

Generalized Fractional Calculus of the H -Function

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Abstract

The paper is devoted to study the generalized fractional calculus of arbitrary complex order for the H -function defined by the Mellin-Barnes integral

$$H_{p,q}^{m,n}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds,$$

where the function $\mathcal{H}_{p,q}^{m,n}(s)$ is a certain ratio of products of Gamma functions with the argument s and the contour $\mathcal{L}$ is specially chosen. The considered generalized fractional integration and differentiation operators contain the Gauss hypergeometric function as a kernel and generalize classical fractional integrals and derivatives of Riemann-Liouville, Erdélyi-Kober type, etc. It is proved that the generalized fractional integrals and derivatives of H -functions are also H -functions but of greater order. In particular, the obtained results define more precisely and generalize known results.

1. Introduction

This paper deals with the H -function $H_{p,q}^{m,n}(z)$. For integers m, n, p, q such that $0 \leq m \leq q$, $0 \leq n \leq p$, for $a_i, b_j \in \mathbb{C}$ with $\mathbb{C}$ of the field of complex numbers and for $\alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty)$ ($i = 1, 2, \dots, p; j = 1, 2, \dots, q$) the H -function $H_{p,q}^{m,n}(z)$ is defined via a Mellin-Barnes type integral in the following way:

$$\begin{aligned} H_{p,q}^{m,n}(z) &\equiv H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] \equiv H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \right. \right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| s \right] z^{-s} ds, \end{aligned} \quad (1.1)$$

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where the contour $\mathfrak{L}$ is specially chosen and

$$\mathcal{H}_{p,q}^{m,n}(s) \equiv \mathcal{H}_{p,q}^{m,n} \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| s \right] = \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^p \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s)}, \quad (1.2)$$

in which an empty product, if it occurs, is taken to be one. Such a function was introduced by S. Pincherle in 1888 and its theory has been developed by Mellin [10], Dixon and Ferrar [2] (see [3, §1.19] in this connection). An interest to the H -function arose again in 1961 when Fox [4] has investigated such a function as a symmetrical Fourier kernel. Therefore this function is sometimes called as Fox's H -function. The theory of this function may be found in [1], [9, Chapter 1], [17, Chapter 2] and [11, 8.8.3].

Classical Riemann-Liouville fractional calculus of real order [17, §2.2] (see (2.1) - (2.6) below) was investigated in [12] - [14], [18] and [11]. The right-sided fractional integrals and derivatives of the H -function (1.1) were studied in [12] - [14] and the results were presented in [18, §2.7], where the case of left-sided fractional differentiation of the H -function was also considered. The left-sided fractional integration of the H -function was given in [11, 2.25.2]. Such results for the generalized fractional calculus operators with the Gauss hypergeometric function as a kernel (see (2.7) - (2.10) below), being introduced by the first author [15], were obtained in [16].

However, some of the results obtained in [12] - [14] (cited in [18]) and [16] can be taken to be more precisely. Moreover, these results were given provided that the parameters $a_i, b_j \in \mathbb{C}$ and $\alpha_i > 0, \beta_j > 0$ ($i = 1, 2, \dots, p; j = 1, 2, \dots, q$) of the H -function satisfy certain conditions. These conditions were based on asymptotic behavior of $H_{p,q}^{m,n}(z)$ at zero and infinity. In [5] we extended such the known asymptotic results for the H -function to more wide class of parameters.

In [7], [8] we have applied the obtained asymptotic estimates in [5] to find the Riemann-Liouville fractional integrals and derivatives of any complex order of the H -function. In particular, we could make more precisely the known results from [12] - [14], [18] and [11].

The present paper is devoted to obtain such type results for the generalized fractional integration and differentiation operators of any complex order with the Gauss hypergeometric function as a kernel. In particular, we give more precisely some of the results from [16] and generalize the results obtained in [7], [8]. The paper is organized as follow. In Section 2 we present classical and generalized fractional calculus operators and some facts from the theory of Gauss hypergeometric function. Sections 3 and 4 contain the result from the theory of the H -function. The existence of $H_{p,q}^{m,n}(z)$ and its asymptotic behavior at zero and infinity is considered in Section 3 and certain reduction and differentiation properties in Section 4. Sections 5 and 6 deal with generalized fractional differentiation of the H -function (1.1). Sections 7 and 8 are devoted to the generalized fractional differentiation of the H -function. Another type of fractional integro-differentiation of the H -function is given in Section 9.

2. Classical and Generalized Fractional Calculus Operators

For $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) > 0$, the Riemann-Liouville left- and right-sided fractional calculus operators are defined as follow [17, §2.3 and §2.4]:

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (x > 0), \quad (2.1)$$

$$(I_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)dt}{(t-x)^{1-\alpha}} \quad (x > 0), \quad (2.2)$$

and

$$\begin{aligned} (D_{0+}^{\alpha} f)(x) &= \left(\frac{d}{dx}\right)^{[\text{Re}(\alpha)]+1} (I_{0+}^{1-\alpha+[\text{Re}(\alpha)]} f)(x) \\ &= \left(\frac{d}{dx}\right)^{[\text{Re}(\alpha)]+1} \frac{1}{\Gamma(1-\alpha+[\text{Re}(\alpha)])} \int_0^x \frac{f(t)}{(x-t)^{\alpha-[\text{Re}(\alpha)]}} dt \quad (x > 0), \end{aligned} \quad (2.3)$$

$$\begin{aligned} (D_{-}^{\alpha} f)(x) &= \left(-\frac{d}{dx}\right)^{[\text{Re}(\alpha)]+1} (I_{-}^{1-\alpha+[\text{Re}(\alpha)]} f)(x) \\ &= \left(-\frac{d}{dx}\right)^{[\text{Re}(\alpha)]+1} \frac{1}{\Gamma(1-\alpha+[\text{Re}(\alpha)])} \int_x^{\infty} \frac{f(t)}{(t-x)^{\alpha-[\text{Re}(\alpha)]}} dt \quad (x > 0), \end{aligned} \quad (2.4)$$

respectively, where the symbol $[\kappa]$ means the integral part of a real number κ , i.e. the largest integer not exceeding κ . In particular, for real $\alpha > 0$, the operators D_{0+}^{α} and D_{-}^{α} take more simple forms

$$\begin{aligned} (D_{0+}^{\alpha} f)(x) &= \left(\frac{d}{dx}\right)^{[\alpha]+1} (I_{0+}^{1-\{\alpha\}} f)(x) \\ &= \left(\frac{d}{dx}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}} dt \quad (x > 0), \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} (D_{-}^{\alpha} f)(x) &= \left(-\frac{d}{dx}\right)^{[\alpha]+1} (I_{-}^{1-\{\alpha\}} f)(x) \\ &= \left(-\frac{d}{dx}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_x^{\infty} \frac{f(t)}{(t-x)^{\{\alpha\}}} dt \quad (x > 0), \end{aligned} \quad (2.6)$$

respectively, where $\{\kappa\}$ stands for the fractional part of κ , i.e. $\{\kappa\} = \kappa - [\kappa]$.

