

A MEAN VALUE INEQUALITY FOR PLURISUBHARMONIC FUNCTIONS ON A COMPACT KÄHLER MANIFOLD

XIAOHUA ZHU

1. Introduction.

Let (ω, M) be a compact Kähler manifold with positive first Chern class $c_1(M) > 0$, where ω is a Kähler form in $2\pi c_1(M)$. Let φ be a Kähler potential function and $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ be its corresponding Kähler form on M . Since the Ricci form of ω_φ represents the class $2\pi c_1(M)$, then for any nonnegative number $\lambda \leq 1$, there is a uniform smooth function $h_{\lambda,\varphi}$ such that

$$\begin{cases} \text{Ric}(\omega_\varphi) = \lambda\omega_\varphi + (1-\lambda)\omega + \sqrt{-1}\partial\bar{\partial}h_{\lambda,\varphi} \\ \int_M e^{h_{\lambda,\varphi}} \omega_\varphi^n = \int_M \omega^n = V. \end{cases} \quad (1.1)$$

When $\lambda = 1$, $h_{\lambda,\varphi} = h_\varphi$ is nothing, just is a Ricci potential function of ω_φ . For a smooth function h on M , we write

$$\text{Ric}^h(\omega_\varphi) = \text{Ric}(\omega_\varphi) - \sqrt{-1}\partial\bar{\partial}h$$

as a modified Ricci curvature with respect to h . Then (1.1) implies

$$\text{Ric}^h(\omega_\varphi) \geq \lambda\omega_\varphi.$$

In this note, we shall prove

Theorem. *Let M be a compact Kähler manifold with positive first Chern class $c_1(M) > 0$. Let ω and $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ be two Kähler forms in $2\pi c_1(M)$. Suppose that the function $h_{\lambda,\varphi}$ defined by (1.1) satisfies*

$$|h_{\lambda,\varphi}| \leq A \quad (1.2)$$

for some constant A . Then for any $\delta > 0$, there is a uniform constant C depending only on the numbers δ, A and the Kähler form ω such that

$$\text{osc}_M \varphi = \sup_M \varphi - \inf_M \varphi \leq C(1 + I(\varphi))^{n+\delta}, \quad (1.3)$$

where

$$I(\varphi) = \frac{1}{V} \int_M \varphi(\omega^n - \omega_\varphi^n).$$

We note that equation (1.1) is just one studied by S.T. Yau for the Calabi's problem when $\lambda = 0$ ([Ya]). He proved that the oscillation of φ is bounded by the metric ω and the C^0 -norm of function $h_{0,\varphi}$. We also note that for some class of functions $h_{\lambda,\varphi}$ inequality (1.3) can be improved as follow,

$$\text{osc}_M \varphi = \sup_M \varphi - \inf_M \varphi \leq C(1 + I(\varphi)), \quad (1.4)$$

where C is a uniform constant depending only on the number A and the Kähler form ω ([Ma], [CTZ]). In general we guess that (1.4) is also true. Inequality (1.4) can be regarded as a generalization of the mean value inequality on a compact Riemannian manifold with positive Ricci curvature ([CP], [BM]).

Some applications of inequality (1.3) have been found in the Kähler geometry, such as in the study of uniqueness of Kähler-Ricci solitons ([TZ1]), and in the proof of convergence of Kähler-Ricci flow ([TZ2]), etc. The proof of Theorem depends on a prior C^0 -estimate for plurisubharmonic functions by using the relative capacity theory which was first found by Klodziej ([Ko]). For a self-containing, we give a brief describing for the relative capacity in the next section.

2. Relative capacity and C^0 -estimate.

In this section, we will use the relative capacity theory for plurisubharmonic functions to derive a C^0 -estimate on certain Monge-Ampère equation.

