

On Radius Problems for Analytic Functions of Koebe Type

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Abstract. By virtue of the extremal function $f(z)$ for $S^*(\alpha)$ which is the class of all starlike functions $f(z)$ of order α having $f(0) = 0$ and $f'(0) = 1$ in the open unit disk U , new function of Koebe type is considered. The object of the present paper is to derive radii for starlikeness of order α , and for convexity of order α for the function of Koebe type. Using the extremal functions for the classes of α -spiral like of order β and of α -convex like of order β , we also consider the analytic function of the generalized Koebe type. Some interesting examples for the theorems are also given with their mapping properties.

1 Introduction

Let A be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. A function $f(z)$ in A is said to be starlike of order α if it satisfies

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha$$

for some α ($0 \leq \alpha < 1$) and all z in U . We denote by $S^*(\alpha)$ the subclass of A consisting of all starlike functions of order α in U . A function $f(z)$ in A is said to be convex of order α if it satisfies

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha$$

for some α ($0 \leq \alpha < 1$) and all z in U . Also we denote by $K(\alpha)$ the subclass of A consisting of functions $f(z)$ which are convex of order α in U . In particular, we denote by $S^*(0) \equiv S^*$ and $K(0) \equiv K$ (cf. Robertson [3]). By Robertson [3], we note that

$$(i) \quad f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \text{ is the extremal function for the class } S^*(\alpha).$$

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$$(ii) f(z) = \begin{cases} \frac{1 - (1 - z)^{2\alpha-1}}{2\alpha - 1} & (\alpha \neq \frac{1}{2}) \\ -\log(1 - z) & (\alpha = \frac{1}{2}) \end{cases}$$

is the extremal function for the class $K(\alpha)$.

If we take $\alpha = 0$ in (i) and (ii), then we see that

$$(iii) f(z) = \frac{z}{(1 - z)^2} \text{ is the extremal function for the class } S^*.$$

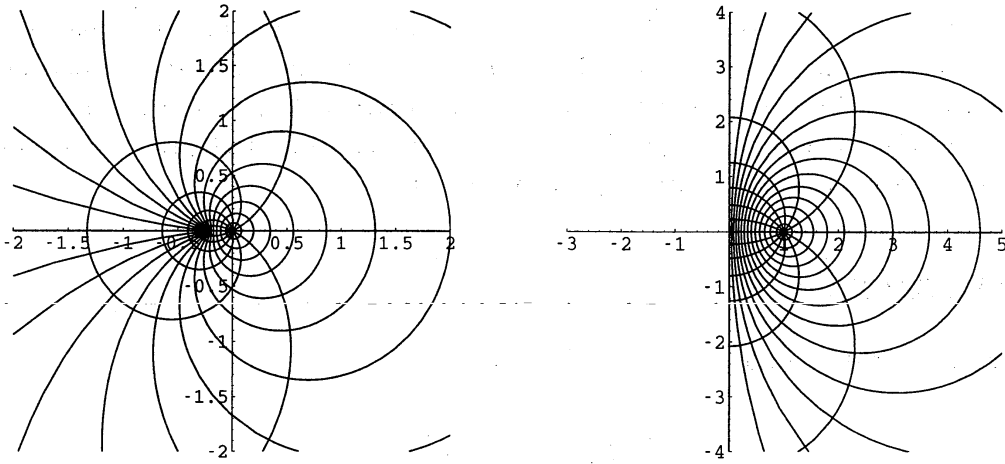


Fig 1.1: Image of $|z| = r$ by $f(z) = \frac{z}{(1 - z)^2}$ (left), $\frac{zf'(z)}{f(z)}$ (right) ($r = 1$ in all cases).

$$(iv) f(z) = \frac{z}{1 - z} \text{ is the extremal function for the class } K.$$

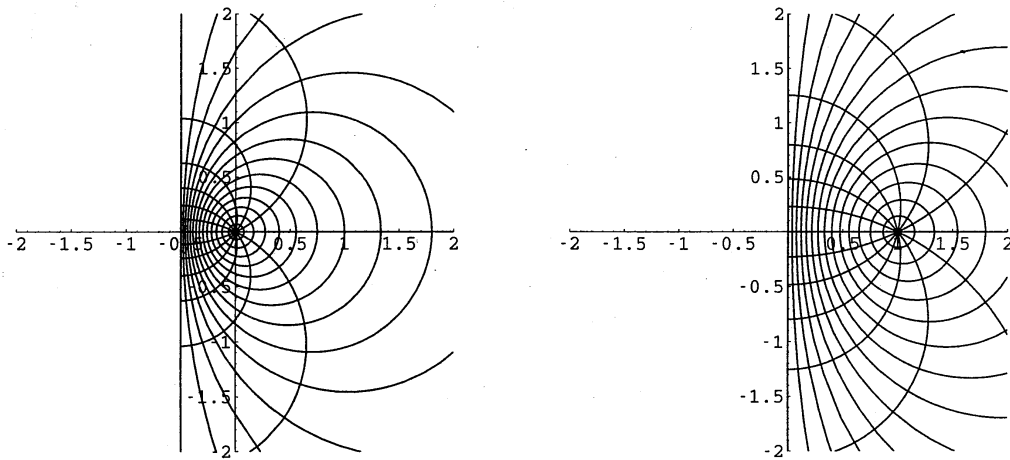


Fig 1.2: Image of $|z| = r$ by $f(z) = \frac{z}{1 - z}$ (left), $1 + \frac{zf''(z)}{f'(z)}$ (right) ($r = 1$ in all cases).

Furthermore, by Marx [2] and Stroh  cker [4] (also by Komatu [1]), we see that K is the subclass of $S^*(\frac{1}{2})$. And by Wilken and Feng [5], $K(\alpha)$ is the subclass of $S^*(\beta(\alpha))$, where

$$\beta(\alpha) = \begin{cases} \frac{2\alpha - 1}{2(1 - 2^{1-2\alpha})} & (\alpha \neq \frac{1}{2}) \\ \frac{1}{2\log 2} & (\alpha = \frac{1}{2}). \end{cases}$$

In view of the previous properties for the classes $S^*(\alpha)$ and $K(\alpha)$, it is very interesting to consider the following analytic function

$$f(z) = \frac{z}{(1 - z)^k} \quad (k \in \mathbb{R})$$

which was called Koebe type. Then by the extremal functions $f(z)$ for the classes $S^*(\alpha)$ and $K(\alpha)$, we know that, in general, this function $f(z)$ is not univalent (so, is not starlike or convex) in U . But, since every analytic function $f(z)$ maps, one-to-one, a small disk onto a small disk, we consider the radius problems for the analytic function $f(z)$ of Koebe type to be starlike and convex of order α .

2 Radii for starlikeness of order α

We derive radii of starlikeness of order α for the function $f(z)$ of Koebe type to be in the class of $S^*(\alpha)$. Our first result is contained in

Theorem 1. *The function $f(z)$ of Koebe type satisfies*

- (1) $k > 2(1 - \alpha) \implies f(z) \in S^*(\alpha)$ for $0 \leq r < \frac{1 - \alpha}{k - (1 - \alpha)}$ ($|z| = r$),
- (2) $0 \leq k \leq 2(1 - \alpha) \implies f(z) \in S^*(\alpha)$ for $0 \leq r < 1$ ($|z| = r$),
- (3) $k < 0 \implies f(z) \in S^*(\alpha)$ for $0 \leq r < \frac{1 - \alpha}{1 - \alpha - k}$ ($|z| = r$).

Proof. By a simple calculation, we have

$$\frac{zf'(z)}{f(z)} = 1 + k \frac{z}{1 - z}.$$

Letting $z = re^{i\theta}$, and we have

$$\begin{aligned} \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) &= \operatorname{Re} \left(1 + k \frac{re^{i\theta}}{1 - re^{i\theta}} \right) \\ &= 1 - k \frac{r^2 - r\cos\theta}{1 + r^2 - 2r\cos\theta}. \end{aligned}$$

We define the function $g(\theta)$ by

$$g(\theta) = \frac{r^2 - r\cos\theta}{1 + r^2 - 2r\cos\theta}.$$

For $k \geq 0$, we calculate maximum $g(\theta)$ to get the minimum value for $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)$. Changing $\cos\theta$ by t ($-1 \leq t \leq 1$) in $g(\theta)$, we get

$$g(t) = \frac{r^2 - rt}{1 + r^2 - 2rt},$$

and then

$$g'(t) = \frac{r(r^2 - 1)}{(1 + r^2 - 2rt)^2}.$$

Thus we know that $g(t)$ is monotone decreasing because $g'(t)$ is non-positive for $0 \leq r < 1$. Therefore, $g(t)$ has maximum value at $t = -1$. It follows from the above that

$$\begin{aligned} \max g(\theta) &= \frac{r^2 + r}{1 + r^2 + 2r} \\ &= \frac{r}{1 + r}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= 1 - k \frac{r^2 - r\cos\theta}{1 + r^2 - 2r\cos\theta} \\ &\geq \frac{1 - (k-1)r}{1 + r} > \alpha \end{aligned}$$

for r satisfying the following inequality

$$1 - \alpha > (k - (1 - \alpha))r. \quad (2.1)$$

We see that if $k > 1 - \alpha$, then

$$0 \leq r < \frac{1 - \alpha}{k - (1 - \alpha)},$$

and if $k > 2(1 - \alpha)$, then

$$\frac{1 - \alpha}{k - (1 - \alpha)} < 1,$$

so, we derive the case (1) in Theorem 1.

If $0 \leq k < 1 - \alpha$, then the inequality (2.1) is always satisfied for all r ($0 \leq r < 1$).

If $1 - \alpha \leq k \leq 2(1 - \alpha)$, then we have the next inequality

$$1 < \frac{1 - \alpha}{k - (1 - \alpha)}.$$

This gives us that $f(z) \in S^*(\alpha)$ for $0 \leq r < 1$. Hence we get the result of case (2) in Theorem 1.

