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ON LINEAR RELATIONS BETWEEN L-VALUES AND ARITHMETIC FUNCTIONS

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1. Results from Siegel

Let $F=\mathbb{Q}(\sqrt{D})$ be a quadratic field with discriminant $D>0$, ring of integers $\mathfrak{o}$ and different $\mathfrak{d}$. For integer $k\geq 2$ and B an ideal class of F, we define the Eisenstein series

$$E_{k,B}(z)=\frac{1}{4}\zeta_{\mathfrak{d}B}(1-k)+\sum_{\substack{\xi\in\mathfrak{d}^{-1}\\ \xi\succ 0}}\sigma_{k-1,\mathfrak{d}B}(\xi\mathfrak{d})q^{\xi}$$

of weight k and level $\mathrm{SL}_2(\mathfrak{o})$, where $z\in\mathfrak{h}^2$ and $q^{\xi}=\exp(2\pi\sqrt{-1}\mathrm{Tr}(\xi z))$. If we let

$$\mathcal{R}E_{k,B}(\tau)=E_{k,B}(\tau,\tau)$$

for $\tau\in\mathfrak{h}$, then $\mathcal{R}E_{k,B}\in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. It is easy to see that

$$\mathcal{R}E_{k,B}(\tau)=\frac{1}{4}\zeta_{\mathfrak{d}B}(1-k)+\sum_{n=1}^{\infty}\left(\sum_{\substack{\xi\in\mathfrak{d}^{-1}\\ \xi\succ 0\\ \mathrm{Tr}(\xi)=n}}\sigma_{k-1,\mathfrak{d}B}(\xi\mathfrak{d})\right)q^{n},$$

where $q^n=\exp(2\pi\sqrt{-1}n\tau)$. Thus by comparing the Fourier coefficients, we have the following results.

Theorem 1 (Siegel). *If $k=2,4$, then*

$$\zeta_B(1-k)=-\frac{B_{2k}}{k}\sum_{\substack{\xi\in\mathfrak{d}^{-1}\\ \xi\succ 0\\ \mathrm{Tr}(\xi)=1}}\sigma_{k-1,B}(\xi\mathfrak{d}).$$

Corollary 1. *We have*

$$\zeta_F(-1)=\frac{1}{60}\sum_{\substack{m\in\mathbb{Z}\\ m^2<D\\ m^2\equiv D \mod 4}}\sigma_1\left(\frac{D-m^2}{4}\right)$$

and

$$\zeta_F(-3) = \frac{1}{120} \sum_{\substack{m\in\mathbb{Z} \\ m^2<D \\ m^2\equiv D \mod 4}} \sigma_3\left(\frac{D-m^2}{4}\right).$$

Today, we want to use the same technique on the Hilbert forms of half-integral weight.

2. Main results

Consider the same F as in the last section. Put

$$\omega = \begin{cases} \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \mod 4, \\ \frac{\sqrt{D}}{2} & \text{otherwise.} \end{cases}$$

Then $\mathfrak{o} = \mathbb{Z} + \omega\mathbb{Z}$. For any two ideals $\mathfrak{b}$ and $\mathfrak{c}$ of F such that $\mathfrak{bc} \subset \mathfrak{o}$, we let

$$\Gamma[\mathfrak{b}, \mathfrak{c}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \,\middle|\, a, d \in \mathfrak{o}, b \in \mathfrak{b}, c \in \mathfrak{c} \right\}$$

and put $\Gamma = \Gamma[\mathfrak{d}^{-1}, 4\mathfrak{d}]$.

Definition 1. *For any $z \in \mathfrak{h}^2$, we put*

$$\theta(z) = \sum_{\xi\in\mathfrak{o}} q^{\xi^2} \quad (z \in \mathfrak{h}^2)$$

The factor of automorphy of weight $1/2$ is given by

$$\tilde{j}(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}$$

where $\gamma \in \Gamma$.

Let $k \geq 0$. A Hilbert modular form of parallel weight $k + 1/2$ and level Γ is a holomorphic function f on $\mathfrak{h}^2$ which satisfies

$$f(\gamma z) = \tilde{j}(\gamma, z)^{2k+1} f(z)$$

for any $\gamma \in \Gamma$. The space of all such forms is denoted by $M_{k+1/2}(\Gamma)$ and the subspace consisting cusp forms in it is denoted by $S_{k+1/2}(\Gamma)$.