First we recall some notations which can be found in [BT]. For any compact subset K of a strictly pseudoconvex domain Ω in $\mathbb{C}^n$, its relative capacity in Ω is defined as

$$\text{cap}(K, \Omega) = \sup \left\{ \int_K (\sqrt{-1} \partial \bar{\partial} u)^n \mid u \in \text{PSH}(\Omega), -1 \leq u < 0 \right\},$$

where $\text{PSH}(\Omega)$ denotes the space of plurisubharmonic functions (abbreviated as psh) in the weak sense. For any open set $U \subset \Omega$, we have

$$\text{cap}(U, \Omega) = \sup \{ \text{cap}(K, \Omega) \mid \text{for any compact } K \subset U \}.$$

The extremal function of K relative to Ω is defined by

$$u_K(z) = \sup \{ u(z) \mid u \in \text{PSH}(\Omega) \cap L^\infty(\Omega), u < 0 \text{ and } u|_K \leq -1 \}.$$

One can show that $u_K^*(z) = \overline{\lim_{z' \rightarrow z} u_K(z')}$ is a psh function. It is called the upper semicontinuous regularization of u_K . A compact set K is said to be regular if $u_K^* = u_K$. Here are some properties of u_K^* (cf. [BT], [AT]):

$$\begin{aligned} u_K^* &\in \text{PSH}(\Omega), \quad -1 \leq u_K^* \leq 0, \quad \lim_{z \rightarrow \partial\Omega} u_K^* = 0, \\ (\sqrt{-1} \partial \bar{\partial} u_K^*)^n &= 0 \quad \text{on } \Omega \setminus K, \\ u_K^* &= -1 \quad \text{on } K, \text{ except on a set of relative capacity zero,} \end{aligned}$$

moreover, we have

$$\text{cap}(K, \Omega) = \int_\Omega (\sqrt{-1} \partial \bar{\partial} u_K^*)^n = \int_K (\sqrt{-1} \partial \bar{\partial} u_K^*)^n. \quad (2.1)$$

Lemma 2.1. *Let Ω be a strictly pseudoconvex domain in C^n and $u < 0$ be a smooth solution of the following complex Monge-Ampère equation on Ω ,*

$$\det(u_{i\bar{j}}) = f.$$

Suppose that u and f satisfy:

$$\begin{aligned} u(p) &> c \quad (p \in \Omega) \quad \text{and} \\ \int_K f dv &\leq B \operatorname{cap}(K, \Omega) \frac{\operatorname{cap}(K, \Omega)^{\frac{1}{n}}}{1 + \operatorname{cap}(K, \Omega)^{\frac{1}{n}}} \end{aligned} \quad (2.2)$$

for any compact subset K of Ω , where B is a uniform constant. If the sets

$$U(s) = \{z | u(z) < s\} \cap \Omega''$$

are non-empty and relatively compact in $\Omega'' \subset \Omega' \subset \subset \Omega$ for any $s \in [S, S + D]$, where S is some number, then there is a uniform constant C , which depends only on $c, D, \delta, \Omega', \Omega$, such that

$$-\inf_{\Omega''} u \leq CB^\delta + D. \quad (2.3)$$

Proof. This lemma is essentially due to [Ko]. For readers' convenience, we will include a proof using an argument from [TZ1]. Put

$$a(s) = \operatorname{cap}(U(s), \Omega) \quad \text{and} \quad b(s) = \int_{U(s)} (\sqrt{-1} \partial \bar{\partial} u)^n.$$

Then we define an increasing sequence $s_0, s_1, \dots, s_N$ by setting $s_0 = S$ and

$$s_j = \sup\{s | a(s) \leq \lim_{t \rightarrow s_{j-1}^+} e a(t)\}$$

for $j = 1, \dots, N$, where N is chosen to be the greatest integer such that $s_N \leq S + D$. By using an argument in Lemma 4.1 of [TZ1], we can prove

$$S + D - s_N \leq (Be)^{\frac{1}{n}} a(S + D)^{\frac{1}{n\delta}}. \quad (2.4)$$

and

$$s_N - S \leq 2(Be)^{\frac{1}{n}} (1 + n\delta) a(S + D)^{\frac{1}{n\delta}}. \quad (2.5)$$