If $k < 0$, letting $k = -j$, we have

$$f(z) = \frac{z}{(1 - z)^{-j}} = z(1 - z)^j \quad (j > 0).$$

Similary, for $k \geq 0$, we have to consider

$$\frac{zf'(z)}{f(z)} = 1 - j \frac{z}{1-z}.$$

This gives that

$$\begin{aligned} \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) &= \operatorname{Re} \left(1 - j \frac{re^{i\theta}}{1-re^{i\theta}} \right) \\ &= 1 + j \frac{r^2 - r\cos\theta}{1+r^2-2r\cos\theta} = 1 + jg(\theta), \end{aligned}$$

where

$$g(\theta) = \frac{r^2 - r\cos\theta}{1+r^2-2r\cos\theta}.$$

When $g(\theta)$ has its minimum value, then $\operatorname{Re}(\frac{zf'(z)}{f(z)})$ becomes minimum. It is easy to check that $g(\theta)$ has the minimum value at $\cos\theta = 1$ because it is monotone decreasing. Hence, we have

$$\begin{aligned} \min g(\theta) &= \frac{r^2 - r}{1+r^2-2r} \\ &= \frac{r}{1-r}. \end{aligned}$$

It follows that

$$\begin{aligned} \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) &= 1 + j \frac{r^2 - r\cos\theta}{1+r^2-2r\cos\theta} \\ &\geq \frac{1 - (j+1)r}{1-r} > \alpha \end{aligned}$$

for r satisfying

$$r < \frac{1-\alpha}{j+1-\alpha} < 1.$$

Noting that $j = -k$, we conclude that

$$0 \leq r < \frac{1-\alpha}{1-\alpha-k},$$

which proves the cases (3) in Theorem 1.

We give some examples of functions $f(z)$ in $S^*(\alpha)$ for Theorem 1.

Example 1.

$$(1) f(z) = \frac{z}{(1-z)^{\frac{3}{4}}} \in S^*\left(\frac{2}{3}\right) \text{ for } 0 \leq r < \frac{4}{5},$$

$$(2) f(z) = \frac{z}{(1-z)^{\frac{1}{3}}} \in S^*\left(\frac{1}{4}\right) \text{ for } 0 \leq r < 1,$$

$$(3) f(z) = \frac{z}{(1-z)^{-5}} \in S^*\left(\frac{1}{6}\right) \text{ for } 0 \leq r < \frac{1}{7}.$$

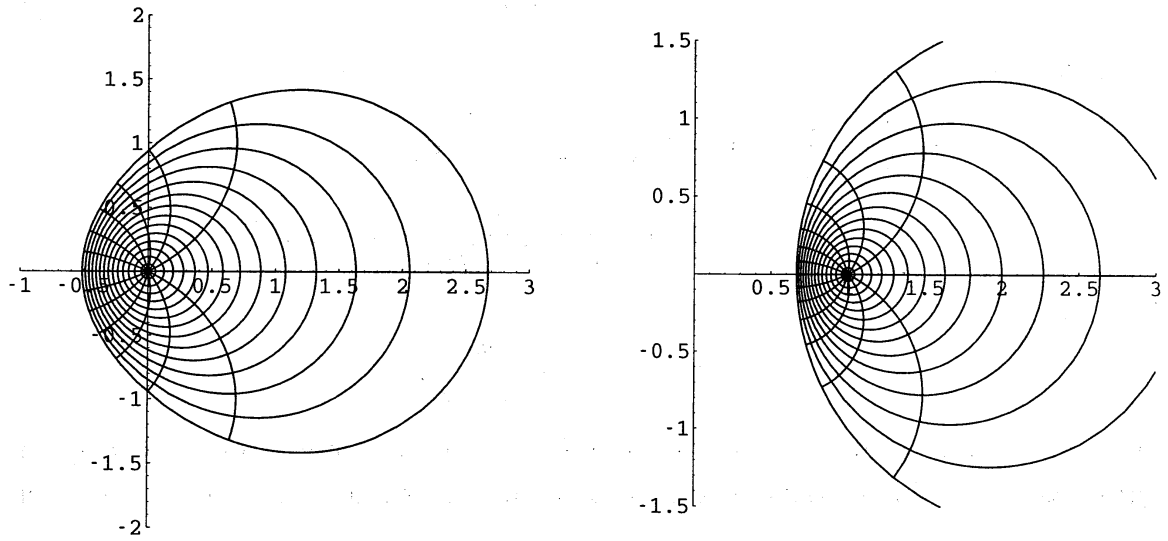


Fig 2.1: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{\frac{3}{4}}}$ (left), $\frac{zf'(z)}{f(z)}$ (right) ($r = \frac{4}{5}$ in all cases).

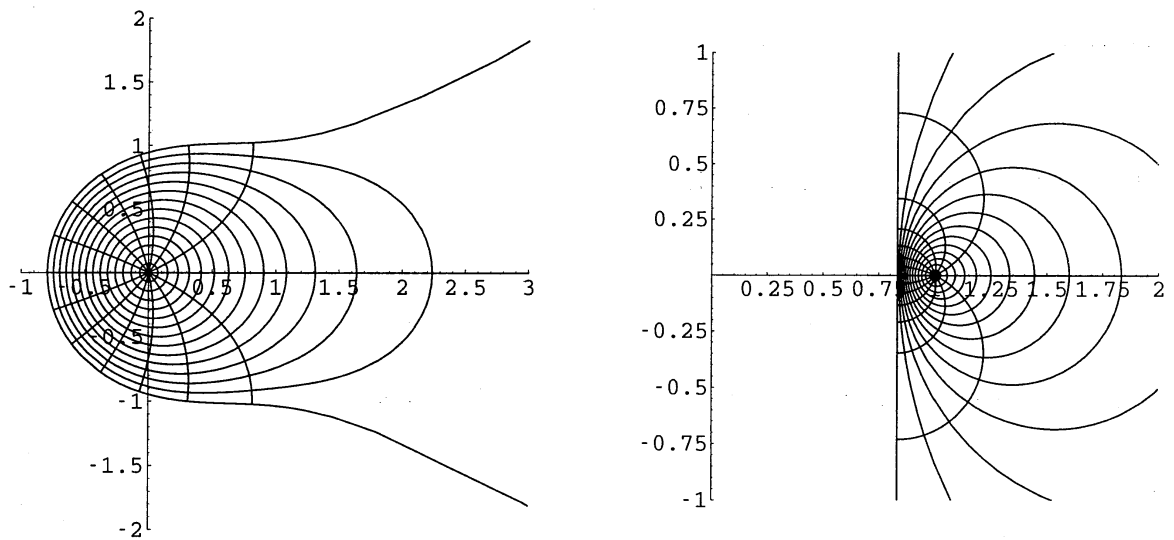


Fig 2.2: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{\frac{1}{3}}}$ (left), $\frac{zf'(z)}{f(z)}$ (right) ($r = 1$ in all cases).

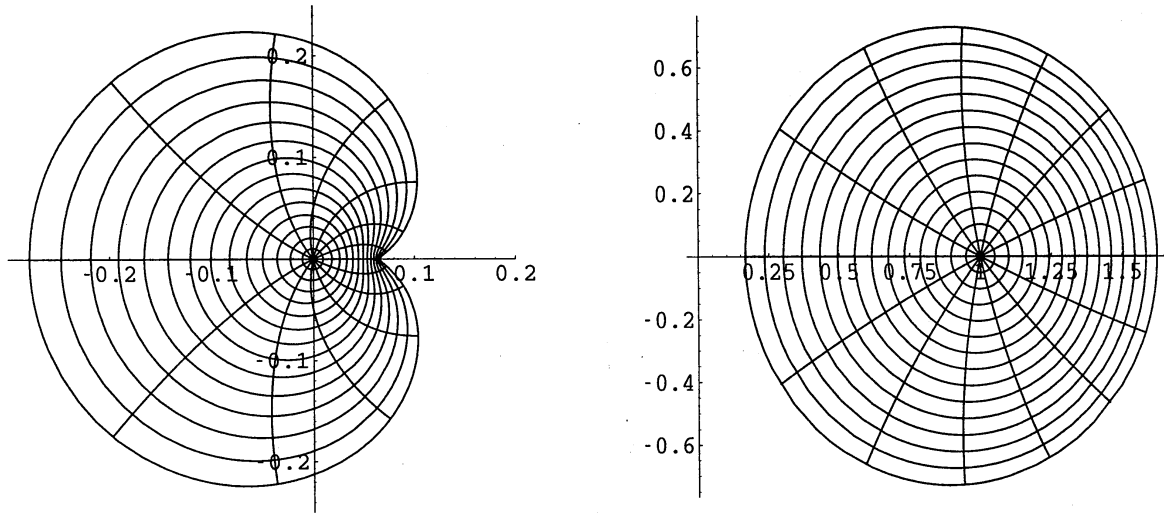


Fig 2.3: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^5}$ (left), $\frac{zf'(z)}{f(z)}$ (right) ($r = \frac{1}{7}$ in all cases).

3 Radii for convexity of order α

Next we discuss the radii of convexity of order α for the function $f(z)$ of Koebe type.

Theorem 2. *The function $f(z)$ of Koebe type satisfies*

(1) $k \geq 1 \Rightarrow f(z) \in K(\alpha)$ for

$$0 \leq r < \frac{(3-\alpha)k - 2(1-\alpha) - \sqrt{k((\alpha^2 - 2\alpha + 5)k - 4(1-\alpha))}}{2(k-1)(k-(1-\alpha))} \quad (|z| = r),$$

(2) $k \leq -1 \Rightarrow f(z) \in K(\alpha)$ for

$$0 \leq r < \frac{2(1-\alpha) - (3-\alpha)k - \sqrt{k((\alpha^2 - 2\alpha + 5)k - 4(1-\alpha))}}{2(1-k)(1-\alpha-k)} \quad (|z| = r).$$

Proof. From Theorem 1, since

$$\frac{zf'(z)}{f(z)} = \frac{1 + (k-1)z}{1-z},$$

we have

$$1 + \frac{zf''(z)}{f'(z)} = (k-1) \frac{z}{1 + (k-1)z} + \frac{1 + kz}{1-z}.$$

Let

$$g_1(z) = \frac{z}{1 + (k-1)z} \quad \text{and} \quad g_2(z) = \frac{1 + kz}{1-z}.$$

Taking $z = re^{i\theta}$, we see that

$$\begin{aligned}\text{Reg}_1(z) &= \text{Re} \left(\frac{re^{i\theta}}{1 + (k-1)re^{i\theta}} \right) \\ &= \frac{r^2(k-1) + r\cos\theta}{1 + r^2(k-1)^2 + 2r(k-1)\cos\theta}\end{aligned}$$

and

$$\begin{aligned}\text{Reg}_2(z) &= \text{Re} \left(\frac{1 + kre^{i\theta}}{1 - re^{i\theta}} \right) \\ &= \frac{1 - r^2k + r(k-1)\cos\theta}{1 + r^2 - 2r\cos\theta}.\end{aligned}$$

Let $h_1(\theta) = \text{Reg}_1(z)$ and $h_2(\theta) = \text{Reg}_2(z)$. Hence we get

$$\text{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) = (k-1)h_1(\theta) + h_2(\theta).$$

If $k \geq 1$, when $h_1(\theta)$ and $h_2(\theta)$ take the minimum values for the same θ , $\text{Re}(1 + \frac{zf''(z)}{f'(z)})$ has its minimum value. After the calculations, we see

$$\min h_1(\theta) = \frac{-r}{1 - r(k-1)} \quad (\cos\theta = -1 \text{ and } r \leq \frac{1}{k-1})$$

and

$$\min h_1(\theta) = \frac{r}{1 + r(k-1)} \quad (\cos\theta = 1 \text{ and } r > \frac{1}{k-1}).$$

Similary,

$$\min h_2(\theta) = \frac{1 - kr}{1 + r} \quad (\cos\theta = -1).$$

It follows that

$$\begin{aligned}\text{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) &\geq (k-1) \frac{-r}{1 - r(k-1)} + \frac{1 - kr}{1 + r} \\ &= \frac{(k-1)^2 r^2 - (3k-2)r + 1}{(1 - (k-1)r)(1 + r)} > \alpha,\end{aligned}$$

where $r \leq \frac{1}{k-1}$.