For $\alpha, \beta, \eta \in \mathbb{C}$ and $x > 0$ the generalized fractional calculus operators are defined by [15]

$$(I_{0+}^{\alpha, \beta, \eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x}\right) f(t) dt \quad (2.7)$$

$$(\operatorname{Re}(\alpha) > 0);$$

$$(I_{0+}^{\alpha, \beta, \eta} f)(x) = \left(\frac{d}{dx}\right)^n (I_{0+}^{\alpha+n, \beta-n, \eta-n} f)(x) \quad (\operatorname{Re}(\alpha) \leq 0; n = [\operatorname{Re}(-\alpha)] + 1); \quad (2.8)$$

$$(I_{-}^{\alpha, \beta, \eta} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t}\right) f(t) dt \quad (2.9)$$

$$(\operatorname{Re}(\alpha) > 0);$$

$$(I_{-}^{\alpha, \beta, \eta} f)(x) = \left(-\frac{d}{dx}\right)^n (I_{-}^{\alpha+n, \beta-n, \eta} f)(x) \quad (\operatorname{Re}(\alpha) \leq 0; n = [\operatorname{Re}(-\alpha)] + 1); \quad (2.10)$$

and

$$\begin{aligned} (D_{0+}^{\alpha, \beta, \eta} f)(x) &\equiv (I_{0+}^{-\alpha, -\beta, \alpha+\eta} f)(x) \\ &= \left(\frac{d}{dx}\right)^n (I_{0+}^{-\alpha+n, -\beta-n, \alpha+\eta-n} f)(x) \quad (\operatorname{Re}(\alpha) > 0; n = [\operatorname{Re}(\alpha)] + 1); \end{aligned} \quad (2.11)$$

$$\begin{aligned} (D_{-}^{\alpha, \beta, \eta} f)(x) &\equiv (I_{-}^{-\alpha, -\beta, \alpha+\eta} f)(x) \\ &= \left(-\frac{d}{dx}\right)^n (I_{-}^{-\alpha+n, -\beta-n, \alpha+\eta} f)(x) \quad (\operatorname{Re}(\alpha) > 0; n = [\operatorname{Re}(\alpha)] + 1). \end{aligned} \quad (2.12)$$

Here ${}_2F_1(a, b; c; z)$ ($a, b, c, z \in \mathbb{C}$) is the Gauss hypergeometric function defined by the series

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \quad (2.13)$$

with

$$(a)_0 = 1, \quad (a)_k = a(a+1) \cdots (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (k \in \mathbb{N}), \quad (2.14)$$

where $\Gamma(z)$ is the Gamma function [3, Chapter I] and $\mathbb{N}$ denotes the set of positive integers.

The series in (2.13) is convergent for $|z| < 1$ and for $|z| = 1$ with $\operatorname{Re}(c - a - b) > 0$, and can be analytically continued into $\{z \in \mathbb{C} : |\arg(1-z)| < \pi\}$ (see [3, Chapter II]).

Since

$${}_2F_1(0, b; c; z) = 1 \quad (2.15)$$

for $\beta = -\alpha$, the generalized fractional calculus operators (2.7), (2.9), (2.11) and (2.12) coincide with the Riemann-Liouville operators (2.1) - (2.4) for $\operatorname{Re}(\alpha) > 0$:

$$(I_{0+}^{\alpha, -\alpha, \eta} f)(x) = (I_{0+}^{\alpha} f)(x), \quad (I_{-}^{\alpha, -\alpha, \eta} f)(x) = (I_{-}^{\alpha} f)(x), \quad (2.16)$$

$$(D_{0+}^{\alpha, -\alpha, \eta} f)(x) = (D_{0+}^{\alpha} f)(x), \quad (D_{-}^{\alpha, -\alpha, \eta} f)(x) = (D_{-}^{\alpha} f)(x). \quad (2.17)$$

According to the relation [3, 2.8(4)]

$${}_2F_1(a, b; a; z) = (1 - z)^{-b}, \quad (2.18)$$

when $\beta = 0$ the operators (2.7) and (2.9) coincide with the Erdélyi-Kober fractional integrals [17, §18.1]:

$$(I_{0+}^{\alpha, 0, \eta} f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\eta f(t) dt \equiv (I_{\eta, \alpha}^+ f)(x) \quad (\alpha, \eta \in \mathbb{C}, \operatorname{Re}(\alpha) > 0), \quad (2.19)$$

$$(I_{-}^{\alpha, 0, \eta} f)(x) = \frac{x^\eta}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\eta-\alpha} f(t) dt \equiv (K_{\eta, \alpha}^- f)(x) \quad (\alpha, \eta \in \mathbb{C}, \operatorname{Re}(\alpha) > 0). \quad (2.20)$$

Therefore the operators (2.7), (2.9) and (2.11), (2.12) are called "generalized" fractional integrals and derivatives, respectively. Moreover, the operators (2.11) and (2.12) are inverse to (2.7) and (2.9):

$$D_{0+}^{\alpha, \beta, \eta} = (I_{0+}^{\alpha, \beta, \eta})^{-1}, \quad D_{-}^{\alpha, \beta, \eta} = (I_{-}^{\alpha, \beta, \eta})^{-1}. \quad (2.21)$$

Fractional calculus operators (2.1), (2.3), (2.5), (2.7), (2.8), (2.11) and (2.2), (2.4), (2.6), (2.9), (2.10), (2.12) are called left-sided and right-sided, respectively [17, §2].