Now suggest that the different $\mathfrak{d}$ has a totally positive generator $\boldsymbol{\delta}$. Put $\Gamma' = \Gamma[\mathfrak{o}, 4\mathfrak{o}]$. A Hilbert modular form of parallel weight $k + 1/2$ and level Γ' is a function with the form

$$f_0(z) = f(\boldsymbol{\delta}^{-1} z)$$

where $f \in M_{k+1/2}(\Gamma)$. It satisfies the automorphic condition

$$f_0(\gamma z) = \tilde{j}_0(\gamma, z)^{2k+1} f_0(z)$$

where $\gamma \in \Gamma'$ and $\tilde{j}_0$ comes from θ_0 as for $\tilde{j}$. The space of all such forms and cusp forms are dentoed by $M_{k+1/2}(\Gamma')$ and $S_{k+1/2}(\Gamma')$, respectively.

Note that the congruence subgroup $\Gamma_0(4) \subset \mathrm{SL}_2(\mathbb{Z})$ can be embedded into Γ' diagonally. One can show that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we have

$$\tilde{j}_0\left(\gamma, (\tau, \tau)\right) = (c\tau + d)\chi_{-4}(d)$$

where $\tau \in \mathfrak{h}$. Thus if we put

$$\mathcal{R}f(\tau) = f_0\left((\tau, \tau)\right)$$

where $f \in M_{k+1/2}(\Gamma)$ and $\tau \in \mathfrak{h}$, then

$$\mathcal{R}f \in M_{2k+1}(\Gamma_0(4), \chi_{-4}).$$

We write $\boldsymbol{\delta}$ in the form

$$\boldsymbol{\delta} = (\alpha + \beta\omega)\sqrt{D}$$

with $\alpha+\beta\omega > 0$ a unit of norm -1. Then if f has the Fourier expansion

$$f(z) = \sum_{\xi \in \mathfrak{o}} c(\xi) q^{\xi} \quad (z \in \mathfrak{h}^2),$$

we can write down the Fourier expansion of $\mathcal{R}f$ explicitly as

$$\mathcal{R}f(\tau) = \sum_{n=0}^{\infty} \left(\sum_{\substack{(a,b) \in \mathbb{Z}^2 \\ a\beta - b\alpha = n}} c(a + b\omega) \right) q^n \quad (\tau \in \mathfrak{h}).$$

If we apply the mapping $\mathcal{R}$ on θ^2, we get the following result.

Corollary 2. *For $m > 0$ and number field K we set*

$$r_{K,m}(x) = \#\left\{ (x_1, x_2, \ldots, x_m) \in \mathfrak{o}_K^m \;\middle|\; x_1^2 + \cdots + x_m^2 = x \right\} \quad (x \in K).$$

Then with the notations given above, we have

$$\sum_{\substack{(a,b) \in \mathbb{Z}^2 \\ a\beta - b\alpha = n}} r_{F,m}(a + b\omega) = r_{\mathbb{Q},2m}(n).$$

From now let us consider the case for Kohnen plus space. The specific spaces were first defined by Kohnen [3] in 1980 and generalized to the case for Hilbert modular forms by Hiraga and Ikeda [2] in 2013. For any $\xi \in F$, we denote by

$$\xi \equiv \square \mod 4$$

if there exists $\lambda \in \mathfrak{o}$ such that $\xi - \lambda^2 \in 4\mathfrak{o}$.

Definition 2. *The Kohnen plus space is a subspace of $M_{k+1/2}(\Gamma)$ defined as*

$$M^+_{k+1/2}(\Gamma) = \left\{ f(z) = \sum_{\xi} c(\xi) q^{\xi} \in M_{k+1/2}(\Gamma) \,\middle|\, c(\xi) = 0 \text{ unless } (-1)^k \xi \equiv \square \mod 4 \right\}.$$

Also we put $S^+_{k+1/2}(\Gamma) = M^+_{k+1/2}(\Gamma) \cap S_{k+1/2}(\Gamma)$. Their images in $M_{k+1/2}(\Gamma')$ under the isomorphism $f \mapsto f_0$ are denoted by $M^+_{k+1/2}(\Gamma')$ and $S^+_{k+1/2}(\Gamma')$, respectively.

We also define the plus spaces contained in $M_{2k+1}(\Gamma_0(4), \chi_{-4})$ and $S_{2k+1}(\Gamma_0(4), \chi_{-4})$.

Definition 3. *We put*

$$M^+_{2k+1}(\Gamma_0(4), \chi_{-4}) = \left\{ h(z) = \sum_{n=0}^{\infty} d(n) q^n \in M_{2k+1}(\Gamma_0(4), \chi_{-4}) \,\middle|\, d(n) = 0 \text{ if } \chi_{-4}(n) = (-1)^{k+1} \right\}$$

and $S^+_{2k+1}(\Gamma_0(4), \chi_{-4}) = M^+_{2k+1}(\Gamma_0(4), \chi_{-4}) \cap S_{2k+1}(\Gamma_0(4), \chi_{-4})$.