However, it was proved in [AT] (or Theorem 1.2.11 in [Ko]) that

$$\operatorname{cap}(\{u < s\} \cap \Omega', \Omega) \leq \frac{c'}{|s|},$$

where c' depends only on c and Ω' . It implies that

$$a(S + D) \leq \frac{c'}{-D - S}. \quad (2.6)$$

Combining (2.4)-(2.6), we get

$$D \leq 2(2 + n\delta)(Be)^{\frac{1}{n}} \left(\frac{c'}{-D - S} \right)^{\frac{1}{n\delta}}.$$

It follows

$$-S \leq c' \left(\frac{2(2 + n\delta)}{D} \right)^{n\delta} e^\delta B^\delta + D,$$

consequently, we have

$$-\inf_{\Omega''} u \leq c' \left(\frac{2(1 + n\delta)}{D} \right)^{n\delta} e^\delta B^\delta + D,$$

so (2.3) is proved. $\square$

Lemma 2.2. *Let Ω be a strictly pseudoconvex domain in C^n and $u < 0$ be a smooth solution of the following complex Monge-Ampère equation on Ω ,*

$$\det(u_{i\bar{j}}) = f.$$

Suppose that u satisfies

$$u(p) > c \quad (p \in \Omega).$$

Define $U(s)$ as in last lemma. If $U(s)$ are non-empty and relatively compact in Ω'' for any $s \in [S, S + D]$ for some S , then for any positive $\delta \leq \delta_0$ and $\epsilon \leq \epsilon_0$, there is a uniform constant $C = C(c, D, \delta_0, \epsilon_0, \Omega', \Omega)$ such that

$$-\inf_{\Omega''} u \leq C \left(\frac{1}{\delta\epsilon} \right)^{n+\delta} \|f\|_{L^{1+\epsilon}(\Omega)}^\delta + D.$$

Proof. Let u_K be the relative extremal function of a regular set K with respect to Ω and $v = \text{cap}^{-\frac{1}{n}}(K, \Omega)u_K$. Then v is a psh function and satisfies

$$\int_{\Omega} (\sqrt{-1} \partial \bar{\partial} v)^n = 1, \quad \text{and} \quad \lim_{z \rightarrow \partial\Omega} v = 0.$$

By Lemma 2.5.1 in [Ko], we have

$$\lambda(U'(s)) \leq c' \exp\{-2\pi|s|\}$$

for some uniform constant c' independent of v , where $\lambda(U'(s))$ is the Lebesgue measure of $U'(s) = \{v < s\}$. It follows that for any $q \geq 1$,

$$\begin{aligned} \int_{\Omega} |v|^q d\mu &\leq |\Omega| + \sum_{i=1}^{\infty} \int_{-s-1 \leq v \leq -s} |v|^q d\mu \\ &\leq |\Omega| + c' \sum_{i=1}^{\infty} (s+1)^q e^{-2\pi s} \\ &\leq |\Omega| + c' e^{4\pi} \int_2^{+\infty} s^q e^{-2\pi s} ds \\ &\leq C_1 2^{q+2} ([q] + 2)! \leq C_1 2^{q+2} (q+2)^{q+2}. \end{aligned} \tag{2.7}$$

On the other hand, for any $\epsilon > 0$ we have

$$\begin{aligned}
& \text{cap}(K, \Omega)^{-1} (1 + \text{cap}^{-1/\delta}(K, \Omega)) \int_K f d\mu \\
& \leq \int_K |v|^n (1 + |v|^{\frac{n}{\delta}}) f d\mu \\
& \leq \int_{\Omega} (|v|^n + |v|^{n(1+\frac{1}{\delta})}) f d\mu \\
& \leq [(\int_{\Omega} |v|^{\frac{n(1+\epsilon)}{\delta}} d\mu)^{\frac{\delta}{1+\epsilon}} + (\int_{\Omega} |v|^{\frac{n(1+\delta)(1+\epsilon)}{\delta}} d\mu)^{\frac{\delta}{1+\epsilon}}] \|f\|_{L^{1+\epsilon}(\Omega)}.
\end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we get

$$\int_{\Omega} f d\mu \leq B \text{cap}(K, \Omega) \frac{\text{cap}(K, \Omega)^{\frac{1}{\delta}}}{1 + \text{cap}(K, \Omega)^{\frac{1}{\delta}}},$$

where

$$\begin{aligned}
B &= 2C_1 2^{\frac{n(1+\delta)}{\delta}+2} \left(\frac{n(1+\delta)(1+\epsilon)}{\delta\epsilon} + 2 \right)^{\frac{n(1+\delta)}{\delta}+2} \|f\|_{L^{1+\epsilon}(\Omega)} \\
&\leq C_2 \left(\frac{C_0}{\delta\epsilon} \right)^{\frac{n(1+\delta)}{\delta}+2} \|f\|_{L^{1+\epsilon}(\Omega)}.
\end{aligned}$$