We give some other properties of ${}_2F_1(a, b; c; z)$ [3, 2.8(46), 2.9(2), 2.10(14)] which will be used in the following calculations:

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c \neq 0, -1, -2, \dots; \operatorname{Re}(c-a-b) > 0); \quad (2.22)$$

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z); \quad (2.23)$$

$$\begin{aligned} {}_2F_1(a, b; a+b; z) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(k!)^2} [2\psi(1+k) - \psi(a+k) + \psi(b+k) \\ &\quad - \log(1-z)] (1-z)^k \quad (|\arg(z)| < \pi; a, b \neq 0, -1, -2, \dots), \end{aligned} \quad (2.24)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the Psi function [3, 1.7].

Formulas (2.22) - (2.24) mean the following asymptotic behavior of ${}_2F_1(a, b; c; z)$ at the point $z = 1$.

Lemma 1. For $a, b, c \in \mathbb{C}$ with $\operatorname{Re}(c) > 0$ and $z \in \mathbb{C}$, there hold the following asymptotic relations near $z = 1$:

$${}_2F_1(a, b; c; z) = O(1) \quad (z \rightarrow 1-) \quad (2.25)$$

for $\operatorname{Re}(c-a-b) > 0$;

$${}_2F_1(a, b; c; z) = O((1-z)^{c-a-b}) \quad (z \rightarrow 1-) \quad (2.26)$$

for $\operatorname{Re}(c-a-b) < 0$; and

$${}_2F_1(a, b; c; z) = O(\log(1-z)) \quad (z \rightarrow 1-) \quad (2.27)$$

for $c - a - b = 0$, $a, b \neq 0, -1, -2, \dots$ and $|\arg(z)| < \pi$.

3. Existence and Asymptotic Behavior of the H -Function

We shall consider the H -function (1.1) provided that the poles

$$b_{jl} = \frac{-b_j - l}{\beta_j} \quad (j = 1, \dots, m; l \in \mathbb{N}_0) \quad (3.1)$$

of the Gamma functions $\Gamma(b_j + \beta_j s)$ and that

$$a_{ik} = \frac{1 - a_i + k}{\alpha_i} \quad (i = 1, \dots, n; k \in \mathbb{N}_0) \quad (3.2)$$

of $\Gamma(1 - a_i - \alpha_i s)$ do not coincide:

$$\alpha_i(b_j + l) \neq \beta_j(a_i - k - 1) \quad (i = 1, \dots, n; j = 1, \dots, m; k, l \in \mathbb{N}_0), \quad (3.3)$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathfrak{L}$ in (1.1) is the infinite contour splitting poles b_{jl} in (3.1) to the left and poles a_{ik} in (3.2) to the right of $\mathfrak{L}$ and has one of the following forms:

- (i) $\mathfrak{L} = \mathfrak{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at the point $-\infty + i\varphi_1$ and terminating at the point $-\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$;
- (ii) $\mathfrak{L} = \mathfrak{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at the point $+\infty + i\varphi_1$ and terminating at the point $+\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$;
- (iii) $\mathfrak{L} = \mathfrak{L}_{i\gamma\infty}$ is a contour starting at the point $\gamma - i\infty$ and terminating at the point $\gamma + i\infty$ with $\gamma \in \mathbb{R} = (-\infty, +\infty)$.

The properties of the H -function $H_{p,q}^{m,n}(z)$ depend on the numbers a^*, Δ, δ and μ which are expressed via p, q , a_i, α_i ($i = 1, 2, \dots, p$) and b_j, β_j ($j = 1, 2, \dots, q$) by the following relations:

$$a^* = \sum_{i=1}^n \alpha_i - \sum_{i=n+1}^p \alpha_i + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j, \quad (3.4)$$

$$\Delta = \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i, \quad (3.5)$$

$$\delta = \prod_{i=1}^p \alpha_i^{-\alpha_i} \prod_{j=1}^q \beta_j^{\beta_j}, \quad (3.6)$$

$$\mu = \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2}. \quad (3.7)$$

Here an empty sum in (3.4), (3.5), (3.7) and an empty product in (3.6), if they occur, are taken to be zero and one, respectively.

The existence of the H -function is given by the following result [6].

Theorem A. Let a^* , Δ , δ and μ be given by (3.4) - (3.7). Then the H -function $H_{p,q}^{m,n}(z)$ defined by (1.1) and (1.2) makes sense in the following cases:

$$\mathfrak{L} = \mathfrak{L}_{-\infty}, \quad \Delta > 0, \quad z \neq 0; \quad (3.8)$$

$$\mathfrak{L} = \mathfrak{L}_{-\infty}, \quad \Delta = 0, \quad 0 < |z| < \delta; \quad (3.9)$$

$$\mathfrak{L} = \mathfrak{L}_{-\infty}, \quad \Delta = 0, \quad \operatorname{Re}(\mu) < -1, \quad |z| = \delta; \quad (3.10)$$

$$\mathfrak{L} = \mathfrak{L}_{+\infty}, \quad \Delta < 0, \quad z \neq 0; \quad (3.11)$$

$$\mathfrak{L} = \mathfrak{L}_{+\infty}, \quad \Delta = 0, \quad |z| > \delta; \quad (3.12)$$

$$\mathfrak{L} = \mathfrak{L}_{+\infty}, \quad \Delta = 0, \quad \operatorname{Re}(\mu) < -1, \quad |z| = \delta; \quad (3.13)$$

$$\mathfrak{L} = \mathfrak{L}_{i\gamma\infty}, \quad a^* > 0, \quad |\arg z| < \frac{a^*\pi}{2}, \quad z \neq 0; \quad (3.14)$$

$$\mathfrak{L} = \mathfrak{L}_{i\gamma\infty}, \quad a^* = 0, \quad \Delta\gamma + \operatorname{Re}(\mu) < -1, \quad \arg z = 0, \quad z \neq 0. \quad (3.15)$$

Remark 1. The results of Theorem A in the cases (3.10), (3.13) and (3.15) are more precisely than those in [11, §8.3.1].

The next statement being followed from the results in [5] characterizes the asymptotic behavior of the H -function at zero and infinity.

Theorem B. Let a^* and Δ be given by (3.4) and (3.5) and let conditions in (3.3) be satisfied.