Hence, we derive the next inequality

$$(k-1)(k - (1 - \alpha))r^2 - ((3 - \alpha)k - 2(1 - \alpha))r + 1 - \alpha > 0. \quad (3.1)$$

From (3.1) and $r \geq 0$,

$$0 \leq r < \frac{(3 - \alpha)k - 2(1 - \alpha) - \sqrt{k((\alpha^2 - 2\alpha + 5)k - 4(1 - \alpha))}}{2(k-1)(k - (1 - \alpha))},$$

which completes the case (1) of Theorem 2.

If $0 \leq k < 1$, when $h_1(\theta)$ is maximum and $h_2(\theta)$ is minimum for the same θ , $\operatorname{Re}(1 + \frac{zf''(z)}{f'(z)})$ becomes minimum. Note that $h_2(\theta)$ is the same as the case (1). For $h_1(\theta)$,

$$\max h_1(\theta) = \frac{r}{1 + r(k-1)} \quad (\cos\theta = 1 \text{ and } r \leq \frac{1}{1-k})$$

or

$$\max h_1(\theta) = \frac{-r}{1 - r(k-1)} \quad (\cos\theta = -1 \text{ and } r > \frac{1}{1-k}).$$

But $f(z)$ is not univalent in this case for $|z| = r$ does not include the origin. Therefore, k does not exist such that this condition is satisfied in this case.

If $k < 0$, letting $k = -j$ in $f(z)$. Similary to $k \geq 0$, we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1-jz}{1-z} - (j+1) \frac{z}{1-(j+1)z}.$$

Let

$$g_3(z) = \frac{1-jz}{1-z} \quad \text{and} \quad g_4(z) = \frac{z}{1-(j+1)z}.$$

By a simple calculation,

$$\begin{aligned} \operatorname{Reg}_3(z) &= \operatorname{Re} \left(\frac{1-jre^{i\theta}}{1-re^{i\theta}} \right) \\ &= \frac{1+r^2j - (j+1)r\cos\theta}{1+r^2-2r\cos\theta} \end{aligned}$$

and

$$\begin{aligned} \operatorname{Reg}_4(z) &= \operatorname{Re} \left(\frac{re^{i\theta}}{1-(j+1)re^{i\theta}} \right) \\ &= \frac{r\cos\theta - r^2(j+1)}{1+r^2(j+1)^2 - 2r(j+1)\cos\theta} \end{aligned}$$

for $z = re^{i\theta}$. And let $h_3(\theta) = \operatorname{Reg}_3(z)$ and $h_4(\theta) = -\operatorname{Reg}_4(z)$. Then we have

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) = h_3(\theta) + (j+1)h_4(\theta).$$

When $h_3(\theta)$ and $h_4(\theta)$ have the minimum values for the same θ , we see that $\operatorname{Re}(1 + \frac{zf''(z)}{f'(z)})$ has its minimum value. After calculations, we know that

$$\min h_3(\theta) = \frac{1+jr}{1+r} \quad (\cos\theta = -1 \text{ and } 0 < j < 1)$$

or

$$\min h_3(\theta) = \frac{1-jr}{1-r} \quad (\cos\theta = 1 \text{ and } j \geq 1).$$

Similary, for $h_4(\theta)$,

$$\min h_4(\theta) = \frac{r}{(j+1)r+1} \quad (\cos\theta = -1 \text{ and } \frac{1}{j+1} < r)$$

or

$$\min h_4(\theta) = \frac{r}{(j+1)r-1} \quad (\cos\theta = 1 \text{ and } \frac{1}{j+1} \geq r).$$

Thus we have to consider two cases for $\cos\theta = -1$ and $\cos\theta = 1$.

If $\cos\theta = -1$, the domain of $f(z)$ is not the simply connected domain because $|z| = r$ does not include the origin. Therefore, $f(z)$ is not univalent. This case is impossible.

If $\cos\theta = 1$, the domain of $f(z)$ is the simply connected domain because $|z| = r$ includes the origin. Hence we have

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) &\geq \frac{1-jr}{1-r} - (j+1) \frac{r}{1-(j+1)r} \\ &= \frac{(j+1)^2 r^2 - (3j+2)r+1}{(1-r)(1-(j+1)r)} > \alpha \quad (j \geq 1 \text{ and } 0 \leq r \leq \frac{1}{j+1}). \end{aligned}$$

In view of the above, we have

$$(j+1)(j+1-\alpha)r^2 - ((3-\alpha)j+2(1-\alpha))r+1-\alpha > 0. \quad (3.2)$$

Solving (3.2) for $r \geq 0$, we obtain that

$$0 \leq r < \frac{(3-\alpha)j+2(1-\alpha) - \sqrt{j((\alpha^2-2\alpha+5)j-4(1-\alpha))}}{2(j+1)(j+1-\alpha)}.$$

Since $j = -k$, this inequality becomes that

$$0 \leq r < \frac{2(1-\alpha) - (3-\alpha)k - \sqrt{k((\alpha^2-2\alpha+5)k-4(1-\alpha))}}{2(1-k)(1-\alpha-k)} \quad (k \leq -1),$$

which gives the case (2) in Theorem 2. Therefore we complete the proof of the theorem.

We give two examples for Theorem 2 as follows.

Example 2.

$$(1) f(z) = \frac{z}{(1-z)^{10}} \in K(\frac{1}{3}) \text{ for } 0 \leq r < \frac{19-\sqrt{235}}{126} = 0.00291293 \dots,$$

$$(2) f(z) = \frac{z}{(1-z)^{-4}} \in K(\frac{1}{7}) \text{ for } 0 \leq r < \frac{23-\sqrt{274}}{85} = 0.0758477 \dots.$$

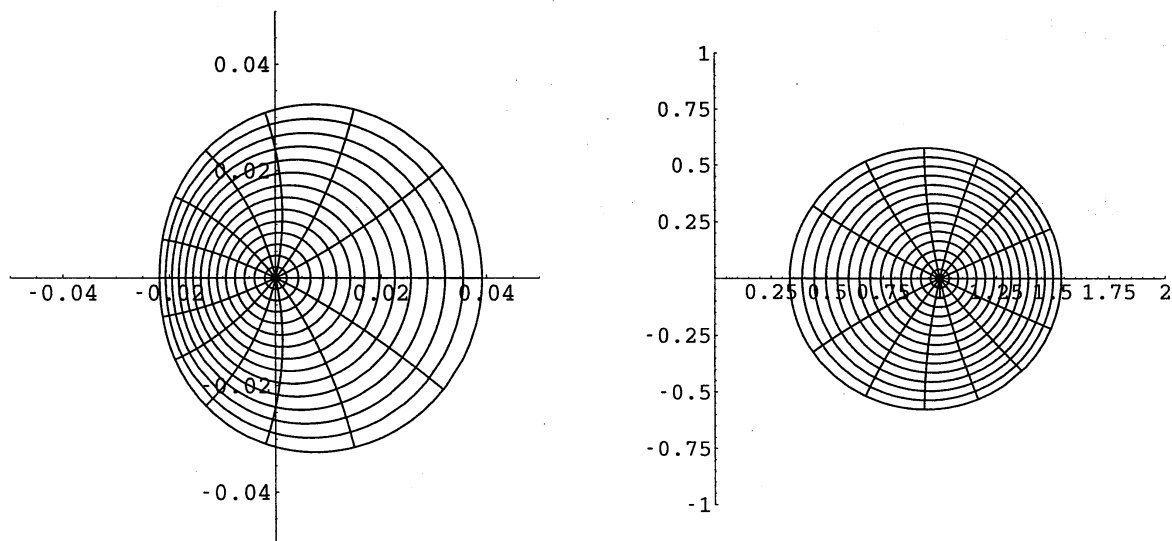


Fig 3.1: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{10}}$ (left), $1 + \frac{zf''(z)}{f'(z)}$ (right)
 (in all cases, $r = \frac{19-\sqrt{235}}{126} = 0.00291293\dots$).

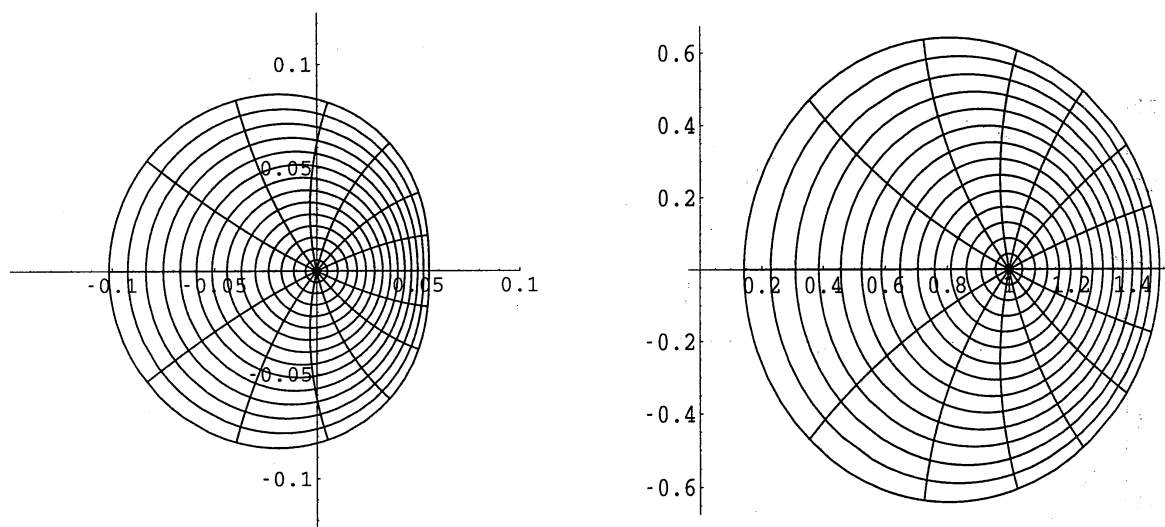


Fig 3.2: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{-4}}$ (left), $1 + \frac{zf''(z)}{f'(z)}$ (right)
 (in all cases, $r = \frac{23-\sqrt{274}}{85} = 0.0758477\dots$).

