

On Regular Factors

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In this paper we consider graphs which may have loops and multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G , respectively. The degree of a vertex x of G is denote by $d_G(x)$. Let S, T be disjoint subsets of $V(G)$. $e_G(S, T)$ is the number of edges which join vertices of S and T , and $e_G(S)$ is the number of edges which join vertices of S . The other notations which are not explained explicitly are due to [3].

Let f be an integer-valued function defined on $V(G)$. A spanning subgraph H of G is called an f -factor of G if $d_H(x) = f(x)$ for any $x \in V(G)$. If f is the constant function taking the value k , the f -factor is said to be a k -factor or a k -regular-factor.

In this paper we shall give a sufficient condition for the existence of regular factors of regular graphs. This condition are related to the edge-connectivity of

graphs. Such a research goes back to Petersen [4]. Various results are obtained by Petersen, Gallai, Plesnik et al., which are listed below. The main theorem in this paper is an extension of Proposition E (Plesnik [6]).

Proposition A. (Petersen [4]) *If G is $2k$ -regular ($k \geq 1$), G has an l -factor for any even integer l such that $0 \leq l \leq 2k$. $\square$*

Proposition B. (Petersen [4]) *If G is 3-regular and 2-edge-connected, G has a 2-factor and a 1-factor. $\square$*

Proposition C. (Berge [1], Plesnik [5]) *If G is r -regular, $(r-1)$ -edge-connected and $|G|$ is even, then G has a 1-factor and an $(r-1)$ -factor. $\square$*

Proposition D. (Gallai [2], Plesnik [6]) *Let G be a $2k$ -regular ($k \geq 1$) and a -edge-connected graph. Then G has an l -factor for any odd integer l satisfying $\frac{1}{a}2k \leq l \leq \frac{a-1}{a}2k$. $\square$*

Proposition E. (Plesnik [6]) *Let G be a $(2k+1)$ -regular ($k \geq 1$) and a -edge-connected graph. Then G has an l -factor for any even integer l satisfying $0 \leq l \leq \frac{a-1}{a}(2k+1)$. $\square$*

Remark that the edge-connectivity of an even-regular graph is even. Hence when r is even, Proposition C is a special case of Proposition D. But if r is odd, Proposition C (and Proposition B) cannot be deduced from Proposition E.

Tutte obtained a criterion for the existence of an f -factor. We use it to obtain our theorems.

Lemma 1. (Tutte [7,8,9]) *A graph G has an f -factor if and only if*

$$h(S, T) + \sum_{t \in T} \{f(t) - d_{G-S}(t)\} \leq \sum_{s \in S} f(s) \quad (1)$$

for any disjoint subsets S, T of $V(G)$, where $h(S, T)$ is the number of components C of $G - (S \cup T)$ such that

$$\sum_{c \in V(C)} f(c) + e_G(V(C), T) \equiv 1 \pmod{2}. \quad \square$$

Here we use (1) in the following form.

$$h(S, T) + e_G(S, T) \leq \sum_{s \in S} f(s) + \sum_{t \in T} d_G(t) - \sum_{t \in T} f(t). \quad (1)'$$

First we improve the evaluation in Proposition E.

Theorem 2. *Let G be a $(2k+1)$ -regular ($k \geq 1$) and α -edge-connected graph.*

If α is even, G has an l -factor for any even integer l satisfying

$$0 \leq l \leq \frac{\alpha}{\alpha+1}(2k+1).$$

To prove theorem 2, we need some notations and lemmas.

Let G be a $(2k+1)$ -regular graph. For $S, T \subset V(G)$ ($S \cap T = \emptyset$) and an integer l , define $\delta(S, T; l)$ by:

$$\delta(S, T; l) = l|S| + (2k - l + 1)|T| - h(S, T) - e_G(S, T).$$

Obviously G has an l -factor if and only if $\delta(S, T; l) \geq 0$ for any $S, T \subset V(G)$

such that $S \cap T = \emptyset$. Let $H(S, T)$ be the set of components C of $G - (S \cup T)$

such that $e_G(V(C), T) \equiv 1 \pmod{2}$. Suppose l be an even integer. Then

$h(S, T) = |H(S, T)|$. Let V_0 be a set of vertices of $G - (S \cup T)$ which do not

belong to components of $H(S, T)$. Define m_1, m_2, n_1, n_2 and N by

$$m_1 = e_G(V_0, S), \quad m_2 = e_G(S), \quad n_1 = e_G(V_0, T), \quad n_2 = e_G(T) \quad \text{and} \quad N = e_G(S, T).$$

Remark that n_1 is even. We shall write $H(S, T) = \{C_1, \dots, C_r\}$ ($r = h(S, T)$) and set $s_i = e_G(V(C_i), S)$ and $t_i = e_G(V(C_i), T)$.