(i) If $\Delta \geq 0$ or $\Delta < 0, a^* > 0$, then the H -function has either of the asymptotic estimates at zero

$$H_{p,q}^{m,n}(z) = O(z^{\varrho^*}) \quad (|z| \rightarrow 0) \quad (3.16)$$

or

$$H_{p,q}^{m,n}(z) = O(z^{\varrho^*} [\log(z)]^{N^*}) \quad (|z| \rightarrow 0), \quad (3.17)$$

with the additional condition $|\arg(z)| < a^*\pi/2$ when $\Delta < 0, a^* > 0$. Here

$$\varrho^* = \min_{1 \leq j \leq m} \left[\frac{\operatorname{Re}(b_j)}{\beta_j} \right], \quad (3.18)$$

and N^* is the order of one of the point b_{j1} in (3.1) to which some other poles of $\Gamma(b_j + \beta_j s)$ ($j = 1, \dots, m$) coincide.

(ii) If $\Delta \leq 0$ or $\Delta > 0, a^* > 0$, then the H -function has either of the asymptotic estimates at infinity

$$H_{p,q}^{m,n}(z) = O(z^{\varrho}) \quad (|z| \rightarrow \infty) \quad (3.19)$$

or

$$H_{p,q}^{m,n}(z) = O\left(z^{\varrho}[\log(z)]^N\right) \quad (|z| \rightarrow \infty), \quad (3.20)$$

with the additional condition $|\arg(z)| < a^* \pi/2$ when $\Delta > 0, a^* > 0$. Here

$$\varrho = \max_{1 \leq i \leq n} \left[\frac{\operatorname{Re}(a_i) - 1}{\alpha_i} \right], \quad (3.21)$$

and N is the order of one of the point a_{ik} in (3.2) in which some other poles of $\Gamma(1 - a_i - \alpha_i s)$ ($i = 1, \dots, n$) coincide.

4. Reduction and Differentiation Properties of the H -Function

In this and next sections we suppose that the conditions for the existence of the H -function given in Theorem A are satisfied.

The following two Lemmas which characterize symmetric and reduction properties of the H -function follow from the definition of the H -function in (1.1) - (1.2).

Lemma 2. *The H -function (1.1) is commutative in the set of pairs $(a_1, \alpha_1), \dots, (a_n, \alpha_n)$; in $(a_{n+1}, \alpha_{n+1}), \dots, (a_p, \alpha_p)$; in $(b_1, \beta_1), \dots, (b_m, \beta_m)$ and in $(b_{m+1}, \beta_{m+1}), \dots, (b_q, \beta_q)$.*

Lemma 3. *If one of (a_i, α_i) ($i = 1, \dots, n$) is equal to one of (b_j, β_j) ($j = m + 1, \dots, q$) (or one of (a_i, α_i) ($i = n + 1, \dots, p$) is equal to one of (b_j, β_j) ($j = 1, \dots, m$)), then the H -function reduces to the lower order one, that is, p, q and n (or m) decrease by unity. Two such results have the forms*

$$H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q-1}, (a_1, \alpha_1) \end{array} \right. \right] = H_{p-1,q-1}^{m,n-1} \left[z \left| \begin{array}{c} (a_i, \alpha_i)_{2,p} \\ (b_j, \beta_j)_{1,q-1} \end{array} \right. \right] \quad (4.1)$$

provided that $n \geq 1$ and $q > m$, and

$$H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (a_i, \alpha_i)_{1,p-1}, (b_1, \beta_1) \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] = H_{p-1,q-1}^{m-1,n} \left[z \left| \begin{array}{c} (a_i, \alpha_i)_{1,p-1} \\ (b_j, \beta_j)_{2,q} \end{array} \right. \right] \quad (4.2)$$

provided that $m \geq 1$ and $p > n$.

The next differentiation formulae follow from the definition of the H -function given in (1.1) - (1.2) and from the functional equation for the Gamma function [3, §1.2(6)]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \quad (4.3)$$

Lemma 4. *There hold the following differentiation formulae for $\omega, c \in \mathbb{C}, \sigma > 0$*

$$\begin{aligned} & \left(\frac{d}{dz} \right)^k \left\{ z^\omega H_{p,q}^{m,n} \left[cz^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right\} \\ &= z^{\omega-k} H_{p+1,q+1}^{m,n+1} \left[cz^\sigma \left| \begin{array}{c} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (k-\omega, \sigma) \end{array} \right. \right], \end{aligned} \quad (4.4)$$

$$\begin{aligned} & \left(\frac{d}{dz} \right)^k \left\{ z^\omega H_{p,q}^{m,n} \left[cz^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right\} \\ &= (-1)^k z^{\omega-k} H_{p+1,q+1}^{m+1,n} \left[cz^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma) \\ (k-\omega, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right. \right]. \end{aligned} \quad (4.5)$$

5. Left-Sided Generalized Fractional Integration of the H -Function

In the following sections we treat the H -function (1.1) - (1.2) with $\mathfrak{L} = \mathfrak{L}_{i\gamma\infty}$ and under the assumptions $a^* > 0$ or $a^* = 0, \Delta\gamma + \operatorname{Re}(\mu) < -1$ for a^*, Δ, μ being given by (3.4), (3.5), (3.7).

Here we consider the left-sided generalized fractional integration $I_{0+}^{\alpha,\beta,\eta}$ defined by (2.7).

Theorem 1. *Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) \neq \operatorname{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \min_{1 \leq j \leq m} \left[\frac{\operatorname{Re}(b_j)}{\beta_j} \right] + \operatorname{Re}(\omega) + \min[0, \operatorname{Re}(\eta - \beta)] + 1 > 0, \quad (5.1)$$

$$\sigma\gamma < \operatorname{Re}(\omega) + \min[0, \operatorname{Re}(\eta - \beta)] + 1. \quad (5.2)$$

Then the generalized fractional integral $I_{0+}^{\alpha,\beta,\eta}$ of the H -function (1.1) exists and the following relation holds:

$$\begin{aligned} & \left(I_{0+}^{\alpha,\beta,\eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x) \\ &= x^{\omega-\beta} H_{p+2,q+2}^{m,n+2} \left[x^\sigma \left| \begin{array}{c} (-\omega, \sigma), (-\omega + \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega + \beta, \sigma), (-\omega - \alpha - \eta, \sigma) \end{array} \right. \right]. \end{aligned} \quad (5.3)$$