By the way, for $-1 < k < 1$ in Theorem 2, we could not specify the bound for the radius r . But we know that every analytic function $f(z)$ in U has the radius r for convexity. For example, the function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^{\frac{1}{2}}},$$

which is the case $k = \frac{1}{2}$, belongs to K for $0 \leq r \leq 0.95$ as follows.

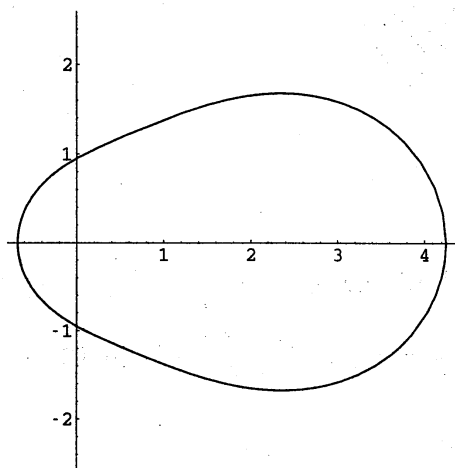


Fig 3.3: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{\frac{1}{2}}}$ ($r = 0.95$).

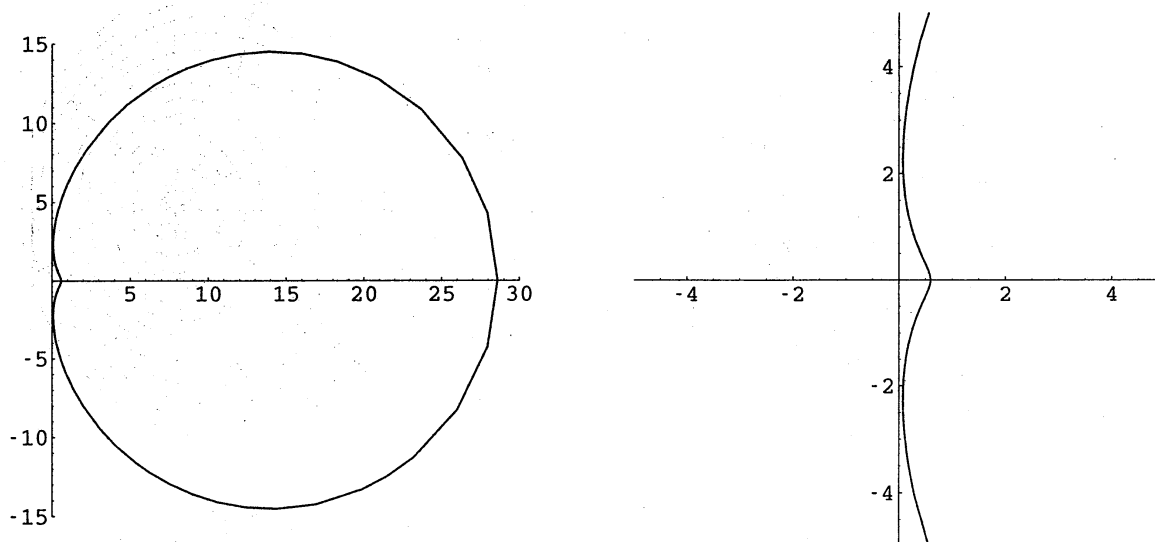


Fig 3.4: Image of $|z| = r$ by $1 + \frac{z f''(z)}{f'(z)}$ ($r = 0.95$).

Thus we give the following problem for convexity of the function $f(z)$ for $-1 < k < 1$.

Problem 1. Find the sharp bound r for the function $f(z)$ of Koebe type to be convex in $|z| < r$.

4 Definitions of $S_\alpha^*(\beta)$ and $K_\alpha(\beta)$

A function $f(z) \in A$ is said to be α -spiral like of order β if it satisfies

$$\operatorname{Re} \left(e^{-i\alpha} \frac{zf'(z)}{f(z)} \right) > \beta$$

for some α ($-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$), β ($0 \leq \beta < \cos \alpha \leq 1$) and all z in U . We denote by $S_\alpha^*(\beta)$ the subclass of A consisting of functions $f(z)$ which are α -spiral like of order β in U . A function $f(z)$ in A is said to be α -convex like of order β if it satisfies

$$\operatorname{Re} \left(e^{-i\alpha} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right) > \beta$$

for some α ($-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$), β ($0 \leq \beta < \cos \alpha \leq 1$) and all z in U . Also we denote by $K_\alpha(\beta)$ the subclass of A consisting of functions $f(z)$ which are α -convex like of order β in U . In particular, we denote by $S_0^*(0) \equiv S^*(0) \equiv S^*$ and $K_0(0) \equiv K(0) \equiv K$.

We can check that the function

$$f(z) = \frac{z}{(1-z)^{2e^{i\alpha}(\cos \alpha - \beta)}} \quad (4.1)$$

is the extremal function for the class $S_\alpha^*(\beta)$. Because, since

$$\frac{zf'(z)}{f(z)} = 1 + 2e^{i\alpha}(\cos \alpha - \beta) \frac{z}{1-z}$$

for the extremal function, we have

$$e^{-i\alpha} \frac{zf'(z)}{f(z)} = e^{-i\alpha} + 2(\cos \alpha - \beta) \frac{z}{1-z}.$$

Note that $w = \frac{z}{1-z}$ maps the unit disk U onto the half domain with $\operatorname{Re}(w) > -\frac{1}{2}$. Therefore, we see that

$$\operatorname{Re} \left(e^{-i\alpha} \frac{zf'(z)}{f(z)} \right) > \cos \alpha - (\cos \alpha - \beta) = \beta.$$

By definitions for the classes $S_\alpha^*(\beta)$ and $K_\alpha(\beta)$, since $f(z) \in K_\alpha(\beta)$ if and only if $zf'(z) \in S_\alpha^*(\beta)$, we calculate the extremal function $f(z)$ for the class $K_\alpha(\beta)$ given by

$$f(z) = \frac{1}{2e^{i\alpha}(\cos \alpha - \beta) - 1} \left(\frac{1}{(1-z)^{2e^{i\alpha}(\cos \alpha - \beta) - 1}} - 1 \right). \quad (4.2)$$

Let us give some examples of functions $f(z)$ in $S_\alpha^*(\beta)$ and $K_\alpha(\beta)$.

Example 3.

$$(1) f(z) = \frac{z}{(1-z)^{\frac{1+\sqrt{3}i}{2}}} \quad (\text{when } \alpha = \frac{\pi}{3}, \beta = 0 \text{ in (4.1)}),$$

$$(2) f(z) = \frac{2-\sqrt{3}-(2\sqrt{3}-1)i}{5-2\sqrt{3}} \left(\frac{1}{(1-z)^{\frac{2-\sqrt{3}+(2\sqrt{3}-1)i}{4}}} - 1 \right) \quad (\text{when } \alpha = \frac{\pi}{6}, \beta = \frac{1}{4} \text{ in (4.2)}).$$

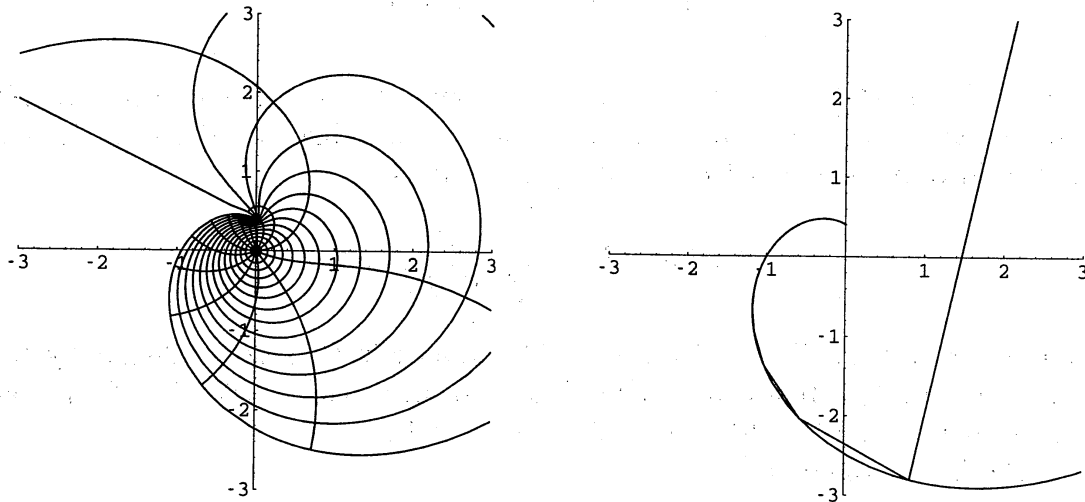


Fig 4.1: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{\frac{1+\sqrt{3}i}{2}}}$ ($0 \leq r \leq 0.95$ (left), $r = 1$ (right)).

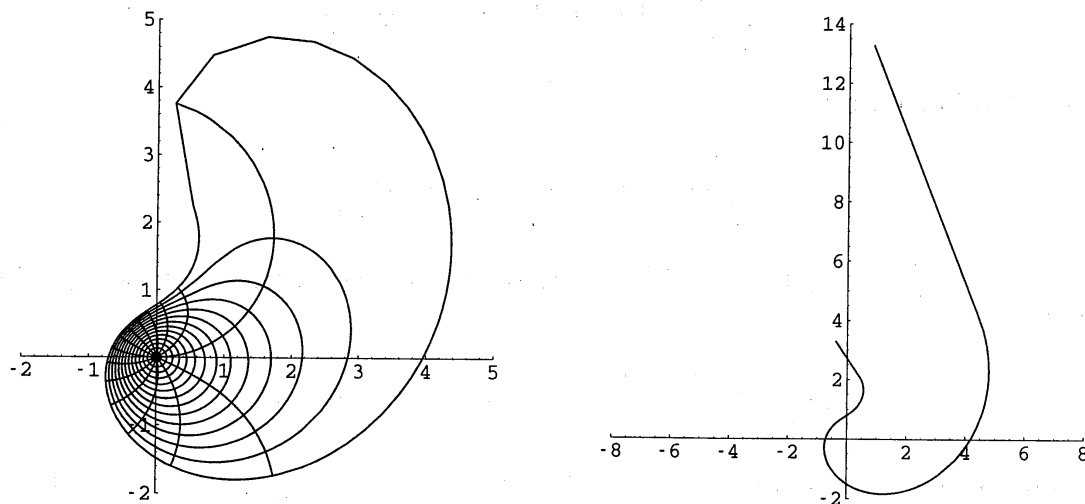


Fig 4.2: Image of $|z| = r$ by $f(z) = \frac{2-\sqrt{3}-(2\sqrt{3}-1)i}{5-2\sqrt{3}} \left(\frac{1}{(1-z)^{\frac{2-\sqrt{3}+(2\sqrt{3}-1)i}{4}}} - 1 \right)$ ($0 \leq r \leq 0.99$ (left), $r = 1$ (right)).