Therefore, it follows from Lemma 2.1 that

$$-\inf_{\Omega''} u \leq C \left(\frac{C_0}{\delta\epsilon} \right)^{n+3n\delta} \|f\|_{L^{1+\epsilon}(\Omega)}^{\delta} + D.$$

Now the lemma follows from replacing $3n\delta$ by δ . $\square$

Proposition 2.1. *Let (M, g) be a compact Kähler manifold and φ be a smooth solution of the following complex Monge-Ampère equation on M ,*

$$\begin{cases} \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}}) f, \\ \sup_M \varphi = 0. \end{cases}$$

Then, for any positive $\delta \leq \delta_0$ and $\epsilon \leq \epsilon_0$, there are two uniform constants C, C' which depending only on g, δ_0, ϵ_0 such that

$$-\inf_M \varphi \leq C \left(\frac{1}{\delta\epsilon} \right)^{n+\delta} \|f\|_{L^{1+\epsilon}(M)}^{\delta} + C'.$$

Proof. This is a direct corollary of Lemma 2.2 (cf. the proof of Proposition 4.1 in [TZ1]). We omit its proof. $\square$

3. Proof of the theorem.

In this section, we use Proposition 2.1 in Section 2 to prove the theorem in Introduction. Note that by using the maximal principle one can reduce (1.1) to a complex Monge-Ampère equation,

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}}) e^{-\lambda\varphi + h_0 - h_{\lambda, \varphi}}, \tag{3.1}$$

where h_0 is a potential function of the Ricci curvature of the metric $\omega_g = \omega$.

Proposition 3.1. *Let φ be a solution of (3.1). Suppose that*

$$|h_{\lambda, \varphi}| \leq A$$

for some constant A . Then for any positive $\delta \leq 1$, there are two uniform constants $C = C(A, g, n, \delta)$ and $C' = C'(A, g, n, \delta)$ such that

$$\text{osc}_M \varphi = \sup_M \varphi - \inf_M \varphi \leq C(I(\varphi))^{n+\delta} + C'.$$

Proposition 3.1 is analogous to Proposition 4.2 in [TZ1] for the complex Monge-Ampère equation which arises from the equation for Kähler-Ricci solitons. We need two lemmas in order to prove Proposition 3.1.

Lemma 3.1 (Poincaré-type inequality). *Let (M, g) be a compact Kähler manifold and h be a smooth function on M . Suppose that the modified Ricci curvature $\text{Ric}^h(\omega_g)$ of ω_g satisfies*

$$\text{Ric}^h(\omega_g) \geq \lambda \omega_g$$

for some number $\lambda > 0$. Let $C^\infty(M, \mathbb{C})$ be the space of complex-valued smooth functions. Then for any $\psi \in C^\infty(M, \mathbb{C})$, we have

$$\int_M |\bar{\partial} \tilde{\psi}|^2 e^h \omega_g^n \geq \int_M |\tilde{\psi}|^2 e^h \omega_g^n, \quad (3.2)$$

where

$$\tilde{\psi} = \psi - \frac{1}{V} \int_M \psi e^h \omega_g^n,$$

and $V = \int_M \omega_g^n$. In particular, for any $\varphi \in C^\infty(M)$, we have

$$\int_M |\bar{\partial} \varphi|^2 e^h \omega_g^n \geq \int_M \varphi^2 e^h \omega_g^n - \frac{1}{V} \left(\int_M \varphi e^h \omega_g^n \right)^2. \quad (3.3)$$