Proof. By (2.7) we have

$$\begin{aligned} & \left(I_{0+}^{\alpha,\beta,\eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x) \\ &= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\omega {}_2F_1 \left(\alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] dt. \end{aligned} \quad (5.4)$$

According to (2.25), (2.26), (3.16) and (3.17), the integrand in (5.4) for any $x > 0$ has the asymptotic estimate at zero

$$\begin{aligned} & (x-t)^{\alpha-1} t^\omega {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) H_{p,q}^{m,n} \left[t^\sigma \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] \\ &= O\left(t^{\omega+\sigma\varrho^*+\min[0, \operatorname{Re}(\eta-\beta)]}\right) \quad (t \rightarrow +0) \end{aligned}$$

or

$$= O\left(t^{\omega+\sigma\varrho^*+\min[0, \operatorname{Re}(\eta-\beta)]} [\log(t)]^{N^*}\right) \quad (t \rightarrow +0).$$

Here ϱ^* is given by (3.18) and N^* is indicated in Theorem B(i). Therefore the condition (5.1) ensures the existence of the integral (5.4).

Applying (1.2), making the change of variable $t = x\tau$, changing the order of integration and taking into account the formula [11, §2.21.1.11]

$$\int_0^x t^{\alpha-1} (x-t)^{c-1} {}_2F_1\left(a, b; c; 1-\frac{t}{x}\right) dt = \frac{\Gamma(c)\Gamma(\alpha)\Gamma(\alpha+c-a-b)}{\Gamma(\alpha+c-a)\Gamma(\alpha+c-b)} x^{\alpha+c-1} \quad (5.5)$$

$$(a, b, c, \alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(c) > 0, \operatorname{Re}(\alpha+c-a-b) > 0),$$

we obtain

$$\begin{aligned} & \left(I_{0+}^{\alpha, \beta, \eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) (x) \\ &= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\omega {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) H_{p,q}^{m,n} \left[t^\sigma \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] dt \\ &= \frac{x^{-\alpha-\beta}}{2\pi i \Gamma(\alpha)} \int_{\mathfrak{L}} \mathcal{H}_{p,q}^{m,n} \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| s \right] ds \int_0^x (x-t)^{\alpha-1} t^{\omega-\sigma s} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) dt \\ &= \frac{x^{\omega-\beta}}{2\pi i} \int_{\mathfrak{L}} \mathcal{H}_{p,q}^{m,n} \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| s \right] \frac{\Gamma(1+\omega-s\sigma)\Gamma(1+\omega-\beta+\eta-s\sigma)}{\Gamma(1+\omega-\beta-s\sigma)\Gamma(1+\omega+\alpha+\eta-s\sigma)} x^{-\sigma s} ds. \quad (5.6) \end{aligned}$$

We note that since $\mathfrak{L} = \mathfrak{L}_{i\gamma\infty}$, $\operatorname{Re}(s) = \gamma$ and therefore the condition (5.2) ensures the existence of the Mellin-Barnes integral above. Hence in view of (1.2)

$$\begin{aligned} & \left(I_{0+}^{\alpha, \beta, \eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) (x) \\ &= x^{\omega-\beta} H_{p+2, q+2}^{m, n+2} \left[x^\sigma \left| \begin{matrix} (-\omega, \sigma), (-\omega+\beta-\eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega+\beta, \sigma), (-\omega-\alpha-\eta, \sigma) \end{matrix} \right. \right]. \quad (5.7) \end{aligned}$$

and in accordance with (1.1) we obtain (5.3) which completes the proof of Theorem 1.

Corollary 1.1. Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \min_{1 \leq j \leq m} \left[\frac{\operatorname{Re}(b_j)}{\beta_j} \right] + \operatorname{Re}(\omega) + 1 > 0, \quad (5.8)$$

$$\sigma\gamma < \operatorname{Re}(\omega) + 1. \quad (5.9)$$

Then the Riemann-Liouville fractional integral I_{0+}^α of the H -function (1.1) exists and the following relation holds:

$$\left(I_{0+}^\alpha t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) (x) = x^{\omega+\alpha} H_{p+1,q+1}^{m,n+1} \left[t^\sigma \left| \begin{matrix} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \alpha, \sigma) \end{matrix} \right. \right]. \quad (5.10)$$

Corollary 1.2. Let $\alpha, \eta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \min_{1 \leq j \leq m} \left[\frac{\operatorname{Re}(b_j)}{\beta_j} \right] + \operatorname{Re}(\omega) + \min[0, \operatorname{Re}(\eta)] + 1 > 0, \quad (5.11)$$

$$\sigma\gamma < \operatorname{Re}(\omega) + \min[0, \operatorname{Re}(\eta)] + 1. \quad (5.12)$$

Then the Erdélyi-Kober fractional integral $I_{\eta,\alpha}^+$ of the H -function (1.1) exists and the following relation holds:

$$\left(I_{\eta,\alpha}^+ t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) (x) = x^\omega H_{p+1,q+1}^{m,n+1} \left[x^\sigma \left| \begin{matrix} (-\omega - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \alpha - \eta, \sigma) \end{matrix} \right. \right]. \quad (5.13)$$

Remark 2. In the case $a^* > 0, \Delta \geq 0$ the relation (5.3) was indicated in [16, (4.2)], but in the assumptions of the result the condition (5.2) of Theorem 1 should be added.

Remark 3. Corollary 1.1 coincides with Theorem 1 in [7]. For real $\alpha > 0$ and $a^* > 0$ the relation (5.10) was indicated in [11, 2.25.2.2], but the conditions of its validity have to be also corrected according to (5.8) and (5.9).

6. Right-Sided Generalized Fractional Integration of the H -Function

In this section we consider the right-sided generalized fractional integration $I_-^{\alpha,\beta,\eta}$ defined by (2.9).

