数理解析研究所講究録
第1581巻 2008年 135-141

FROBENIUS TWISTS, MORITA EQUIVALENCES AND QUANTUM COMPLETE INTERSECTIONS

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Kyoto, August 31, 2007

1. Frobenius twists

Let p be a prime number, let k be an algebraically closed, perfect field of characteristic p, and let $q = p^a$ be a power of p. If A is a k-algebra, we can define the Frobenius twist of A, denoted $A^{(q)}$, as follows. The underlying ring is the same, but we endow it with a new action of the scalars in k via the Frobenius map on k: for $\lambda \in k$ and $x \in A$, the new action is given by $\lambda \cdot x = \lambda^{\frac{1}{q}} x$.

Definition 1.1. We say that a k-algebra A is *defined over* $\mathbb{F}_q$ if there is a k-vector space basis of A such that all the structure constants lie in $\mathbb{F}_q$.

Clearly an algebra A defined over $\mathbb{F}_q$ satisfies $A \cong A^{(q)}$. Conversely, we have the following [8, Lemma 2.1].

Lemma 1.2. *If A is finite dimensional over k, then $A \cong A^{(q)}$ as k-algebras if and only if A is defined over $\mathbb{F}_q$.*

It follows from this lemma that a finite dimensional algebra A is Morita equivalent to $A^{(q)}$ as a k-algebra if and only if the basic algebra of A is defined over $\mathbb{F}_q$.

We define several invariants of a finite dimensional k-algebra A as follows.

Definition 1.3. The *Frobenius number* of A is the least a such that $A^{(p^a)}$ is isomorphic to A. Then $A^{(p^d)}$ is isomorphic to A if and only if d is divisible by a. The *Morita Frobenius number* is the least a such that $A^{(p^a)}$ is Morita equivalent to A, and the *derived Frobenius number* is the least a such that $A^{(p^a)}$ is derived equivalent to A.

In general, if A has only one simple module, then its Frobenius number and its Morita Frobenius number coincide, but the derived Frobenius number may be smaller. However, if A is a symmetric algebra and A has only one simple module, then by a result of Roggenkamp and

Zimmermann the derived Frobenius number of A is equal to its Morita Frobenius number.

2. Frobenius twists of blocks of finite groups

For blocks of a finite group, then there is another way to realize the Frobenius twist, via Galois conjugation via Galois conjugation inside the group algebra. Namely, if G is a finite group, we write $\sigma\colon kG \to kG$ for the map induced by the Frobenius automorphism of k. This is defined by

$$\sigma\Big(\sum_{g\in G}\alpha_g g\Big) = \sum_{g\in G}\alpha_g^p g.$$

The map σ is an isomorphism of rings, but not of k-algebras. Since the blocks of kG are in one to one correspondence with primitive idempotents of the center of kG, σ permutes the blocks of kG. We say that blocks related by powers of σ are *Galois conjugate*. For any block B of kG, σ defines an isomorphism of k-algebras between $B^{(p)}$ and $\sigma(B)$ as k-algebras. In particular, for any block B of a group algebra kG, the Frobenius twist $B^{(p)}$ can be regarded as a Galois conjugate block of the same group algebra kG via this isomorphism.

Morita Frobenius numbers play a role in the various finiteness conjectures in block theory. In what follows, if d is a non-negative integer and B is a block of kG for some finite group G, we will say that B is a d-block if B has defect d.

One of the main outstanding problems in modular representation theory is the following conjecture of Donovan.

Conjecture 2.1. (Donovan) Let d be a non-negative integer. Up to Morita equivalence there are only finitely many k-algebras that occur as d-blocks.

Donovan also conjectured a weaker form of the above which is motivated by Brauer's Problem 22.

Conjecture 2.2. Let d be a non-negative integer. There are only finitely many possibilities for the Cartan entries of d-blocks.

The following conjecture predicts that the Morita Frobenius number of a block should be bounded by the defect of the block.

Conjecture 2.3. For a non-negative integer d, there is an integer m_d, depending only on d such that the Morita Frobenius of any d-block is at most m_d.

It is easy to see that any two Galois conjugate blocks of a finite group have the same defect. Hence, Conjecture 2.1, implies both Conjecture 2.2 and Conjecture 2.3. It turns out that Conjecture 2.3 is exactly the gap between Conjectures 2.1 and 2.2.

Theorem 2.4. *Conjectures 2.2 and 2.3 are together equivalent to Conjecture 2.1.*

This theorem is proved in [8]. For G a finite group, if B is the principal block of kG, then $B = \sigma(B)$ whence the Morita Frobenius number of B is 1. Hence a consequence of Theorem 2.4 is that for principal blocks Conjecture 2.2 and Conjecture 2.1 are equivalent.