Proof. Let L be the linear differential operator on $C^\infty(M, \mathbb{C})$ defined by

$$L\psi = \Delta\psi + \langle \bar{\partial}h, \bar{\partial}\bar{\psi} \rangle, \quad \text{for } \psi \in C^\infty(M, \mathbb{C}),$$

where Δ denotes the Laplacian operator of g . Then L is elliptic and self-adjoint with respect to the following Hermitian inner product:

$$(\psi, \psi')_h = \int_M \psi \bar{\psi}' e^h \omega_g^n, \quad \text{for } \psi, \psi' \in C^\infty(M, \mathbb{C}),$$

namely,

$$(L\psi, \psi')_h = (\psi, L\psi')_h.$$

It follows that all eigenvalues of L are real. Denote by $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_i \leq \dots$ the sequence of eigenvalues of L and by ψ_i ($i = 0, 1, 2, \dots$) the corresponding sequence of

eigenfunctions with the property: $(\psi_i, \psi_j)_h = \delta_{ij}$, for any i, j . Note that ψ_0 is constant. Then $\{\psi_i\}$ is a complete orthonormal basis of the space $W^{1,2}(M, \mathbb{C})$ with respect to the weighted L^2 -norm $(\cdot, \cdot)_h$.

Let ψ be one of eigenfunctions of λ_1 , i.e.,

$$\Delta\psi + \langle \bar{\partial}h, \bar{\partial}\bar{\psi} \rangle = -\lambda_1\psi.$$

Then integrating by parts and using

$$\begin{aligned} & \lambda_1 \int_M \psi_i \bar{\psi}_i e^{h\omega_g^n} \\ &= - \int_M (\Delta\psi + \langle \bar{\partial}h, \bar{\partial}\bar{\psi} \rangle + \lambda_1\psi)_i \bar{\psi}_i e^{h\omega_g^n} \\ &= - \int_M (\psi_{j\bar{j}i} \bar{\psi}_i + \lambda_1 \psi_i \bar{\psi}_i) e^{h\omega_g^n} - \int_M (h_{j\bar{i}} \psi_j \bar{\psi}_i + h_{j\bar{j}} \psi_{ij} \bar{\psi}_i) e^{h\omega_g^n} \\ &= \int_M (R_{i\bar{j}} - \lambda \delta_{i\bar{j}} - h_{i\bar{j}}) \bar{\psi}_i \psi_j e^{h\omega_g^n} + \int_M \psi_{ij} \bar{\psi}_{ij} e^{h\omega_g^n}. \end{aligned}$$

Thus we prove that $\lambda_1 \geq \lambda$. So (3.2) holds, so does (3.3). $\square$

Lemma 3.2. *Let (ω, M) be a compact Kähler manifold and h be a smooth function on M . Let*

$$\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$$

be a Kähler form associated to a Kähler potential function φ so that the modified Ricci curvature $\text{Ric}^h(\omega_\varphi)$ of ω_φ satisfies

$$\text{Ric}^h(\omega_\varphi) \geq \lambda\omega_\varphi$$

for some constant $\lambda > 0$. Then there are two uniformly $c_0, C > 0$ depending only $\|h\|_{C^0(M)}$ and the metric ω such that

$$\int_M \exp\left\{-\frac{c_0\lambda}{I(\varphi)}(\varphi - \sup_M \varphi)\right\} \omega_\varphi^n \leq C.$$

Proof. As in [TZ1], we will use an iteration argument to prove this lemma. Without loss of generality, we may assume $I(\varphi) > 1$.

Let $\bar{\varphi} = \varphi - \sup_M \varphi$. Then for any $p > 0$, we have

$$\begin{aligned} & \int_M (-\bar{\varphi})^p (\omega_\varphi^n - \omega_\varphi^{n-1} \wedge \omega) \\ &= \frac{\sqrt{-1}}{2\pi} \int_M (-\bar{\varphi})^p \partial\bar{\partial}(\bar{\varphi}) \wedge \omega_\varphi^{n-1} \\ &= p \frac{\sqrt{-1}}{2\pi} \int_M (-\bar{\varphi})^{p-1} (-\partial\bar{\varphi}) \wedge (-\bar{\partial}\bar{\varphi}) \wedge \omega_\varphi^{n-1} \\ &= \frac{4p}{n(p+1)^2} \int_M |\bar{\partial}(-\bar{\varphi})|^{\frac{p+1}{2}}|^2 \omega_\varphi^n. \end{aligned}$$