Theorem 2. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) \neq \operatorname{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \max_{1 \leq i \leq n} \left[\frac{\operatorname{Re}(a_i) - 1}{\alpha_i} \right] + \operatorname{Re}(\omega) < \min[\operatorname{Re}(\beta), \operatorname{Re}(\eta)], \quad (6.1)$$

$$\sigma \gamma > \operatorname{Re}(\omega) - \min[\operatorname{Re}(\beta), \operatorname{Re}(\eta)]. \quad (6.2)$$

Then the generalized fractional integral $I_-^{\alpha, \beta, \eta}$ of the H -function (1.1) exists and the following relation holds:

$$\begin{aligned} & \left(I_-^{\alpha, \beta, \eta} t^\omega H_{p, q}^{m, n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{array} \right. \right] \right) (x) \\ &= x^{\omega - \beta} H_{p+2, q+2}^{m+2, n} \left[x^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1, p}, (-\omega, \sigma), (-\omega + \alpha + \beta + \eta, \sigma) \\ (-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1, q} \end{array} \right. \right]. \end{aligned} \quad (6.3)$$

Proof. By (2.9) we have

$$\begin{aligned} & \left(I_-^{\alpha, \beta, \eta} t^\omega H_{p, q}^{m, n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{array} \right. \right] \right) (x) \\ &= \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{\omega-\alpha-\beta} {}_2F_1 \left(\alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right) H_{p, q}^{m, n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{array} \right. \right] dt. \end{aligned} \quad (6.4)$$

Due to (2.25), (2.26), (3.19) and (3.20), the integrand in (6.4) for any $x > 0$ has the asymptotic at infinity

$$\begin{aligned} & (t-x)^{\alpha-1} t^{\omega-\alpha-\beta} {}_2F_1 \left(\alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right) H_{p, q}^{m, n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{array} \right. \right] \\ &= O \left(t^{\omega - \min[\operatorname{Re}(\beta), \operatorname{Re}(\eta)] - 1 + \sigma \varrho} \right) \quad (t \rightarrow +\infty) \end{aligned}$$

or

$$= O \left(t^{\omega - \min[\operatorname{Re}(\beta), \operatorname{Re}(\eta)] - 1 + \sigma \varrho} [\log(t)]^N \right) \quad (t \rightarrow +\infty).$$

Here ϱ is given by (3.21) and N is indicated in Theorem B(ii). Therefore the condition (6.1) ensures the existence of the integral (6.4). Applying (1.2), making the change $t = 1/\tau$ and using (5.5), we obtain

$$\begin{aligned} & \left(I_-^{\alpha, \beta, \eta} t^\omega H_{p, q}^{m, n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{array} \right. \right] \right) \left(\frac{1}{x} \right) \\ &= \frac{1}{\Gamma(\alpha)} \int_{1/x}^\infty \left(t - \frac{1}{x} \right)^{\alpha-1} t^{\omega-\alpha-\beta} {}_2F_1 \left(\alpha + \beta, -\eta; \alpha; 1 - \frac{1}{tx} \right) H_{p, q}^{m, n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{array} \right. \right] dt \end{aligned}$$

$$\begin{aligned}
&= \frac{x^{1-\alpha}}{2\pi i \Gamma(\alpha)} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| s \right] \tau^{\sigma s} ds \\
&\quad \cdot \int_0^x (x-\tau)^{\alpha-1} \tau^{\beta-\omega-1+\sigma s} {}_2F_1 \left(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x} \right) d\tau \\
&= \frac{x^{-\omega+\beta}}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| s \right] \frac{\Gamma(-\omega + \beta + \sigma s) \Gamma(-\omega + \eta + \sigma s)}{\Gamma(-\omega + \sigma s) \Gamma(-\omega + \alpha + \beta + \eta + \sigma s)} x^{\sigma s} ds. \quad (6.5)
\end{aligned}$$

Since $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$, $\operatorname{Re}(s) = \gamma$ and therefore the condition (6.2) guarantees the existence of the Mellin-Barnes integral above. Replacing in (6.5) x by $1/x$, we obtain (6.3).

Corollary 2.1. Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \max_{1 \leq i \leq n} \left[\frac{\operatorname{Re}(a_i) - 1}{\alpha_i} \right] + \operatorname{Re}(\omega) + \operatorname{Re}(\alpha) < 0, \quad (6.6)$$

$$\sigma\gamma > \operatorname{Re}(\omega) + \operatorname{Re}(\alpha). \quad (6.7)$$

Then the Riemann-Liouville fractional integral I_-^α of the H -function (1.1) exists and the following relation holds:

$$\left(I_-^\alpha t^\omega H_{p,q}^{m,n} \left[t^\sigma \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \right) (x) = x^{\omega+\alpha} H_{p+1,q+1}^{m+1,n} \left[x^\sigma \middle| \begin{matrix} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma) \\ (-\omega - \alpha, \sigma), (b_j, \beta_j)_{1,q} \end{matrix} \right]. \quad (6.8)$$

Corollary 2.2. Let $\alpha, \eta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \max_{1 \leq i \leq n} \left[\frac{\operatorname{Re}(a_i) - 1}{\alpha_i} \right] + \operatorname{Re}(\omega) < \operatorname{Re}(\eta), \quad (6.9)$$

$$\sigma\gamma > \operatorname{Re}(\omega) - \operatorname{Re}(\eta). \quad (6.10)$$

Then the Erdélyi-Kober fractional integral $K_{\eta,\alpha}^-$ of the H -function (1.1) exists and the following relation holds:

$$\left(K_{\eta,\alpha}^- t^\omega H_{p,q}^{m,n} \left[t^\sigma \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \right) (x) = x^\omega H_{p+1,q+1}^{m,n+1} \left[x^\sigma \middle| \begin{matrix} (a_i, \alpha_i)_{1,p}, (-\omega + \eta + \alpha, \sigma) \\ (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{matrix} \right]. \quad (6.11)$$

Remark 4. In the case $\alpha^* > 0, \Delta \geq 0$ the relation of the form (6.3) was indicated in [16, (4.3)]. But it includes a mistake and should be replaced by (6.3) with the conditions (6.1) and (6.2).

Remark 5. Corollary 2.1 coincides with Theorem 2 in [7]. For real $\alpha > 0$ and $\alpha^* > 0$ the relation (6.8) was indicated in [18, (2.5)], but the conditions of its validity have to be also corrected in accordance with (6.6) and (6.7).

7. Left-Sided Generalized Fractional Differentiation of the H -Function

Now we treat the left-sided generalized fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ given by (2.11).