As we give extremal functions for the classes $S_\alpha^*(\beta)$ and $K_\alpha(\beta)$, our functions in Example 3 show that it is interesting for us to introduce the analytic function of generalized Koebe type by

$$f(z) = \frac{z}{(1-z)^{ke^{i\alpha}}}$$

for some $k \in \mathbb{R}$ and α ($0 \leq \alpha < 2\pi$).

If $k = 2(\cos\alpha - \beta)$, then $f(z)$ becomes the extremal function of the class $S_\alpha^*(\beta)$.

5 Radii for α -spiral likeness of order β

We discuss the radii of α -spiral like of order β for the function $f(z)$ of the generalized Koebe type.

Theorem 3. *The function $f(z)$ of the generalized Koebe type satisfies*

$$(1) \quad k > 2(\cos\alpha - \beta) \implies f(z) \in S_\alpha^*(\beta) \quad \text{for} \quad 0 \leq r < \frac{\cos\alpha - \beta}{k - (\cos\alpha - \beta)} \quad (|z| = r),$$

$$(2) \quad 0 \leq k \leq 2(\cos\alpha - \beta) \implies f(z) \in S_\alpha^*(\beta) \quad \text{for} \quad 0 \leq r < 1 \quad (|z| = r),$$

$$(3) \quad k < 0 \implies f(z) \in S_\alpha^*(\beta) \quad \text{for} \quad 0 \leq r < \frac{\cos\alpha - \beta}{\cos\alpha - \beta - k} \quad (|z| = r).$$

Proof. By a simple calculation, we have

$$\frac{zf'(z)}{f(z)} = 1 + k \frac{e^{i\alpha}z}{1-z},$$

which gives

$$e^{-i\alpha} \frac{zf'(z)}{f(z)} = e^{-i\alpha} + k \frac{z}{1-z}.$$

Letting $w = \frac{z}{1-z}$, we have $z = \frac{w}{w+1}$. Since $|z|^2 \leq |r|^2$ for $|z| \leq |r|$, we have

$$|z|^2 = \left| \frac{w}{w+1} \right|^2 \leq |r|^2.$$

After a simple calculation, we have

$$|w|^2 \leq |w+1|^2 r^2,$$

which implies

$$\left| w - \frac{r^2}{1-r^2} \right|^2 \leq \frac{r^2}{(1-r^2)^2}.$$

Hence we have

$$\left| w - \frac{r^2}{1-r^2} \right| \leq \frac{r}{1-r^2}. \quad (5.1)$$

Now, we can calculate the maximum and minimum values of $\operatorname{Re}(w)$ from (5.1) as follows:

$$\begin{aligned} \max \operatorname{Re}(w) &= \frac{r^2}{1-r^2} + \frac{r}{1-r^2} \\ &= \frac{r}{1-r} \end{aligned}$$

and

$$\begin{aligned} \min \operatorname{Re}(w) &= \frac{r^2}{1-r^2} - \frac{r}{1-r^2} \\ &= -\frac{r}{1+r}. \end{aligned}$$

For $k \geq 0$, to get the minimum value of $\operatorname{Re}(e^{-i\alpha} \frac{zf'(z)}{f(z)})$, we take the minimum value of $\operatorname{Re}(w)$. Then we see that

$$\begin{aligned} \operatorname{Re} \left(e^{-i\alpha} \frac{zf'(z)}{f(z)} \right) &= \cos\alpha + k \operatorname{Re}(w) \\ &\geq \cos\alpha - k \frac{r}{1+r} > \beta. \end{aligned}$$

Since

$$(\cos\alpha - \beta)(1+r) - kr > 0,$$

or

$$(\cos\alpha - \beta - k)r + \cos\alpha - \beta > 0,$$

r satisfies the following inequality

$$(k - (\cos\alpha - \beta))r - (\cos\alpha - \beta) < 0. \quad (5.2)$$

We see that if $k > \cos\alpha - \beta$, then

$$r < \frac{\cos\alpha - \beta}{k - (\cos\alpha - \beta)}$$

and if $k > 2(\cos\alpha - \beta)$, then

$$\frac{\cos\alpha - \beta}{k - (\cos\alpha - \beta)} < 1.$$

So, we derive the case (1) in Theorem 3.

If $0 \leq k < \cos\alpha - \beta$, then the inequality (5.2) is always satisfied for all r ($0 \leq r < 1$).

If $\cos\alpha \leq k \leq 2(\cos\alpha - \beta)$, then we have the next inequality

$$1 < \frac{\cos\alpha - \beta}{k - (\cos\alpha - \beta)}.$$

This gives us that $f(z) \in S_{\alpha}^*(\beta)$ for $0 \leq r < 1$. Hence we get the result of the case (2) in Theorem 3.

For $k < 0$, to get the minimum value of $\operatorname{Re}(e^{-i\alpha} \frac{zf'(z)}{f(z)})$, we take the maximum value of $\operatorname{Re}(w)$. In this case, we have

$$\begin{aligned} \operatorname{Re}\left(e^{-i\alpha} \frac{zf'(z)}{f(z)}\right) &= \cos\alpha + k \operatorname{Re}(w) \\ &\geq \cos\alpha + k \frac{r}{1-r} > \beta. \end{aligned}$$

Similary, for $k \geq 0$, since

$$(\cos\alpha - \beta)(1 - r) + kr > 0,$$

or

$$(k - (\cos\alpha - \beta))r + \cos\alpha - \beta > 0,$$

we have

$$r < \frac{\cos\alpha - \beta}{\cos\alpha - \beta - k}$$

for r satisfying

$$r < \frac{\cos\alpha - \beta}{\cos\alpha - \beta - k} < 1.$$

Thus we have

$$0 \leq r < \frac{\cos\alpha - \beta}{\cos\alpha - \beta - k},$$

which gives the result of the case (3) in Theorem 3. The proof of Theorem 3 is completed.

We give some examples of functions $f(z)$ in $S_{\alpha}^*(\beta)$ for Theorem 3.

Example 4.

$$(1) f(z) = \frac{z}{(1-z)^{3e^{i\frac{\pi}{3}}}} \in S_{\frac{\pi}{3}}^*\left(\frac{1}{3}\right) \quad \text{for } 0 \leq r < \frac{1}{17} = 0.0588235\dots,$$

$$(2) f(z) = \frac{z}{(1-z)^{\frac{1}{2}e^{i\frac{\pi}{4}}}} \in S_{\frac{\pi}{4}}^*\left(\frac{1}{5}\right) \quad \text{for } 0 \leq r < 1,$$

$$(3) f(z) = \frac{z}{(1-z)^{-7e^{i\frac{\pi}{6}}}} \in S_{\frac{\pi}{6}}^*\left(\frac{2}{5}\right) \quad \text{for } 0 \leq r < \frac{5\sqrt{3}-4}{5\sqrt{3}+66} = 0.0624195\dots$$

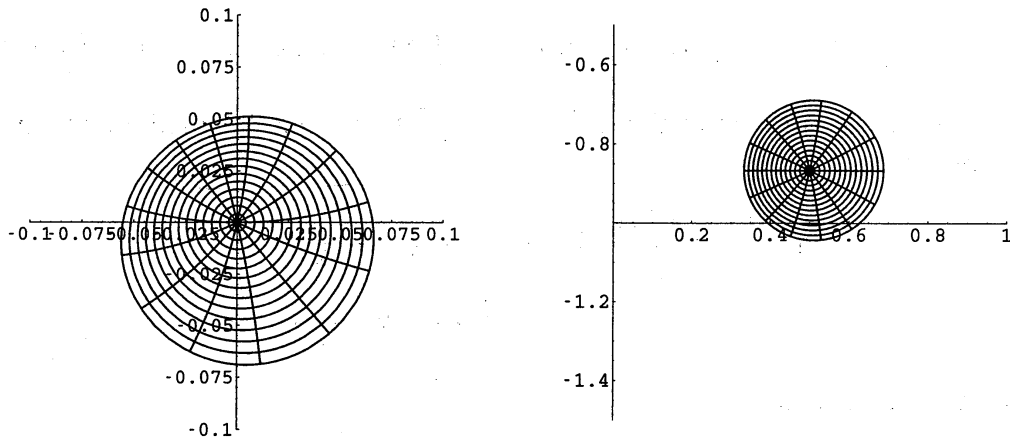


Fig 5.1: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{3e^{i\frac{\pi}{3}}}}$ (left), $e^{-i\frac{\pi}{3}} \frac{zf'(z)}{f(z)}$ (right)
(in all cases, $r = \frac{1}{17} = 0.0588235 \dots$).

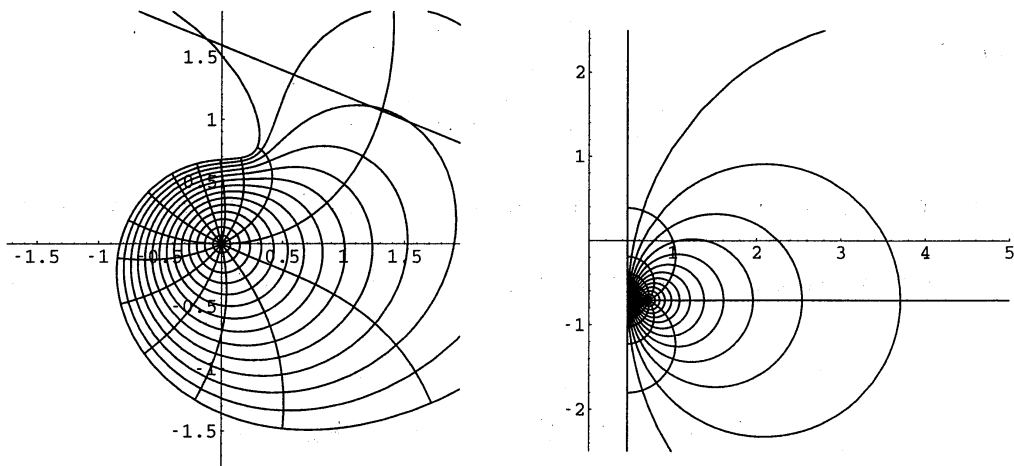


Fig 5.2: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{\frac{1}{2}e^{i\frac{\pi}{4}}}}$ (left), $e^{-i\frac{\pi}{4}} \frac{zf'(z)}{f(z)}$ (right) (in all cases, $r = 1$).