It follows

$$\int_M |\bar{\partial}(-\bar{\varphi})^{\frac{p+1}{2}}|^2 \omega_\varphi^n \leq \frac{n(p+1)^2}{4p} \int_M (-\bar{\varphi})^p \omega_\varphi^n.$$

Applying Lemma 3.1 to function $(-\bar{\varphi})^{\frac{p+1}{2}}$ in the case of the metric ω_φ , we have

$$\int_M |\bar{\partial}(-\bar{\varphi})^{\frac{p+1}{2}}|^2 e^h \omega_\varphi^n \geq \lambda \int_M (-\bar{\varphi})^{p+1} e^h \omega_\varphi^n - \frac{1}{V} \left(\int_M (-\bar{\varphi})^{(p+1)/2} e^h \omega_\varphi^n \right)^2.$$

Thus by using the Hölder inequality, we get

$$\begin{aligned} & \int_M (-\bar{\varphi})^{p+1} e^h \omega_\varphi^n \\ & \leq \frac{c}{\lambda} p \int_M (-\bar{\varphi})^p e^h \omega_\varphi^n \\ & \quad + \frac{1}{\lambda V} \int_M (-\bar{\varphi})^p e^h \omega_\varphi^n \cdot \int_M (-\bar{\varphi}) e^h \omega_\varphi^n, \end{aligned}$$

and consequently

$$\int_M (-\bar{\varphi})^{p+1} \omega_\varphi^n \leq \frac{c'}{\lambda} [p \int_M (-\bar{\varphi})^p \omega_\varphi^n + \frac{1}{V} \int_M (-\bar{\varphi})^p \omega_\varphi^n \cdot \int_M (-\bar{\varphi}) \omega_\varphi^n], \quad (3.4)$$

where c, c' are uniform constants.

By the mean-value inequality, we have

$$\sup_M \varphi \leq V^{-1} \int_M \varphi \omega^n + C.$$

It follows

$$\begin{aligned} \int_M (-\bar{\varphi}) \omega_\varphi^n &= V \sup_M \varphi + \int_M (-\varphi) \omega_\varphi^n \\ &\leq \int_M \varphi (\omega_g^n - \omega_\varphi^n) + C V \\ &\leq a I(\varphi), \end{aligned}$$

where a is a uniform constant. Thus inserting this inequality into (3.4), we get

$$\int_M (-\bar{\varphi})^{p+1} \omega_\varphi^n \leq \frac{ac'}{\lambda} (p + I(\varphi)) \int_M (-\bar{\varphi})^p \omega_\varphi^n. \quad (3.5)$$

Iterating (3.5), we have

$$\begin{aligned} & \int_M (-\bar{\varphi})^{p+1} \omega_\varphi^n \\ & \leq 2 \left(\frac{ac'}{\lambda} I(\varphi) \right)^p (p+1)! \int_M (-\bar{\varphi}) \omega_\varphi^n \leq \left(\frac{ac'}{\lambda I(\varphi)} \right)^{p+1} (p+1)!. \end{aligned}$$

Now choosing $\varepsilon < \frac{\lambda}{ac'I(\varphi)}$, we obtain

$$\begin{aligned} & \int_M \exp\{-\varepsilon\bar{\varphi}\}\omega_\varphi^n \\ &= \sum_{p=0}^{+\infty} \frac{\varepsilon^p}{p!} \int_M (-\bar{\varphi})^p \omega_\varphi^n \\ &\leq \sum_{p=0}^{+\infty} \left(\frac{\varepsilon ac'}{\lambda} I(\varphi)\right)^p \\ &\leq \frac{1}{1 - \frac{ac'\varepsilon}{\lambda} I(\varphi)}. \end{aligned}$$