Theorem 3. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\alpha + \beta + \eta) \neq 0$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \min_{1 \leq j \leq m} \left[\frac{\operatorname{Re}(b_j)}{\beta_j} \right] + \operatorname{Re}(\omega) + \min[0, \operatorname{Re}(\alpha + \beta + \eta)] + 1 > 0, \quad (7.1)$$

$$\sigma \gamma < \operatorname{Re}(\omega) + \min[0, \operatorname{Re}(\alpha + \beta + \eta)] + 1. \quad (7.2)$$

Then the generalized fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ of the H -function (1.1) exists and the following relation holds:

$$\begin{aligned} & \left(D_{0+}^{\alpha,\beta,\eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x) \\ &= x^{\omega+\beta} H_{p+2,q+2}^{m,n+2} \left[x^\sigma \left| \begin{array}{c} (-\omega, \sigma), (-\omega - \eta - \alpha - \beta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \beta, \sigma), (-\omega - \eta, \sigma) \end{array} \right. \right]. \end{aligned} \quad (7.3)$$

Proof. Let $n = [\operatorname{Re}(\alpha)] + 1$. From (2.11) we have

$$\begin{aligned} & \left(D_{0+}^{\alpha,\beta,\eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x) \\ &= \left(\frac{d}{dx} \right)^n \left(I_{0+}^{-\alpha+n, -\beta-n, \alpha+\eta-n} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x), \end{aligned} \quad (7.4)$$

which exists according to Theorem 1 with α, β and η being replaced by $-\alpha + n, -\beta - n$ and $\alpha + \eta - n$, respectively. Then we find

$$\begin{aligned} & \left(D_{0+}^{\alpha,\beta,\eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x) \\ &= \left(\frac{d}{dx} \right)^n x^{\omega+\beta+n} H_{p+2,q+2}^{m,n+2} \left[x^\sigma \left| \begin{array}{c} (-\omega, \sigma), (-\omega - \alpha - \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \beta - n, \sigma), (-\omega - \eta, \sigma) \end{array} \right. \right]. \end{aligned} \quad (7.5)$$

Taking into account the differentiation formula (4.4) we have

$$\begin{aligned} & \left(D_{0+}^{\alpha, \beta, \eta} t^\omega H_{p, q}^{m, n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{array} \right. \right] \right) (x) \\ &= x^{\omega + \beta} H_{p+3, q+3}^{m, n+3} \left[x^\sigma \left| \begin{array}{c} (-\omega - \beta - n, \sigma), (-\omega, \sigma), (-\omega - \alpha - \beta - \eta, \sigma), (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q}, (-\omega - \beta - n, \sigma), (-\omega - \eta, \sigma), (-\omega - \beta, \sigma) \end{array} \right. \right], \quad (7.6) \end{aligned}$$

and Lemma 2 and the reduction relation (4.1) imply (7.3), which completes the proof of theorem.

Corollary 3.1. *Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy the conditions in (5.8) and (5.9). Then the Riemann-Liouville fractional derivative D_{0+}^α of the H -function (1.1) exists and the following relation holds:*

$$\left(D_{0+}^\alpha t^\omega H_{p, q}^{m, n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{array} \right. \right] \right) (x) = x^{\omega - \alpha} H_{p+1, q+1}^{m, n+1} \left[x^\sigma \left| \begin{array}{c} (-\omega, \sigma), (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q}, (-\omega + \alpha, \sigma) \end{array} \right. \right]. \quad (7.7)$$

Remark 6. For real $\alpha > 0$ and $\alpha^* > 0$ the relation (7.3) was given in [18, (2.7.13)], but the conditions of its validity have to be corrected in accordance with (7.1) and (7.2).

Remark 7. Corollary 3.1 coincides with Theorem 3 in [7].

8. Right-Sided Generalized Fractional Differentiation of the H -Function

Here we deal with the right-sided generalized fractional derivative $D_-^{\alpha, \beta, \eta}$ given by (2.12).

Theorem 4. *Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\alpha + \beta + \eta) + [\operatorname{Re}(\alpha)] + 1 \neq 0$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \max_{1 \leq i \leq n} \left[\frac{\operatorname{Re}(a_i) - 1}{\alpha_i} \right] + \operatorname{Re}(\omega) + \max[\operatorname{Re}(\beta) + [\operatorname{Re}(\alpha)] + 1, -\operatorname{Re}(\alpha + \eta)] < 0, \quad (8.1)$$

$$\sigma \gamma > \operatorname{Re}(\omega) + \max[\operatorname{Re}(\beta) + [\operatorname{Re}(\alpha)] + 1, -\operatorname{Re}(\alpha + \eta)]. \quad (8.2)$$

Then the generalized fractional derivative $D_-^{\alpha, \beta, \eta}$ of the H -function (1.1) exists and the following relation holds:

$$\begin{aligned} & \left(D_-^{\alpha, \beta, \eta} t^\omega H_{p, q}^{m, n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{array} \right. \right] \right) (x) \\ &= (-1)^{[\operatorname{Re}(\alpha)] + 1} x^{\omega + \beta} H_{p+2, q+2}^{m+2, n} \left[x^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1, p}, (-\omega, \sigma), (-\omega - \beta + \eta, \sigma) \\ (-\omega - \beta, \sigma), (-\omega + \alpha + \eta, \sigma), (b_j, \beta_j)_{1, q} \end{array} \right. \right]. \quad (8.3) \end{aligned}$$

Proof. Let $n = [\operatorname{Re}(\alpha)] + 1$. Owing to (2.12) we have

$$\begin{aligned} & \left(D_-^{\alpha, \beta, \eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x) \\ &= \left(-\frac{d}{dx} \right)^n \left(I_-^{\alpha+n, -\beta-n, \alpha+\eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x), \end{aligned} \quad (8.4)$$

which exists according to Theorem 2 with α, β and η being replaced by $-\alpha + n, -\beta - n$ and $\alpha + \eta$, respectively. Then applying the differentiation formula (4.5), similarly to (7.5), (7.6), we find in view of the reduction formula (4.2) that

$$\begin{aligned} & \left(D_-^{\alpha, \beta, \eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x) \\ &= \left(-\frac{d}{dx} \right)^n x^{\omega+\beta+n} H_{p+2,q+2}^{m+2,n} \left[x^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega - \beta + \eta, \sigma) \\ (-\omega - \beta - n, \sigma), (-\omega + \alpha + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right. \right] \\ &= (-1)^n x^{\omega+\beta} H_{p+3,q+3}^{m+3,n} \left[x^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega - \beta + \eta, \sigma), (-\omega - \beta - n, \sigma) \\ (-\omega - \beta, \sigma), (-\omega - \beta - n, \sigma), (-\omega + \alpha + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right. \right], \end{aligned}$$

which implies the formula (8.3).