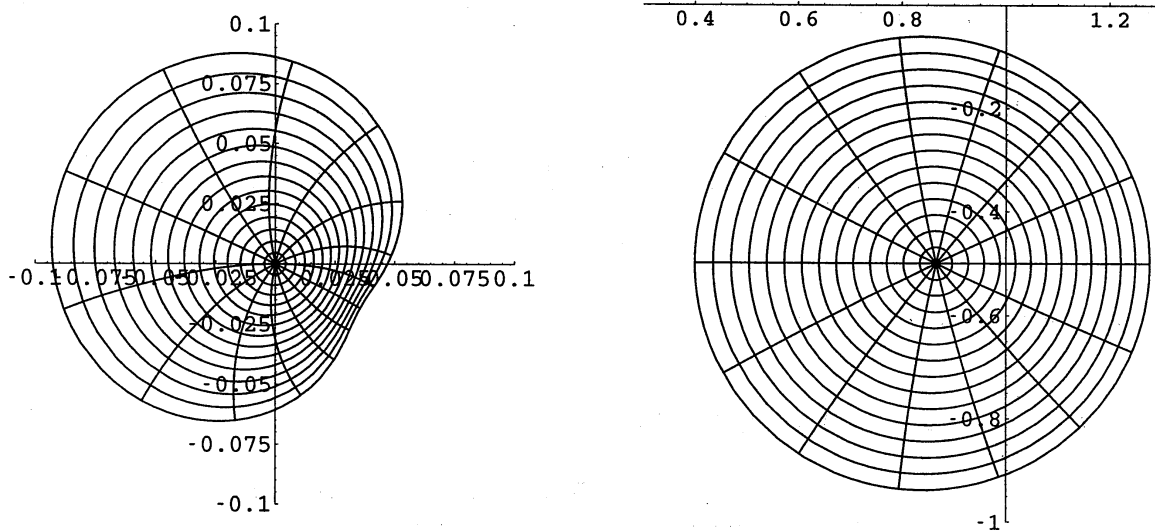


Fig 5.3: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{-7e^{i\frac{\pi}{6}}}}$ (left), $e^{-i\frac{\pi}{6}} \frac{zf'(z)}{f(z)}$ (right)
(in all cases, $r = \frac{5\sqrt{3}-4}{5\sqrt{3}+66} = 0.0624195 \dots$).

6 Radii for starlikeness of order β

Next we discuss the radii of starlikeness of order β for the function $f(z)$ of the generalized Koebe type.

Theorem 4. *The function $f(z)$ of the generalized Koebe type satisfies*

(1) *for $k \geq 0$ and $\cos \alpha \geq 0$,*

(i) $k \neq 0$, $\frac{1-\beta}{\cos \alpha} \implies f(z) \in S^*(\beta)$ for

$$0 \leq r < \frac{k - \sqrt{k^2 - 4k(1-\beta)\cos \alpha + 4(1-\beta)^2}}{2(k\cos \alpha + \beta - 1)} \quad (|z| = r),$$

(ii) $k = \frac{1-\beta}{\cos \alpha} \implies f(z) \in S^*(\beta)$ for $0 \leq r < \cos \alpha$ ($|z| = r$),

(iii) $k = 0 \implies f(z) \in S^*(\beta)$ for $0 \leq r < 1$ ($|z| = r$),

(2) *for $k \geq 0$ and $\cos \alpha < 0$,*

(i) $k \neq 0 \implies f(z) \in S^*(\beta)$ for

$$0 \leq r < \frac{k - \sqrt{k^2 - 4k(1-\beta)\cos \alpha + 4(1-\beta)^2}}{2(k\cos \alpha + \beta - 1)} \quad (|z| = r),$$

(3) for $k < 0$ and $\cos\alpha \geq 0$,

(i) $k < 0 \implies f(z) \in S^*(\beta)$ for

$$0 \leq r < \frac{k + \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(1-\beta - k\cos\alpha)} \quad (|z| = r),$$

(4) for $k < 0$ and $\cos\alpha < 0$,

(i) $k \neq \frac{1-\beta}{\cos\alpha} \implies f(z) \in S^*(\beta)$ for

$$0 \leq r < \frac{k + \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(1-\beta - k\cos\alpha)} \quad (|z| = r),$$

(ii) $k = \frac{1-\beta}{\cos\alpha} \implies f(z) \in S^*(\beta)$ for $0 \leq r < -\cos\alpha$ ($|z| = r$).

Proof. From Theorem 3, we have

$$\frac{zf'(z)}{f(z)} = 1 + k \frac{e^{i\alpha}z}{1-z},$$

Letting $w = \frac{e^{i\alpha}z}{1-z}$, that is, $z = \frac{w}{w + e^{i\alpha}}$, we have

$$|z|^2 = \left| \frac{w}{w + e^{i\alpha}} \right|^2 \leq |r|^2 \quad (|z| \leq r).$$

After calculations, we have

$$|w|^2 \leq |w + e^{i\alpha}|^2 r^2,$$

that is

$$\left| w - \frac{r^2}{1-r^2} e^{i\alpha} \right|^2 \leq \frac{r^2}{(1-r^2)^2}.$$

Hence, we have

$$\left| w - \frac{r^2}{1-r^2} e^{i\alpha} \right| \leq \frac{r}{1-r^2}. \quad (6.1)$$

Now, we have to calculate the maximum and minimum values of $\operatorname{Re}(w)$ from (6.1). Note that

$$\begin{aligned} \max \operatorname{Re}(w) &= \operatorname{Re} \left(\frac{r^2}{1-r^2} e^{i\alpha} \right) + \frac{r}{1-r^2} \\ &= \frac{r^2 \cos\alpha}{1-r^2} + \frac{r}{1-r^2} = \frac{r(r \cos\alpha + 1)}{1-r^2} \end{aligned}$$

and

$$\begin{aligned}\min \operatorname{Re}(w) &= \operatorname{Re} \left(\frac{r^2}{1-r^2} e^{i\alpha} \right) - \frac{r}{1-r^2} \\ &= \frac{r^2 \cos \alpha}{1-r^2} - \frac{r}{1-r^2} = \frac{r(r \cos \alpha - 1)}{1-r^2}.\end{aligned}$$

For $k \geq 0$ and $\cos \alpha \geq 0$, to get the minimum value of $\operatorname{Re}(\frac{zf'(z)}{f(z)})$, we take the minimum value of $\operatorname{Re}(w)$. Hence, we have

$$\begin{aligned}\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) &= 1 + k \operatorname{Re}(w) \\ &= 1 + kr \frac{r \cos \alpha - 1}{1-r^2} > \beta.\end{aligned}$$

By a simple calculation, we have

$$(1 - \beta)(1 - r^2) + kr(r \cos \alpha - 1) > 0,$$

or

$$(k \cos \alpha + \beta - 1)r^2 - kr + 1 - \beta > 0. \quad (6.2)$$

If $k \cos \alpha + \beta - 1 > 0$, we get, from (6.2),

$$r < \frac{k - \sqrt{k^2 - 4k(1 - \beta) \cos \alpha + 4(1 - \beta)^2}}{2(k \cos \alpha + \beta - 1)},$$

and

$$\frac{k + \sqrt{k^2 - 4k(1 - \beta) \cos \alpha + 4(1 - \beta)^2}}{2(k \cos \alpha + \beta - 1)} < r.$$

Since

$$\begin{aligned}&\sqrt{k^2 - 4k(1 - \beta) \cos \alpha + 4(1 - \beta)^2} \\ &= \sqrt{(k - 2(1 - \beta) \cos \alpha)^2 + 4(1 - \beta)^2 - 4(1 - \beta)^2 \cos^2 \alpha} \\ &= \sqrt{(k - 2(1 - \beta) \cos \alpha)^2 + 4(1 - \beta)^2 \sin^2 \alpha} \geq 0\end{aligned}$$

for all k and α , we consider the following inequality

$$\frac{k - \sqrt{k^2 - 4k(1 - \beta) \cos \alpha + 4(1 - \beta)^2}}{2(k \cos \alpha + \beta - 1)} > 0. \quad (6.3)$$

To be satisfied the inequality (6.3), the following inequality

$$k - \sqrt{k^2 - 4k(1 - \beta) \cos \alpha + 4(1 - \beta)^2} > 0,$$

should be satisfied. After calculations, we have

$$4(1 - \beta)(k \cos \alpha + \beta - 1) > 0.$$

The last inequality is always satisfied because $k\cos\alpha + \beta - 1 > 0$ in this case. Thus the inequality (6.3) is always satisfied. Therefore,

$$0 \leq r < \frac{k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} \quad (k > \frac{1 - \beta}{\cos\alpha}).$$

Finally, we calculate for k such that next inequality is satisfied

$$\frac{k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} < 1. \quad (6.4)$$

Since the inequality (6.4) implies that

$$k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2} < 2(k\cos\alpha + \beta - 1),$$

we see

$$4k(1 - \cos\alpha)(k\cos\alpha + \beta - 1) > 0.$$

The last inequality is always satisfied because $k\cos\alpha + \beta - 1 > 0$ in this case. Thus, the inequality (6.4) is always satisfied. Therefore, we derive

$$0 \leq r < \frac{k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} \quad (k > \frac{1 - \beta}{\cos\alpha}).$$

If $k\cos\alpha + \beta - 1 < 0$, we have from (6.2),

$$\frac{k + \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} < r < \frac{k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)}.$$

Since

$$\frac{k + \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} < 0$$

in this case, we have to have

$$\frac{k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} > 0.$$

This inequality shows that

$$4(1 - \beta)(k\cos\alpha + \beta - 1) < 0.$$

The last inequality is always satisfied in this case. Similary to the case $k\cos\alpha + \beta - 1 > 0$, we have to check that

$$\frac{k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} < 1,$$

which implies that

$$4k(1 - \cos\alpha)(k\cos\alpha + \beta - 1) < 0.$$

The last inequality is always satisfied in this case. Hence, we have

$$0 \leq r < \frac{k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} \quad (0 \leq k < \frac{1 - \beta}{\cos\alpha}).$$

Therefore, we derive the result of the case (i) of Theorem 4 - (1).

If $k\cos\alpha + \beta - 1 = 0$, we have, from (6.2),

$$-kr + 1 - \beta > 0,$$

or

$$r < \frac{1 - \beta}{k} = \cos\alpha.$$

We get the result of the case (ii) of Theorem 4 - (1).

