

Y_{555} and related topics

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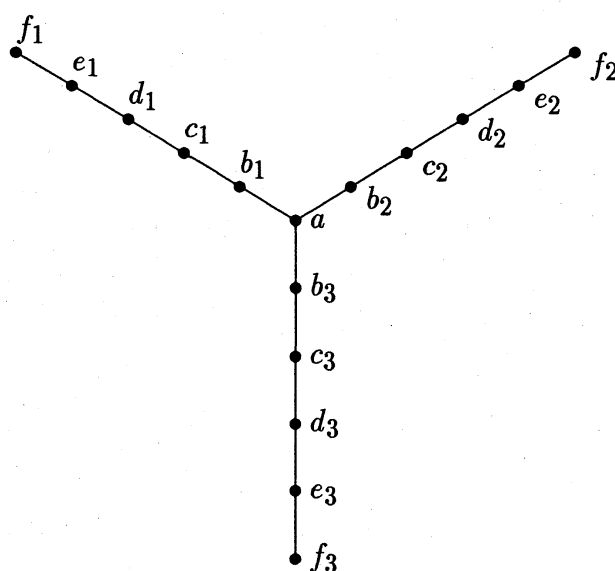
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In this note, we will introduce some idea to study the Y_{555} group given in the paper

[CP] J.H.Conway and A.D.Pritchard : Hyperbolic reflections for the Bimonster and $3Fi_{24}$
in "Groups, Combinatorics and Geometry", Cambridge, 1992

and will report some observations together with some questions.

We denote the Monster simple group by $\mathbb{M}$, and the wreath product $\mathbb{M} \wr 2$ is called the Bimonster. The following diagram is called Y_{555} , and is regarded as a Coxeter diagram which gives 16 generators and some relations among them.



First we will collect some theorems on the presentation of the Bimonster.

Theorem A.

$$\mathbb{M} \wr 2 \cong \langle Y_{555}, (ab_1c_1ab_2c_2ab_3c_3)^{10} = 1 \rangle$$

Theorem B. Suppose that the group G is a minimal group that possesses an S_5 -subgroup S whose centralizer is isomorphic to S_{12} in which a 7 point stabilizer is conjugate to S .

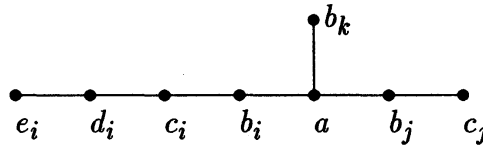
Then $G \cong S_{17}$ or the Bimonster $\mathbb{M} \wr 2$.

Theorem C.

$$\mathbb{M} \wr 2 \cong \langle Y_{555}, f_i = f_{ij}(i, j = 1, 2, 3, i \neq j) \rangle$$

$$f_{ij} = (ab_i c_i d_i b_j c_j b_k)^9, \{i, j, k\} = \{1, 2, 3\}$$

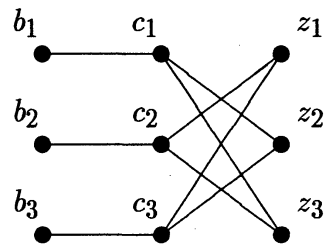
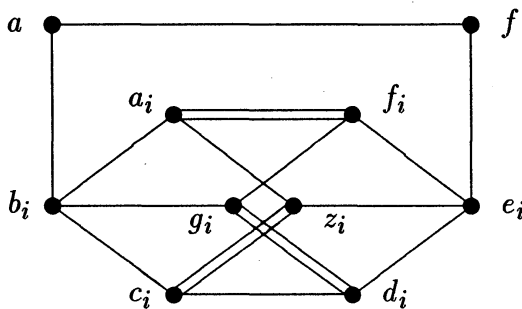
Remark. f_{ij} corresponds with the root of the highest height of the E_8 -lattice:



We will call such a relation an E_8 -relation.

Theorem D. (The 26 node theorem)

The bimonster $\mathbb{M} \wr 2$ contains 26 involutions, including the generators in Y_{555} , satisfying the Coxeter relations of the incidence graph of the projective plane of order 3.



