ has real rank zero.
- (4) $C^*(\mathbf{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ has stable rank one.
- (5) The order on projections over $C^*(\mathbf{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ is determined by traces.

- (6) $C^*(\mathbf{Z}, A_\theta, \alpha_{\theta, \gamma, d, f})$ satisfies the local approximation property of Popa [46] (is a Popa algebra in the sense of Definition 1.2 of [8]).

The following problems are still open.

Question 3.15. Let $\theta, \gamma \in \mathbf{R}$ and suppose that $1, \theta, \gamma$ are linearly independent over $\mathbf{Q}$.

Does $\alpha_{\theta, \gamma, d, f}$ have the Rokhlin property?

Question 3.16. Let $\theta, \gamma \in \mathbf{R}$ and suppose that $1, \theta, \gamma$ are linearly independent over $\mathbf{Q}$.

Does $C^*(\mathbf{Z}, A_\theta, \alpha_{\theta, \gamma, d, f})$ have tracial rank zero?

Recently, Lin and Phillips ([33]) proves that when $h_{\gamma, d, f}$ has uniquely ergodicity, then $C^*(\mathbf{Z}, S^1 \times S^1, \alpha_{h_{\gamma, d, f}})$ has tracial rank zero, where $\alpha_{h_{\gamma, d, f}} : C(S^1 \times S^1) \rightarrow C(S^1 \times S^1)$ by $\alpha_{h_{\gamma, d, f}}(g) = g \circ h_{\gamma, d, f}$.

The method of proof of Theorem 3.13 applies to other example as well. For example, in a series of papers [35], [36], [37], [38], [54], Milnes and Walters have studied the simple quotients of the C^* -algebras of certain discrete subgroups of nilpotent Lie groups of dimension up to five, which are a kind of generalization of the irrational rotation algebras, which occur when the Lie group is the three dimensional Heisenberg group. Since each of these is the crossed product of a simple C^* -algebra (the C^* -algebra of an ordinary minimal Furstenberg transformation on $S^1 \times S^1$) by an automorphism with the tracial Rokhlin property, we can conclude that these algebras have stable rank one and real rank zero, and that the order on projections over them is determined by traces.

4 Tracial Rokhlin property: Finite group G case

We begin with Izumi's definition of the Rokhlin property. To emphasize the difference, we call it the strict Rokhlin property here.

Definition 4.1. Let A be a unital C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A . We say that α has the *strict Rokhlin property* if for every finite set $F \subset A$, and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

- (1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
- (2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- (3) $\sum_{g \in G} e_g = 1$.

We note that if α is approximately inner, requiring $\sum_{g \in G} e_g = 1$ forces $[1_A] \in K_0(A)$ to be divisible by the order of G , and therefore rules out many C^* -algebras of interest.

The following might be well known.

Theorem 4.2. ([41]) Let A be a unital AI-algebra (resp. AT-algebra), and let $\alpha \in \text{Aut}(A)$ be an automorphism which satisfies $\alpha^n = \text{id}_A$ and such that the action of $\mathbf{Z}/n\mathbf{Z}$ generated by α has the strict Rokhlin property. Then $C^*(\mathbf{Z}/n\mathbf{Z}, A, \alpha)$ is a simple unital AI-algebra (resp. AT-algebra).

We now give the definition of the tracial Rokhlin property. The difference is that we do not require that $\sum_{g \in G} e_g = 1$, only that $1 - \sum_{g \in G} e_g$ be “small” in a tracial sense. Of course, $\sum_{g \in G} e_g = 1$ is allowed, in which case Conditions (3) and (4) in the definition are vacuous.

Definition 4.3. ([45]) Let A be an infinite dimensional simple unital C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A . We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are nonzero mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

- (1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
- (2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- (3) With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x .
- (4) With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

Note that when A is finite, the condition (4) in Definition 4.3 is unnecessary ([45]).

When α is an action of a simple C^* -algebra A with tracial rank zero by a finite group G , the crossed product $C^*(G, A, \alpha)$ has also tracial rank zero ([45]). Moreover we have the following:

Theorem 4.4. ([41]) Let A be a simple unital C^* -algebra with $\text{TR}(A) \leq 1$. Suppose that $\alpha \in \text{Aut}(A)$ has the tracial Rokhlin property with $\alpha^n = 1$. Then

$$\text{TR}(C^*(\mathbf{Z}/n\mathbf{Z}, A, \alpha)) \leq 1.$$

Related to Question 3.5 we have an example of an action with tracial Rokhlin property which does not have strictly Rokhlin property.

Definition 4.5. For and nuclear C^* -algebra A , we let $\varphi_A: A \otimes A \rightarrow A \otimes A$ denote the flip automorphism, determined by $\varphi_A(a \otimes b) = b \otimes a$ for $a, b \in A$.

Recall that a C^* -algebra A is subhomogeneous if every irreducible representation of B is finite dimensional. Further recall that an ASH-algebra is a C^* -algebra A such that for every finite set $F \subset A$ and every $\varepsilon > 0$, there is a unital subhomogeneous $B \subset A$ such that $\text{dist}(a, B) < \varepsilon$ for every $a \in F$.

Proposition 4.6. ([41]) Let A be a unital ASH-algebra. Then the action of $\mathbf{Z}/2\mathbf{Z}$ generated by φ_A does not have the strictly Rokhlin property.

In particular, the flip on a simple AH-algebra with (very) slow dimension growth never generates an action with the Rokhlin property.

Proposition 4.7. ([41]) Let A be a simple unital C^* -algebra which is approximately divisible in the sense of [3]. Then the flip φ_A on any symmetric tensor product $A \otimes A$ generates an action of $\mathbb{Z}/2\mathbb{Z}$ with the tracial Rokhlin property.

Therefore, the flip on $A_\theta \otimes A_\theta$ has the tracial Rokhlin property, but does not have the strictly Rokhlin property.

Finally, we give an example of an automorphism on UHF algebra of period 2 which does not have the tracial Rokhlin property.

Example 4.8. Elliott constructed an automorphism α of period 2 on UHF such that $C^*(\mathbb{Z}/2\mathbb{Z}, UHF, \alpha)$ has real rank one (i.e. its tracial rank is not 0). Hence from Theorem 2.3 and [45] we know that α does not have the tracial Rokhlin property. Note that the dual action of α has the strictly Rokhlin property ([41]).

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