If $k = 0$, we have from (6.2),

$$(\beta - 1)r + 1 - \beta > 0$$

which shows $r < 1$.

Therefore, we get the result of the case (iii) of Theorem 4 - (1). The proof of Theorem 4 - (1) is completed.

For $k \geq 0$ and $\cos\alpha < 0$, similary to the case (1), we derive the inequality (6.2). In this condition, $k\cos\alpha + \beta - 1$ is always non-positive. Noting that

$$0 \leq r < \frac{k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)},$$

we see

$$\frac{k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} < 1$$

if $k \neq 0$, and $0 \leq r < 1$ if $k = 0$. Thus we get the result of the case (2) of Theorem 4.

For $k < 0$ and $\cos\alpha \geq 0$, to get the minimum value of $\operatorname{Re}(\frac{zf'(z)}{f(z)})$, we need the maximum value of $\operatorname{Re}(w)$. Indeed, we have

$$\begin{aligned} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= 1 + k \operatorname{Re}(w) \\ &= 1 + kr \frac{r\cos\alpha + 1}{1 - r^2} > \beta. \end{aligned}$$

After calculations, we have

$$(1 - \beta)(1 - r^2) + kr(r\cos\alpha + 1) > 0,$$

that is,

$$(k\cos\alpha + \beta - 1)r^2 + kr + 1 - \beta > 0. \quad (6.5)$$

With this condition, similary to the case (2) of Theorem 4, $k\cos\alpha + \beta - 1$ is always positive. Solving (6.5) for r , we obtain

$$\frac{-k + \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} < r < \frac{-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)}.$$

We note that

$$\frac{-k + \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} < 0$$

and

$$\frac{-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} > 0.$$

This gives us that

$$4(1 - \beta)(k\cos\alpha + \beta - 1) < 0.$$

The last inequality is always satisfied in this condition. Finally, we calculate for k such that next inequality is satisfied

$$\frac{-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} < 1. \quad (6.6)$$

By (6.6), we have

$$-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2} > 2(k\cos\alpha + \beta - 1),$$

so

$$4k(1 + \cos\alpha)(k\cos\alpha + \beta - 1) > 0.$$

The last inequality is always satisfied in this condition. Thus, the inequality (6.6) is also always satisfied. Therefore, we derive

$$\begin{aligned} 0 \leq r &< \frac{-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} \\ &= \frac{k + \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(1 - \beta - k\cos\alpha)}, \end{aligned}$$

which is the result of the case (3) of Theorem 4.

For $k < 0$ and $\cos\alpha < 0$, we derive the inequality (6.5). If $k\cos\alpha + \beta - 1 > 0$, then we have, from (6.5),

$$r < \frac{-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)},$$

and

$$\frac{-k + \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} < r.$$

Therefore the following inequality

$$\frac{-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} > 0 \quad (6.7)$$

is satisfied. Since the inequality (6.7) implies

$$-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2} > 0,$$

we have

$$4(1 - \beta)(k\cos\alpha + \beta - 1) > 0.$$

This last inequality is always satisfied in this case. Then, we calculate for k such that the inequality

$$\frac{-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} < 1$$

is satisfied. Noting that

$$-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2} < 2(k\cos\alpha + \beta - 1),$$

we have

$$4k(1 + \cos\alpha)(k\cos\alpha + \beta - 1) < 0.$$

The last inequality is always satisfied in this case. Hence, we have

$$\begin{aligned} 0 \leq r &< \frac{-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} \\ &= \frac{k + \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(1 - \beta - k\cos\alpha)} \quad \left(k < \frac{1 - \beta}{\cos\alpha}\right). \end{aligned}$$

If $k\cos\alpha + \beta - 1 < 0$, we have, from (6.5),

$$\frac{-k + \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} < r < \frac{-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)}.$$

By the same manner as in the previous cases, we have

$$\frac{-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} > 0,$$

which implies

$$4(1 - \beta)(k\cos\alpha + \beta - 1) < 0.$$

The last inequality is always satisfied in this case. Thus, we have

$$\begin{aligned} 0 \leq r &< \frac{-k - \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(k\cos\alpha + \beta - 1)} \\ &= \frac{k + \sqrt{k^2 - 4k(1 - \beta)\cos\alpha + 4(1 - \beta)^2}}{2(1 - \beta - k\cos\alpha)} < 1 \quad \left(\frac{1 - \beta}{\cos\alpha} < k < 0\right). \end{aligned}$$

Therefore, we derive the result of the case (i) of Theorem 4 - (4).

If $k\cos\alpha + \beta - 1 = 0$, we have, from (6.5),

$$kr + 1 - \beta > 0,$$

or

$$r < -\frac{1 - \beta}{k} = -\cos\alpha.$$

We get the result of the case (ii) of Theorem 4 - (4). And the proof of Theorem 4 is completed.

We give some examples for Theorem 4 as follows.

Example 5.

$$(1) f(z) = \frac{z}{(1 - z)^{5e^{i\frac{\pi}{3}}}} \in S^*\left(\frac{1}{7}\right) \quad \text{for } 0 \leq r < \frac{35 - \sqrt{949}}{23} = 0.182355 \dots,$$

$$(2) f(z) = \frac{z}{(1 - z)^{4e^{i\frac{5}{8}\pi}}} \in S^*\left(\frac{1}{2}\right) \quad \text{for } 0 \leq r < \frac{-4 + \sqrt{17 + 4\sqrt{3}}}{1 + 4\sqrt{3}} = 0.112465 \dots,$$

$$(3) f(z) = \frac{z}{(1 - z)^{-6e^{i\frac{\pi}{4}}}} \in S^*\left(\frac{1}{3}\right) \quad \text{for } 0 \leq r < \frac{3(-6 + \sqrt{\frac{340}{9} + 8\sqrt{2}})}{4 + 18\sqrt{2}} = 0.102513 \dots,$$

$$(4) f(z) = \frac{z}{(1 - z)^{-\frac{2}{3}e^{i\frac{2}{3}\pi}}} \in S^*\left(\frac{3}{7}\right) \quad \text{for } 0 \leq r < \frac{-7 + \sqrt{109}}{5} = 0.688061 \dots,$$

$$(5) f(z) = \frac{z}{(1 - z)^{\frac{5}{2(1 + \sqrt{5})}e^{i\frac{\pi}{5}}}} \in S^*\left(\frac{3}{8}\right) \quad \text{for } 0 \leq r < \cos\frac{\pi}{5} = 0.809017 \dots,$$

$$(6) f(z) = \frac{z}{(1 - z)^{-\frac{5}{4}e^{i\frac{2}{3}\pi}}} \in S^*\left(\frac{3}{8}\right) \quad \text{for } 0 \leq r < -\cos\frac{2}{3}\pi = 0.5.$$

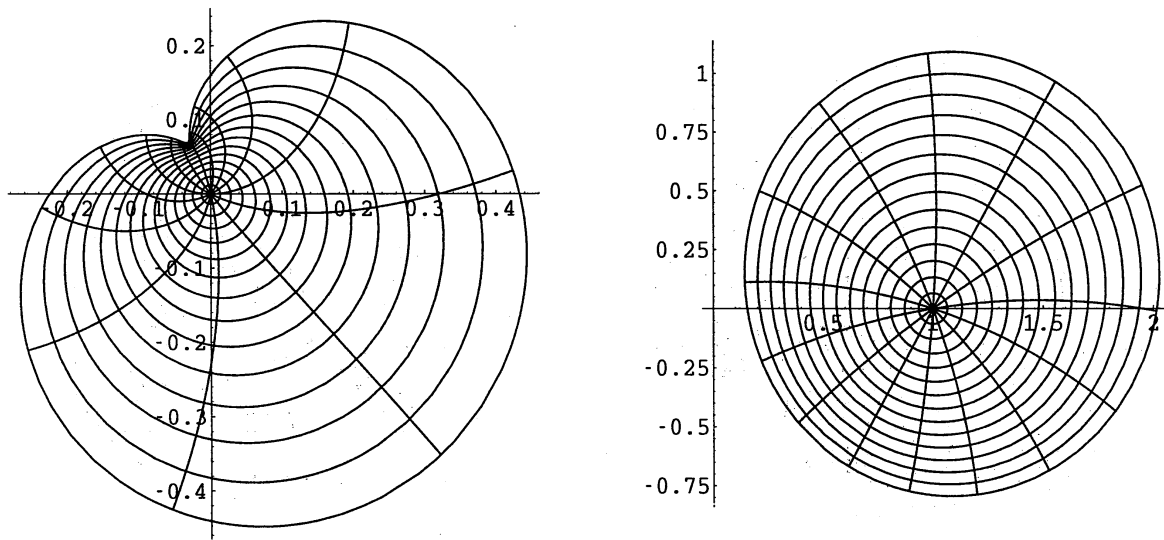


Fig 6.1: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{5e^{i\frac{\pi}{3}}}}$ (left), $\frac{zf'(z)}{f(z)}$ (right)
 (in all cases, $r = \frac{35-\sqrt{949}}{23} = 0.182355\dots$).

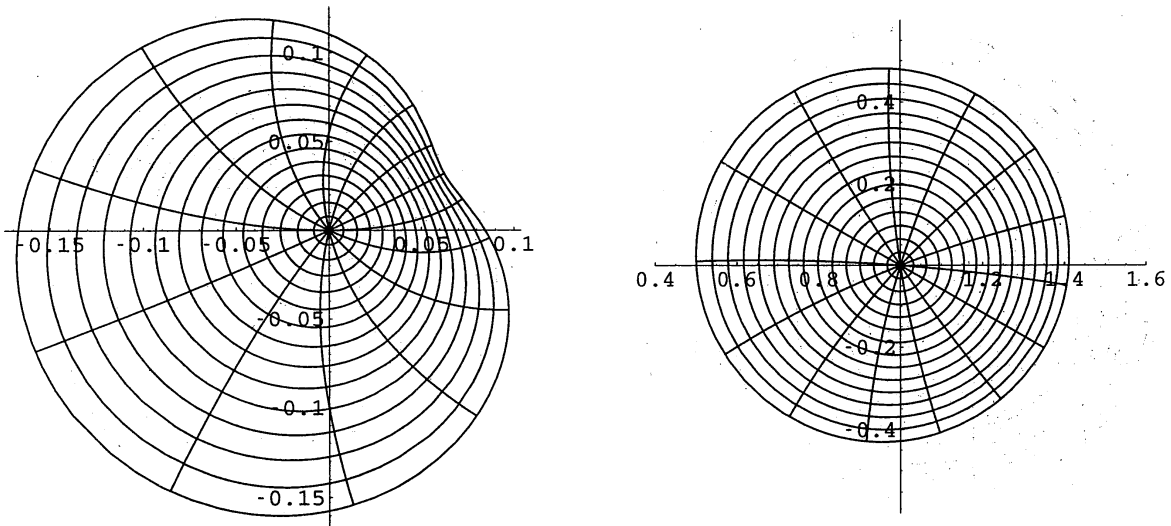


Fig 6.2: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{4e^{i\frac{5}{8}\pi}}}$ (left), $\frac{zf'(z)}{f(z)}$ (right)
 (in all cases, $r = \frac{-4+\sqrt{17+4\sqrt{3}}}{1+4\sqrt{3}} = 0.112465\dots$).