In [CP], Conway and Pritchard defined the Monster roots, which are some vectors defined in the 16 dimensional space with the 19 coordinates

$$v = \begin{pmatrix} a & b & c & d & e & f \\ g & h & i & j & k & l & t \\ m & n & o & p & q & r \end{pmatrix}$$

with the quadratic form

$$a^2 + b^2 + \dots + q^2 + r^2 - t^2$$

and the 3 relations

$$\begin{cases} a + b + c + d + e + f = t \\ g + h + i + j + k + l = t \\ m + n + o + p + q + r = t. \end{cases}$$

For a vector x , the reflection r_x is

$$r_x : y \rightarrow y - \frac{2 \langle y, x \rangle}{\langle x, x \rangle} x,$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

The *fundamental Monster roots* are the vectors

$$a, b_i, c_i, d_i, e_i, f_i \ (i = 1, 2, 3)$$

given in Table 1. (In general, the term 'root' means a vector of squared length 2.) We denote by Π the set of the fundamental Monster roots. The reflections r_x ($x \in \Pi$) satisfy the relation given by the diagram Y_{555} .

The (infinite) group G is defined by

$$G = \langle r_x \mid x \in \Pi \rangle.$$

Then by Theorem A, there exists some normal subgroup N of G such that G/N is isomorphic to the bimonster $\mathbb{M} \wr 2$.

The *Monster roots* are the vectors in the G -orbit Π^G . We will define an equivalence relation between the Monster roots.

Definition.

$$x \doteq y \ (x, y \in \Pi^G) \iff \bar{r}_x = \bar{r}_y \in G/N \cong \mathbb{M} \wr 2$$

The following equivalence is a key of [CP].

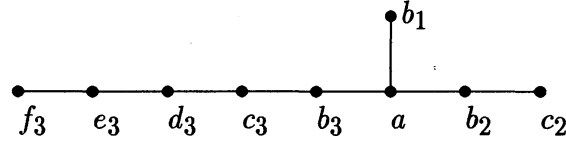
Proposition 1. (Theorem 2 of [CP])

$$v = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 & 2 & 2 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \doteq \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & \bar{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = b_1$$

Notice that the sum of the vectors

$$v + b_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 & 2 & 2 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = f_3 + 2e_3 + 3d_3 + 4c_3 + 5b_3 + 6a + 4b_2 + 2c_2 + 3b_1,$$

and this equals the primitive isotropic vector of the following $\tilde{E}_8$ -lattice.



Hence Proposition 1 is equivalent to

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 & 2 & 2 & 6 \\ 0 & 2 & 1 & 1 & 1 & 1 \end{pmatrix} \doteq \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \bar{1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This means $f_{32} = f_3$, one of the E_8 -relations in Theorem C.

Now a simple question is "What does happen on an E_6 - or E_7 -diagram in Y_{555} ?"

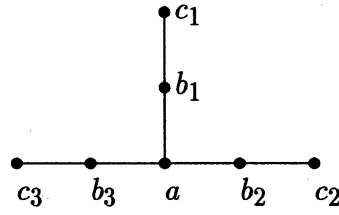
The following equivalence (we call it an E_6 -relation) is proved in [CP].

Proposition 2.(Theorem 4 of [CP])

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 5 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{pmatrix} \doteq \begin{pmatrix} 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 & 4 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{pmatrix}$$

The sum equals $3 \times$ (the primitive isotropic vector of $\tilde{E}_6$ -lattice).

$$\begin{pmatrix} 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 & 9 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{pmatrix}$$



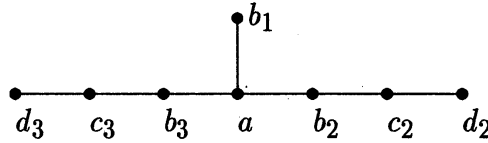
Similarly the following " E_7 -relation" can be proved by using the relations in [CP].

Proposition 3.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{pmatrix} \doteq \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{pmatrix}$$

The sum equals $2 \times$ (the primitive isotropic vector of $\tilde{E}_7$ -lattice).

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 2 & 2 & 2 & 8 \\ 0 & 0 & 2 & 2 & 2 & 2 \end{pmatrix}$$



We will explain the above relations by using the orders of the products of some reflections.

Let $\tilde{X} = X \cup \{e\}$ be a subset of Π such that the diagram of $\tilde{X}$ is an affine diagram $\tilde{E}_n, n = 6, 7$ or 8 , and the diagram of X is a spherical diagram E_n . Then we write $e = e(\tilde{X})$ and denote by $h = h(X)$ the root of the highest height of X .