Lemma 3.

$$(2k+1)\delta(S, T; l) = lm_1 + 2lm_2 + (2k-l+1)n_1 + 2(2k-l+1)n_2 \\ + \sum_{i=1}^r (ls_i + (2k-l+1)t_i - (2k+1)).$$

Proof. Considering the sum of degrees of the vertices in S ,

$$(2k+1)|S| = \sum_{i=1}^r s_i + m_1 + 2m_2 + N.$$

Similarly,

$$(2k+1)|T| = \sum_{i=1}^r t_i + n_1 + 2n_2 + N.$$

From these two equations and the definition of $\delta(S, T; l)$, we obtain Lemma 3.

□

Proof of Theorem 2.

If $0 \leq l \leq \frac{a}{a+1}(2k+1)$, $lm_1 + 2lm_2 + (2k-l+1)n_1 + 2(2k-l+1)n_2 \geq 0$. In case

$S \cup T = \phi$, it is obvious that $(2k+1)\delta(S, T; l) \geq 0$. We shall show that

$\psi(s_i, t_i) \geq 0$ if $S \cup T \neq \phi$, where $\psi(s_i, t_i) = ls_i + (2k-l+1)t_i - (2k+1)$. From the

assumptions, we obtain the following.

$$s_i \geq 0. \tag{2}$$

$$t_i \equiv 1 \pmod{2}, \text{ especially } t_i \geq 1. \tag{3}$$

$$s_i + t_i \geq a. \tag{4}$$

If $s_i \geq 1$, $\psi(s_i, t_i) \geq l + (2k-l+1) - (2k+1) = 0$, since $t_i \geq 1$. Hence we may confine

ourselves to the case $s_i = 0$. Assume $(2k-l+1)t_i - (2k+1) < 0$, then

$$t_i < \frac{2k+1}{2k-l+1} \leq a+1,$$

since $0 \leq l \leq \frac{a}{a+1}(2k+1)$. Hence $t_i \leq a$. But since t_i is odd and a is even,

$t_i \leq a-1$. This contradicts (4). Therefore we obtain $\delta(S, T; l) \geq 0$ and G has an l -factor.

□

Next we assure that the evaluation of theorem 2 is the best.

Theorem 4. *For any integer k and even integer a satisfying $0 \leq a \leq 2k+1$, there exists a $(2k+1)$ -regular and a -edge-connected graph which does not have an l -factor for any even integer l satisfying $l > \frac{a}{a+1}(2k+1)$.*

Proof. We shall construct a graph $G(a, k)$ which has the required property. $G(a, k)$ has subgraphs $H_i (1 \leq i \leq 2k+1)$ and $J_j (1 \leq j \leq a+1)$. First we define H_i and J_j and then construct $G(a, k)$.

(i) $H_i (1 \leq i \leq 2k+1)$.

H_i has $(2k+3)$ vertices, say

$$V(H_i) = \{x_{i,1}, \dots, x_{i,2k+3}\}.$$

Let $\tilde{H}_i$ be a complete graph with the vertex set $V(H_i)$ and P_i be a path defined by $P_i = x_{i,1} x_{i,2} \dots x_{i,a+2}$. We define $E(H_i)$ as follows.

$$E(H_i) = E(\tilde{H}_i) - E(P_i) - x_{i,a+3}x_{i,a+4} \\ - x_{i,a+5}x_{i,a+6} - \dots - x_{i,2k+1}x_{i,2k+2} - x_{i,2k+3}x_{i,1}.$$

H_i has the following properties.

$$d_{H_i}(x_{i,j}) = \begin{cases} 2k & 1 \leq j \leq a+1 \\ 2k+1 & j \geq a+2 \end{cases}$$

H_i is $2k$ -edge-connected.

(ii) $J_j (1 \leq j \leq a+1)$

J_j is a complete bipartite graph with partite sets

$$Y_j = \{y_{j,1}, \dots, y_{j,2k+1}\} \text{ and } Z_j = \{z_{j,1}, \dots, z_{j,2k}\}.$$

Obviously $d_{J_j}(y_{j,l}) = 2k$, $d_{J_j}(z_{j,l}) = 2k+1$ and J_j is $2k$ -edge-connected.