Put $c_0 = \frac{1}{ac'}$. Then (3.1) is proved. $\square$

Proof of Proposition 3.1. By [Ya], we may assume $\lambda > 0$. Let $\tilde{\varphi} = \varphi - \sup_M \varphi$. Then (3.1) becomes

$$\begin{cases} \det(g_{i\bar{j}} + \tilde{\varphi}_{i\bar{j}}) = \det(g_{i\bar{j}})f, \\ \sup_M \tilde{\varphi} = 0, \end{cases} \quad (3.6)$$

where $f = e^{h-h_{\lambda,\varphi}-\lambda\varphi}$. Since $h_{\lambda,\varphi}$ is uniformly bounded, we have

$$0 < c_1 \leq \int_M e^{-\lambda\varphi} \omega_g^n \leq c_2 \quad (3.7)$$

for some uniform constants c_1 and c_2 . This implies

$$\sup_M (\lambda\varphi) \geq -C \text{ and } \inf_M (\lambda\varphi) \leq C. \quad (3.8)$$

By (3.8) and Lemma 3.2, we have

$$\begin{aligned} & \int_M \exp\left\{-\left(1 + \frac{c_0}{I(\varphi)}\right)\lambda\varphi\right\} \omega_g^n \\ &\leq e^{c_0 C} \int_M \exp\left\{-\frac{c_0}{I(\varphi)}(\lambda\varphi - \sup_M \lambda\varphi) - \lambda\varphi\right\} \omega_g^n \\ &= e^{c_0 C} \int_M \exp\left\{-\frac{\lambda c_0}{I(\varphi)}(\varphi - \sup_M \varphi) - \lambda\varphi\right\} \omega_g^n \\ &\leq C_1 \int_M \exp\left\{-\frac{\lambda c_0}{I(\varphi)}(\varphi - \sup_M \varphi)\right\} \omega_\varphi^n \leq C_2. \end{aligned}$$

It follows

$$\|f\|_{L^{1+\frac{c_0}{I(\varphi)}}(M)} \leq C_3.$$

Thus, applying Proposition 2.1 to equation (3.6), we see that for any $\delta > 0$ there are uniform constants C_4 and C_5 only depending on δ such that

$$\sup_M \varphi - \inf_M \varphi = -\inf_M \tilde{\varphi} \leq C_4 I(\varphi)^{n+\delta} + C_5.$$

$\square$

The theorem follows from Proposition 3.1.

REFERENCES

- [AT] Alexander, H.J. and Taylor, B.A., Comparison of two capacities in $\mathbb{C}^n$, Math. Z., 186 (1984), 407-417.
- [BM] Bando, S. and Mabuchi, T., Uniqueness of Kähler-Einstein metrics modula connected group actions, Algebraic Geometry, Adv. Studies in Pure Math., Sendai, Japan, 10 (1987), 11-40.
- [BT] Bedford, E. and Taylor, B.A., A new capacity for plursubharmonic functions, Acta Math., 149 (1982), 1-40.
- [CP] Cheng, S. Y. and Peter, L., Heat kernal estimates and lower bound of eigenvalues, Comm. Math. Helv., 56 (1981), 327-338.
- [CTZ] Cao, H.T, Tian, G. and Zhu, X.H., Kähler-Ricci solitons on compact complex manifolds with $c_1(M) > 0$, Geom. Funct. Anal., 15 (2005), 697-619.
- [Ko] Kolodziej, S., The complex Monge-Ampère equation, Acta Math., 180 (1998), 69-117.
- [Ma] Mabuchi, T., Heat kernel estimates and the Green functions on the multiplier Hermitian manifolds, Tohoku Math. J., 54 (2002), 259-275.
- [TZ1] Tian, G. and Zhu, X.H., Uniqueness of Kähler-Ricci solitons, Acta Math., 184 (2000), 271-305.
- [TZ2] Tian, G. and Zhu, X.H., Convergence of Kähler-Ricci folw, Preprint, 2005.
- [Y] Yau, S.T., On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure Appl. Math., 31 (1978), 339-411.

DEPARTMENT OF MATHEMATICS, PEKING UNIVERSITY, BEIJING, 100871, CHINA

E-mail address: xhzhzhu@math.pku.edu.cn