Corollary 4.1. Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \max_{1 \leq i \leq n} \left\{ \frac{\operatorname{Re}(a_i) - 1}{\alpha_i} \right\} + \operatorname{Re}(\omega) - \{\operatorname{Re}(\alpha)\} + 1 < 0, \quad (8.5)$$

$$\sigma \gamma + \operatorname{Re}(\omega) - \{\operatorname{Re}(\alpha)\} + 1 > 0. \quad (8.6)$$

Then the Riemann-Liouville fractional derivative D_-^α of the H -function (1.1) exists and there holds the relation:

$$\left(D_-^\alpha t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x) \quad (8.7)$$

$$= (-1)^{[\operatorname{Re}(\alpha)]+1} x^{\omega-\alpha} H_{p+1,q+1}^{m+1,n} \left[x^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma) \\ (-\omega + \alpha, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right. \right]. \quad (8.8)$$

Remark 8. The relation of the form (8.7) with real $\alpha > 0$ and $a^* > 0$ was proved in [13, formula (14a)] (see also [12], [14, (2.2)] and [18, (2.7.9)]). But such a formula contains mistakes and should be replaced by (8.7) with the condition (8.5) and (8.6).

Remark 9. When $\alpha = k \in \mathbb{N}$, the relations (7.7) and (8.7) coincide with (4.4) and (4.5), respectively.

9. Generalized Fractional Integro-Differentiation of the H -Function

Here we investigate the generalized fractional integro-differentiation operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_-^{\alpha,\beta,\eta}$ given by (2.8) and (2.10). The following statements are proved similarly to Theorems 3 and 4 by using the relations (2.8) and (2.10), Theorems 1 and 2, and the properties of the H -function in Sections 3 and 4.

Theorem 5. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) \leq 0, \operatorname{Re}(\beta) \neq \operatorname{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \min_{1 \leq j \leq m} \left[\frac{\operatorname{Re}(b_j)}{\beta_j} \right] + \operatorname{Re}(\omega) + \min[0, \operatorname{Re}(\eta - \beta)] + 1 > 0, \quad (9.1)$$

$$\sigma\gamma < \operatorname{Re}(\omega) + \min[0, \operatorname{Re}(\eta - \beta)] + 1. \quad (9.2)$$

Then the generalized fractional integro-differentiation $I_{0+}^{\alpha,\beta,\eta}$ of the H -function (1.1) exists and there holds the relation

$$\begin{aligned} & \left(I_{0+}^{\alpha,\beta,\eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x) \\ &= x^{\omega-\beta} H_{p+2,q+2}^{m,n+2} \left[x^\sigma \left| \begin{array}{c} (-\omega, \sigma), (-\omega - \eta + \beta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega + \beta, \sigma), (-\omega - \alpha - \eta, \sigma) \end{array} \right. \right]. \end{aligned} \quad (9.3)$$

Theorem 6. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) \leq 0, \operatorname{Re}(\beta) + [\operatorname{Re}(\alpha)] - 1 \neq \operatorname{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \dots, p; j = 1, \dots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \max_{1 \leq i \leq n} \left[\frac{\operatorname{Re}(a_i) - 1}{\alpha_i} \right] + \operatorname{Re}(\omega) < \min[\operatorname{Re}(\beta) - [\operatorname{Re}(-\alpha)] - 1, \operatorname{Re}(\eta)], \quad (9.4)$$

$$\sigma\gamma > \operatorname{Re}(\omega) - \min[\operatorname{Re}(\beta) - [\operatorname{Re}(-\alpha)] - 1, \operatorname{Re}(\eta)]. \quad (9.5)$$

Then the generalized fractional integro-differentiation $I_-^{\alpha,\beta,\eta}$ of the H -function (1.1) exists and there holds the relation

$$\begin{aligned} & \left(I_-^{\alpha,\beta,\eta} t^\omega H_{p,q}^{m,n} \left[t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x) \\ &= x^{\omega-\beta} H_{p+2,q+2}^{m+2,n} \left[x^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega + \alpha + \beta + \eta, \sigma) \\ (-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right. \right]. \end{aligned} \quad (9.6)$$

Remark 10. The relation (9.3) with $\alpha^* > 0, \Delta \geq 0$ was indicated in [16, (4.2)], but conditions of its validity have to be corrected in accordance with (9.1) and (9.2).

Remark 11. The relations (9.3) and (9.6) for the fractional integro-differentiation operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_-^{\alpha,\beta,\eta}$, defined in (2.8) and (2.10) for $\alpha \in \mathbb{C}, \operatorname{Re}(\alpha) \leq 0$ coincide with that (5.3) and (6.3) for the fractional integration operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_-^{\alpha,\beta,\eta}$, defined in (2.7) and (2.9) for $\alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0$. Though the conditions for validity of (5.3) and (9.3) in Theorems 1 and 5 have the same form, that of (6.3) and (9.6) presented in Theorems 2 and 4 are slightly different.

In conclusion we note that, as it was mentioned in Remarks 2, 4 and 10, the relations (5.3), (6.3) and (9.3) for generalized calculus operator $I_{0+}^{\alpha,\beta,\eta}$ were already known in the case $\alpha^* > 0, \Delta \geq 0$. Further, Remarks 3, 5, 6 and 8 indicate that the relations (5.10) and (6.8) for the Riemann-Liouville fractional integrals $I_{0+}^\alpha, I_-^\alpha$ and (7.3) and (8.7) for the fractional derivative D_{0+}^α in the case real $\alpha > 0$ and $\alpha^* > 0$ were established. However, the H -function's asymptotic estimates (3.16), (3.17) at zero and (3.19), (3.20) at infinity allow us to prove such results under more general assumptions $\alpha^* > 0$ and $\alpha^* = 0, \Delta\gamma + \operatorname{Re}(\mu) < -1$.

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