(iii) Joining $x_{i,j}$ in $V(H_i)$ with $y_{j,i}$ in Y_j ($1 \leq i \leq 2k+1, 1 \leq j \leq a+1$), we obtain $G(a, k)$. Obviously $G(a, k)$ is $(2k+1)$ -regular and $\min\{a+1, 2k\}$ -edge-connected. Hence G is a -edge-connected (since $0 \leq a \leq 2k+1$ and a is even).

Let $S = \bigcup_{j=1}^{a+1} Z_j$, $T = \bigcup_{j=1}^{a+1} Y_j$. We shall show $\delta(S, T; l) < 0$ if $l > \frac{a}{a+1}(2k+1)$

and l is even. In this case,

$$H(S, T) = \{H_1, \dots, H_{2k+1}\}$$

$$m_1 = m_2 = n_1 = n_2 = 0, s_i = 0, t_i = a+1$$

So by Lemma 9, $\delta(S, T; l) = (2k-l+1)(a+1) - (2k+1)$. It is easy to show that

$(2k-l+1)(a+1) - (2k+1) < 0$ if $l > \frac{a}{a+1}(2k+1)$. Therefore G does not have

an l -factor. $\square$

If a is odd, the bound of Proposition E is the best.

Theorem 5. For any integer k and odd integer a satisfying $1 \leq a \leq 2k+1$, there exists a $(2k+1)$ -regular and a -edge-connected graph which does not have an l -factor for any even integer l satisfying $l > \frac{a-1}{a}(2k+1)$.

Proof. If $a = 2k+1$, then $l > \frac{a-1}{a}(2k+1) = 2k$ and the theorem follows.

We consider the case $a \leq 2k-1$. Since a is odd, $G(a-1, k)$, defined in the

proof of theorem 9, is $2k$ -regular, a -edge-connected and does not have an l -factor for any even integer l satisfying $l > \frac{a-1}{a}(2k+1)$. $\square$

When the regularity of graphs is even and the edge-connectivity is odd, Proposition D gives the best bound.

Theorem 6. *For any integer k and any even integer a satisfying $1 \leq a \leq 2k$, there exists a $2k$ -regular and a -edge-connected graph G with even number of vertices which does not have an l -factor for any odd integer l such that $l < \frac{1}{a}2k$ or $l > \frac{a-1}{a}2k$.*

Proof. If a $2k$ -regular graph G has an l -factor H for some l such that $l > \frac{a-1}{a}2k$, $G-H$ is a $(2k-l)(< \frac{1}{a}2k)$ -factor of G . Hence it suffices to show that some graph G does not have an l -factor for any l such that $l < \frac{1}{a}2k$. Clearly the result follows if $a=2k$. If $a \leq 2k-2$, we construct a graph $G'(a, k)$ as follows. $G'(a, k)$ has $H'_i (a \leq i \leq 2k)$ and $J'_j (1 \leq j \leq a)$ as subgraphs.

(i) $H'_i (1 \leq i \leq 2k)$

$V(H'_i) = \{x_{i,1}, \dots, x_{i,2k+1}\}$. Let $\tilde{H}'_i$ be a complete graph with the vertex set $V(H'_i)$. Then we define $E(H'_i)$ by

$$E(H'_i) = E(\tilde{H}'_i) - x_{i,1}x_{i,2} - x_{i,3}x_{i,4} - \dots - x_{i,a-1}x_{i,a}.$$

(ii) $J'_j (1 \leq j \leq a)$

J'_j is a complete bipartite graph with partite sets $Y'_j = \{y_{j,1}, \dots, y_{j,2k}\}$ and $Z'_j = \{z_{j,1}, \dots, z_{j,2k-1}\}$.