Then the sum $e + h$ is a primitive isotropic vector. Moreover $\langle e, h \rangle = -2$, and $r_e(h) = h + 2e$.

The E_6 -relation is equivalent to $h + 2e \doteq e + 2h$ and

$$h + 2e \doteq e + 2h \iff r_e(h) = r_h(e) \iff |r_e r_h| = 3.$$

The E_7 -relation is equivalent to $h + 2e \doteq h$ and we have

$$h + 2e \doteq h \iff r_e(h) = h \iff |r_e r_h| = 2.$$

The E_8 -relation is also equivalent to $h \doteq e$ and this means

$$h \doteq e \iff r_e = r_h \iff |r_e r_h| = 1.$$

Remark. The numbers 3, 2, 1 are the determinants of E_6, E_7, E_8 -lattice. *Is there any mathematical background ?*

Theorem C shows that

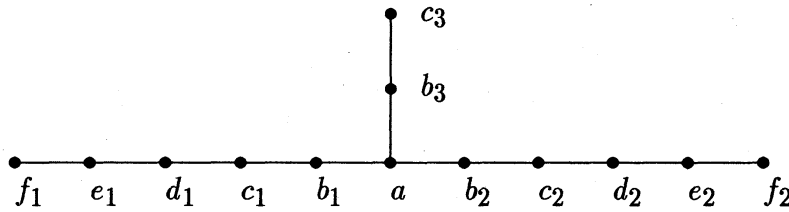
$$\mathbb{M} \wr 2 \cong \langle Y_{555}, E_8\text{-relations}(\forall E_8 \subset Y_{555}) \rangle.$$

The following is a natural question.

Question.

$$\langle Y_{555}, E_7, E_6\text{-relations}(\forall E_7, E_6 \subset Y_{555}) \rangle = ?$$

Remark. The orthogonal group $O(11, 3)$ contains



satisfying the E_6 -relation and $f_1 = f_{13}, f_2 = f_{23}$, (and $(ab_1c_1ab_2c_2ab_3c_3)^{30} = 1$).

It is known that

$$3.Fi_{24} \cong \langle Y_{552}, f_1 = f_{13} = f_{12}, f_2 = f_{23} = f_{21} \rangle.$$

Finally we consider the affine diagrams $\tilde{X}$ contained in 26 node system given in Theorem D. The 26 vectors are listed in Table 1 ([CP]). By using them, we can easily calculate the orders $|r_{e(\tilde{X})}r_{h(X)}|$ in $G/N \cong \mathbb{M} \wr 2$.

The cases (1)-(3) are contained in Y_{555} , the set Π of fundamental Monster roots. We treated them in Propositions 1-3.

The cases (4)-(13) are not contained in Y_{555} . There is no diagram which gives a new relation.

An interesting fact is that for any X, Y of (4)-(13),

$$|r_{e(\tilde{X})}r_{h(X)}| \leq |r_{e(\tilde{Y})}r_{h(Y)}| \iff c(X) \geq c(Y)$$

where we denote by $c(X)$ the Coxeter number of X .

Our final question is "Is there any mathematical background ?"

(Table 2) The affine diagrams $\tilde{X}$ and the order $|r_{e(\tilde{X})}r_{h(X)}|$

(1)	$\tilde{E}_6$	:	3		E_6 -relation
(2)	$\tilde{E}_7$	:	2		E_7 -relation
(3)	$\tilde{E}_8$	:	1		E_8 -relation
(4)	$\tilde{D}_4$	:	3	$\iff$	(1)
(5)	$\tilde{A}_5$	:	3	$\iff$	(1)
(6)	$\tilde{D}_5$	:	3	$\iff$	(1)
(7)	$\tilde{A}_7$	:	2	$\iff$	(2)
(8)	$\tilde{D}_6$	:	2	$\iff$	(2)
(9)	$\tilde{A}_9$	:	2	$\iff$	(2)
(10)	$\tilde{E}_6$	:	2	$\iff$	(2)
(11)	$\tilde{A}_{11}$	:	1	$\iff$	(3)
(12)	$\tilde{D}_8$	:	1	$\iff$	(3)
(13)	$\tilde{E}_7$	:	1	$\iff$	(3)