In [4], Olaf Düvel has shown that in order to prove Conjecture 2.2, it suffices to prove that it holds for blocks of quasi-simple groups. There is at present no such reduction theorem for Conjecture 2.3. However, as the following results suggest, Morita Frobenius numbers are small in many cases.

Theorem 2.5. *Let G be a finite group and B be a block of kG.*

(a) If G is a finite symmetric or alternating group or a double cover of a finite symmetric or alternating group by a group of order 2, then B has Morita Frobenius number 1.

(b) If G is a finite general linear group $GL_n(q)$ or a finite general unitary group, $GU_n(q)$, where p does not divide q, then B has Morita Frobenius number 1.

(c) If $G = \mathbf{G}^F$, where $\mathbf{G}$ is a connected reductive group whose derived subgroup is simple, and F is the Frobenius map on $\mathbf{G}$ corresponding to an $\mathbb{F}_q$-structure on $\mathbf{G}$ and if B contains an (ordinary) unipotent character of $\mathbf{G}^F$, then B has Morita Frobenius number at most 2. Further, if in addition, $\mathbf{G}$ is one of the classical type A, B , C or D, then B has Morita Frobenius number 1.

(d) If B is of finite or tame representation type and either the defect groups of B are not generalized quaternion groups or the number of isomorphism classes of modular simple representations of G in B is different from 2, then B has Morita Frobenius number 1.

(e) If G is a sporadic simple group and $p = 2$, then the Morita Frobenius number of B is 1.

Proof. Let χ be an ordinary irreducible character of G in B. Then ${}^{\sigma}B$ contains an algebraic conjugate of χ. Hence, the Morita Frobenius number of B is at most $[\mathbb{Q}(\chi) : \mathbb{Q}]$ (and is in fact is a divisor of $[\mathbb{Q}(\chi) : \mathbb{Q}]$), where $\mathbb{Q}(\chi)$ is the smallest subfield (of $\mathbb{C}$) containing $\{\chi(g) : g \in G\}$. Further, if every algebraic conjugate of χ is also in B, then the

Morita Frobenius number of B is 1. Also, note that if for each algebraic conjugate χ' of χ there is an automorphism ϕ of G such that $\chi' = \chi \circ \phi$, then B and ${}^{\sigma}B$ are isomorphic as k-algebras whence the Morita Frobenius number of B is 1.

Since all irreducible characters of a finite symmetric group are rational valued, the assertion of the theorem follows when G is a finite symmetric group. Any pair of algebraically conjugate characters of an alternating group are permuted by the natural action of the symmetric group, hence the assertion of the theorem is valid if G is an alternating group. If G is a double cover of a finite symmetric or alternating group, and if B has non-zero defect, then B contains every algebraic conjugate of χ, so the result follows in this case as well. This proves (a).

Now suppose G is as in (c) and let χ be a unipotent irreducible character of G in B. Then $\mathbb{Q}(\chi) \leq 2$ and if $\mathbf{G}$ is of classical type then $\mathbb{Q}(\chi) = 1$ (see [6, Table 1, Propositiion 5.6] and the proofs thereof). Now (c) is immediate from the above remarks.

Now suppose G is a finite general linear group $GL_n(q)$ or a finite general unitary group $GU_n(q)$ for some prime power q not divisible by p. By [3, Théorém 11.8], B is Morita equivalent to a unipotent block (that is a block containing a unipotent ordinary irreducible character) of a direct product of finite general linear or unitary groups. Unipotent characters of such direct products are rational valued thus proving (b).

Next, we prove (d). If B has cyclic defect groups, then the basic algebras of B are Brauer tree algebras and these all have $\mathbb{F}_p$-forms. If B is of tame representation type, and either the defect groups of B are not generalized quaternion groups or the number of isomorphism classes of modular simple representations of G in B is different from 2, then it follows by Erdmann's work [5] on tame blocks that the basic algebras of B again have $\mathbb{F}_p$-forms (the case that B has generalized quaternion defect groups and two isomorphism classes of modular simples has to be excluded since there are some unknown structure constants in the description of the basic algebra in this case).

Finally suppose that $p = 2$ and that G is a sporadic simple group. Let D be a defect group of B. By [10], either D is cyclic or B is the unique block of kG having D as defect group. This proves (e).