(iii) Combining $x_{i,j}$ of $V(H_i)$ with $y_{j,i}$ of Y'_j ($1 \leq i \leq 2k, 1 \leq j \leq a$), we obtain $G'(a, k)$. Clearly $|G|$ is even and it is easy to show that G is $2k$ -regular and a -edge-connected. Set

$$\delta'(S, T; l) = l|S| + (2k - l)|T| - h(S, T) - e_G(S, T)$$

for $S, T \subset V(G)$ ($S \cap T = \emptyset$). If $\delta'(S, T; l) < 0$ for some disjoint subsets S, T of $V(G)$, then G does not have an l -factor. If we set $S = \bigcup_{j=1}^a Y'_j$ and $T = \bigcup_{j=1}^a Z'_j$, we obtain $\delta(S, T; l) = al - 2k$ in the same way as in the proof of theorem 9, and $al - 2k < 0$ if $l < \frac{1}{a}2k$. Hence G does not have an l -factor for any even integer l such that $l < \frac{1}{a}2k$ or $l > \frac{a-1}{a}2k$. $\square$

From Propositions A, D, E, Theorems 2, 4, 5, 6, the best possible bounds of regularity of regular factors in regular graphs are obtained. In these theorems, only the regularity and edge-connectivity of graphs are given. If more properties about graphs are assumed, better bounds can be obtained. For example, graphs which are not 2-edge-connected can have 1-factors and 2-factors.

Proposition F. (Berge [1]) *If G is 3-regular and all of bridges of G are on the same elementary path, then G has a 1-factor and 2-factor.*

We extend the above theorem.

Theorem 7. *Let G be a $(2k+1)$ -regular and $(a-1)$ -edge-connected graph ($a \geq 1$). Suppose there are disjoint subsets $S_1, \dots, S_r$ of $V(G)$ satisfying:*

$$V(G) = \bigcup_{i=1}^r S_i$$

$\langle S_i \rangle_G$ is a -edge-connected.

$$e_G(S_i, S_{i+1}) = a-1 \quad (1 \leq i \leq r-1),$$

$$e_G(S_i, S_j) = 0 \quad \text{if } |i-j| > 1.$$

If a is even, then G has an l -factor for any even integer l satisfying

$$0 \leq l \leq \frac{a}{a+1}(2k+1) \quad (\text{cf. Theorem 2}).$$

Proof.

Suppose G does not have an l -factor for some even integer l such that

$$0 \leq l \leq \frac{a}{a+1}(2k+1). \quad \text{Then there exist some } S, T \subset V(G), \quad S \cap T = \emptyset, \quad \text{such that}$$

$$\delta(S, T; l) < 0. \quad \text{Hence by Lemma 3, } \sum_{i=1}^r \psi(s_i, t_i) < 0. \quad \text{From the assumptions of}$$

this theorem at most two pairs (s_i, t_i) satisfy the condition $s_i + t_i = a-1$ and

$s_j + t_j \geq a$ for the other pairs (s_j, t_j) . If $s_j + t_j \geq a$, $\psi(s_j, t_j) \geq 0$. (See the proof of

Theorem 7.). Hence we may assume $\psi(s_j, t_j) \geq 0$ ($1 \leq j \leq r-2$), $s_r + t_r = a-1$ and

$\psi(s_r, t_r) < 0$. Then $s_r = 0$ and $t_r = a-1$. If $\psi(s_{r-1}, t_{r-1}) \geq 0$,

$$\begin{aligned} 0 &> (2k+1)\delta(S, T; l) \geq \psi(s_r, t_r) = (2k-l+1)(a-1) - (2k+1) \\ &> 2k+1, \end{aligned}$$

since $2k-l+1, a-1 > 0$. This is a contradiction, because $\delta(S, T; l)$ is an integer.

Thus $\psi(s_{r-1}, t_{r-1}) < 0$, $s_{r-1} + t_{r-1} = a-1$ and

$\psi(s_{r-1}, t_{r-1}) = \psi(s_r, t_r) = (2k-l+1)(a-1) - (2k+1)$. Therefore

$$0 > (2k+1)\delta(S, T; l) > -2(2k+1)$$

and $\delta(S, T; l) = -1$. On the other hand, $\psi(s_i, t_i) \equiv 0 \pmod{2}$, since l is even and

t_i is odd. Hence $\delta(S, T; l) \equiv n_1 \pmod{2}$. But by the Definition of n_1 , n_1 is an

even integer. This is a contradiction. $\square$

By the same argument the following theorem can be obtained.

Theorem 8. Let G be a $(2k+1)$ -regular and $(a-1)$ -edge-connected graph ($a \geq 1$). $\{S_1, \dots, S_r\}$ is the partition of $V(G)$ which has the same properties as in the previous theorem. If a is odd, then G has an l -factor for any even integer l satisfying $0 \leq l \leq \frac{a-1}{a}(2k+1)$ (cf. Proposition E). $\square$

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