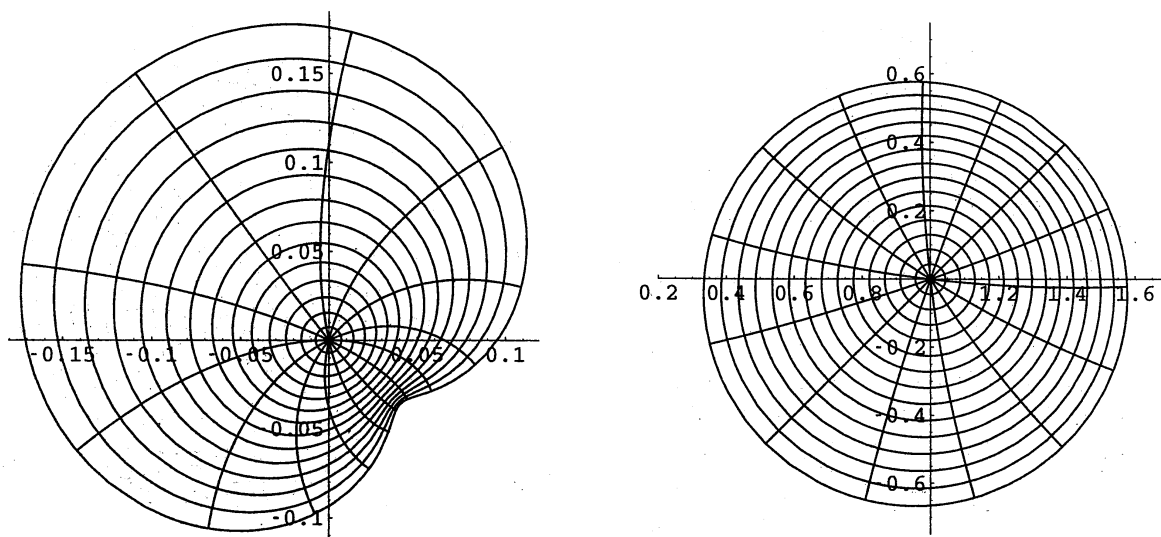


Fig 6.3: Image of $|z| = r$ by $f(z) = \frac{z}{(1 - z)^{-6e^{i\frac{\pi}{4}}}}$ (left), $\frac{zf'(z)}{f(z)}$ (right)

(in all cases, $r = \frac{3(-6 + \sqrt{\frac{340}{9} + 8\sqrt{2}})}{4 + 18\sqrt{2}} = 0.102513 \dots$).

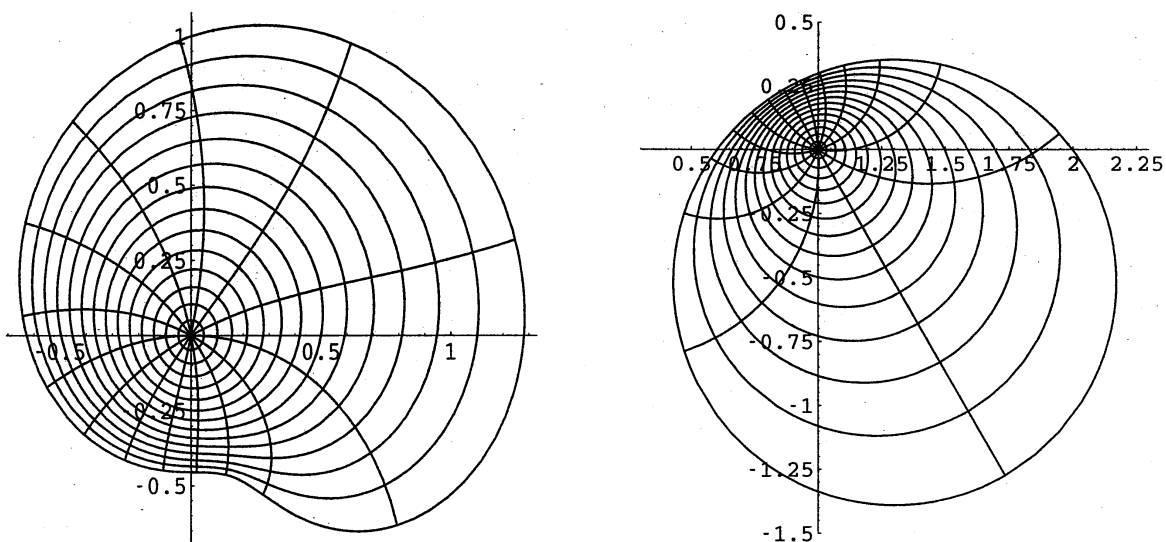


Fig 6.4: Image of $|z| = r$ by $f(z) = \frac{z}{(1 - z)^{-\frac{2}{3}e^{i\frac{2}{3}\pi}}}$ (left), $\frac{zf'(z)}{f(z)}$ (right)

(in all cases, $r = \frac{-7 + \sqrt{109}}{5} = 0.688061 \dots$).

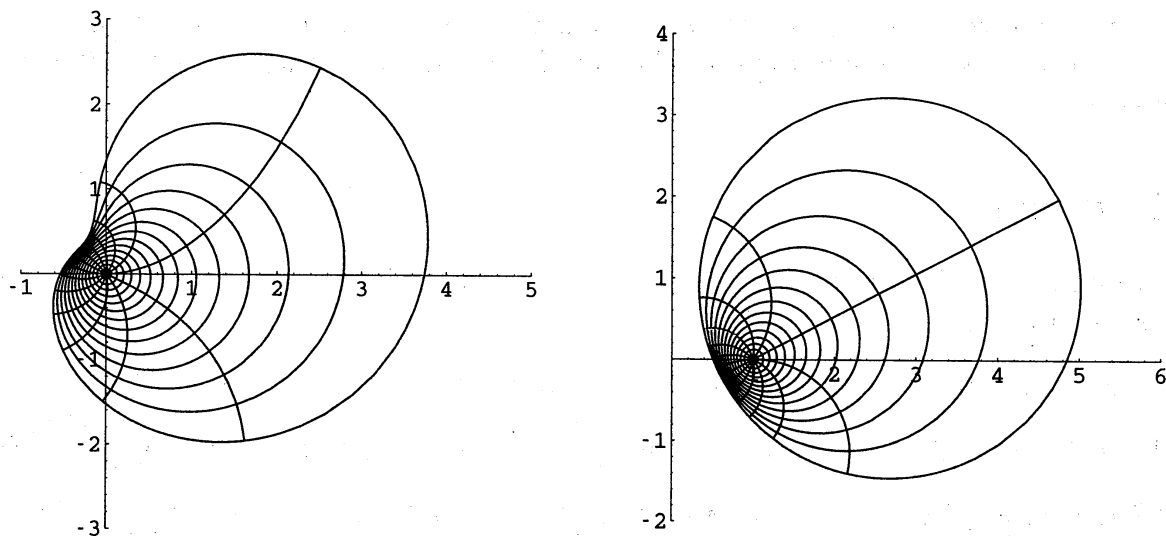


Fig 6.5: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{\frac{5}{2}} e^{i\frac{\pi}{5}}}$ (left), $\frac{zf'(z)}{f(z)}$ (right)
(in all cases, $r = \cos \frac{\pi}{5} = 0.809017 \dots$).

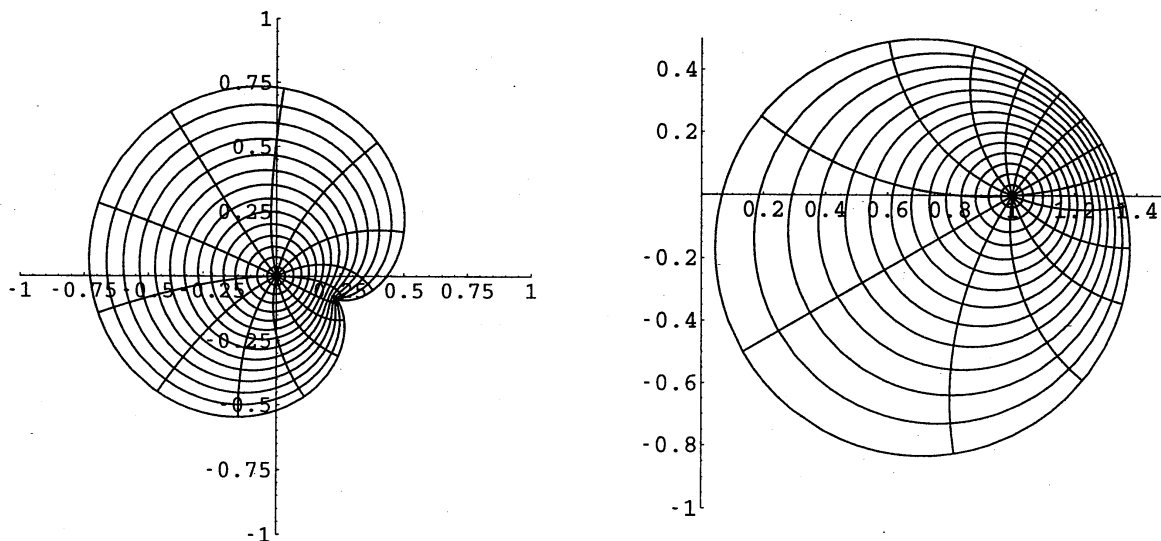


Fig 6.6: Image of $|z| = r$ by $f(z) = \frac{z}{(1-z)^{-\frac{5}{4}} e^{i\frac{2}{3}\pi}}$ (left), $\frac{zf'(z)}{f(z)}$ (right)
(in all cases, $r = -\cos \frac{2}{3}\pi = 0.5$).

Remark 1. Finally, we have to say that we can not find the sharp bound of the radius r for the classes $K_\alpha(\beta)$ and $K(\beta)$, because it is not so easy to calculate. But, as we mention before, the analytic function has the property that it maps, one-to-one, a small disk onto a small disk. Therefore, this problem to find the sharp bound of the radius r for the classes $K_\alpha(\beta)$ and $K(\beta)$ is remained.

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