3. Quantum complete intersections

Definition 3.1. A square matrix $\mathbf{q} = (q_{i,j})_{1 \leq i,j \leq r}$, $q_{i,j} \in k$, is called a *commutation matrix* over k if

$$q_{i,j}q_{j,i} = q_{i,i} = 1 \quad \text{for all} \quad 1 \leq i, j \leq r.$$

The *quantum symmetric algebra* $k_{\mathbf{q}}[X_1, \ldots, X_r]$ is defined to be the quotient of the free (tensor) algebra by the commutation relations given by the matrix $\mathbf{q}$:

$$k_{\mathbf{q}}[X_1, \ldots, X_r] := k\langle X_1, \ldots, X_r\rangle/(X_iX_j - q_{i,j}X_jX_i).$$

The *quantum complete intersection algebra* $A_{\mathbf{q}}[X_1, \ldots, X_r]$ is defined to be

$$A_{\mathbf{q}}[X_1, \ldots, X_r] := k_{\mathbf{q}}[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p).$$

Benson and Green [1] showed that many quantum complete intersections arise as basic algebras of blocks of finite groups having one simple module. Let L be an abelian p'-group and consider a central extension

$$1 \to Z \to H \to L \to 1$$

of L by a cyclic p'-group $Z = \langle z\rangle$. Let N be an elementary abelian p-group on which L acts faithfully. Then N is naturally an $\mathbb{F}_pL$-module. Let ϕ be the character of kL on the extension $k \otimes_{\mathbb{F}_p} N$. Since L is an abelian p'-group and k is algebraically closed, ϕ decomposes as a sum of one dimensional kL-modules,

$$\phi = \bigoplus_1^r \phi_i, \tag{3.2}$$

where r is the rank of N.

Inflate the action of L on N to an action of H with Z acting trivially, and let $G = N \rtimes H$ be the semidirect product. The blocks of kG are in one to one correspondence with the characters of Z. Let χ be a faithful irreducible k-character of Z and let

$$b = \frac{1}{|Z|}\sum_{i=1}^{|Z|}\chi(z^{-i})z^i$$

be the corresponding central idempotent in kG. Then b is a block idempotent of kG. Since N is in the kernel of every simple kG-module, the simple kGb-modules are in one to one correspondence with the simple constituents of $\mathrm{Ind}_Z^H \chi$. In particular kGb has one isomorphism type of simple module if and only if $\mathrm{Ind}_Z^H \chi$ is a direct sum of isomorphic irreducibles. This happens if and only if $kHb \cong \mathrm{Mat}_d(k)$, where d is the dimension of the corresponding irreducible, and in this case, $|L| = \dim_k \mathrm{Mat}_d(k) = d^2$ is a square. In this situation the basic algebra of kGb is a quantum complete intersection.

Theorem 3.3. *Suppose that* $kHb \cong \mathrm{Mat}_d(k)$. *Then*

(i) *The map*

$$L \to \mathrm{Hom}(L, k^\times) \qquad l \mapsto (t \mapsto \chi([\tilde{l}, \tilde{t}])), \qquad l, t \in L$$

(where $\tilde{l}$ and $\tilde{t}$ are any lifts of l and t respectively in H) is an isomorphism.

(ii) *For each i, $1 \leq i \leq r$ let m_i be the element of L corresponding through* (i) *to the character ϕ_i appearing in equation* (3.2). *For $1 \leq i, j \leq r$, set $q_{i,j} = \chi[\tilde{m}_j, \tilde{m}_i]$ and set $\mathbf{q} = (q_{i,j})$. Then the basic algebra of kGb is isomorphic to $A_{\mathbf{q}}[X_1, \ldots, X_r]$.*

The above result follows from [7, Theorem 4.2] and the proof thereof.

Remark. The above result along with Külshammer's structure theory for blocks with normal defect group implies the following: If Broué's abelian defect group conjecture is true, then any block of a finite group algebra having one isomorphism class of simple modules, elementary abelian defect groups, and abelian inertial quotient is Morita equivalent to a quantum complete intersection.

Theorem 3.4. [2, Theorem 1.5,Remark 3.3 Examples 5.1 and 5.2]
(i) If L, H, N, G, and b satisfy the hypothesis of Theorem 3.3, then the Morita Frobenius number of the block kGb is at most 2.
(ii) For each prime p, there exist L, H, N, G, and b satisfying the hypotheses of Theorem 3.3 such that such that the Morita Frobenius number of kGb is exactly 2.

Corollary 3.5. *A block kGb as described in Theorem 3.4(ii) also has derived Frobenius number 2.*

Remarks (i) As of yet, the blocks as in Theorem 3.4(ii) are the only known examples of blocks of finite groups whose Morita Frobenius number is greater than 1. In particular, we do not know of any examples of blocks whose Morita Frobenius number is at least three.

(ii) By [?, Theorem 2.1] any pair of Galois conjugate blocks are isotypic (with all signs +1), hence Corollary 3.5 provides examples of pairs isotypic blocks which are not derived equivalent.

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