For $k > 0$ being odd, the space $S^+_{2k+1}(\Gamma_0(4), \chi_{-4})$ was defined by Kojima [4] in 1982 and shown to be isomorphic to $M_{2k+2}(\Gamma^2(\mathcal{O}))$, the space of Hermitian modular forms of weight $2k+2$ and degree 2. The plus spaces of odd weights can be described exactly in the sense taking certain linear combinations of the normalized Hecke eigenforms of the whole space as a basis.

Theorem 2. *Let $k \geq 0$. We have*

$$\mathcal{R}\left(M^+_{k+1/2}(\Gamma)\right) \subset M^+_{2k+1}(\Gamma_0(4), \chi_{-4})$$

and

$$\mathcal{R}\left(S^+_{k+1/2}(\Gamma)\right) \subset S^+_{2k+1}(\Gamma_0(4), \chi_{-4})$$

This theorem can be proved in an elementary way, but can also be proved in a representation theoretical way, which reflects the nature of both plus spaces of half-integral and integral weights more.

As an application, we apply $\mathcal{R}$ on the Eisenstein series in $M^+_{k+1/2}(\Gamma)$, which was introduced in [5]. Let χ be a character of the ideal class group $Cl(F)$ of F. The Eisenstein series in $M^+_{k+1/2}(\Gamma)$ with respect to

χ is given by

$$E_{k+1/2,\chi}(z) = L_F(1-2k, \bar{\chi}^2) + \sum_{\substack{(-1)^k\xi \equiv \square \mod 4 \\ \xi \succ 0}} \mathcal{H}_k(\xi, \chi) q^\xi$$

where

$$\mathcal{H}_k(\xi,\chi) = \chi(\mathcal{D}_{(-1)^k\xi}) L_F(1-k, \chi_{(-1)^k\xi}\chi) \\ \times \sum_{\mathfrak{a} \mid \mathfrak{f}_{(-1)^k\xi}} \mu_F(\mathfrak{a}) \chi_{(-1)^k\xi}(\mathfrak{a}) \chi(\mathfrak{a}) \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{a})^{k-1} \sigma_{F,2k-1,\chi^2}(\mathfrak{f}_{(-1)^k\xi}\mathfrak{a}^{-1}).$$

Here $\mathcal{D}_x$ and χ_x are the relative discriminant and the quadratic character associated to $F(\sqrt{x})/F$, respectvely, and $\mathfrak{f}_x$ is the integral ideal such that ${\mathfrak{f}_x}^2\mathcal{D}_x = (x)$. Further more, μ_F is the Möbius function with respect to F and

$$\sigma_{F,m,\chi'}(\mathfrak{b}) = \sum_{\mathfrak{r} \mid \mathfrak{b}} \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{r})^m \chi'(\mathfrak{r}).$$

The Eisenstein series given above is a generalization of the one given by Cohen in [1], whose Fourier coefficients are called generalized Hurwitz class numbers.

The two Eisenstein series in $M_{2k+1}(\Gamma_0(4), \chi_{-4})$ are given by

$$E_{2k+1,\chi_{-4}}(\tau) = 1 + \frac{2}{L(-2k, \chi_{-4})} \sum_{n=1}^{\infty} \sigma_{2k,\chi_{-4}}(n) q^n$$

and

$$F_{2k+1,\chi_{-4}}(\tau) = \frac{(-1)^k 2}{L(-2k, \chi_{-4})} \sum_{n=1}^{\infty} \sigma'_{2k,\chi_{-4}}(n) q^n$$

where

$$\sigma'_{2k,\chi_{-4}}(n) = \sum_{r \mid n} r^{2k} \chi_{-4}(n/r).$$

The series $F_{2k+1,\chi_{-4}}$ is the normalized image of $E_{2k+1,\chi_{-4}}$ under the Fricke involution. By comparison of the constant term at the cusps of $\Gamma_0(4)$, we have the following result.

Theorem 3. *For $k \geq 0$, we have*

$$\mathcal{R}G_{k+1/2,\chi} - L_F(1-2k, \bar{\chi}^2)(E_{2k+1,\chi_{-4}} + (-1)^k F_{2k+1,\chi_{-4}}) \in S_{2k+1}(\Gamma_0(4), \chi_{-4}).$$

In particular, since $S_3(\Gamma_0(4), \chi_{-4}) = 0$, we have

$$\sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ a\beta - b\alpha = n}} \mathcal{H}_1(a+b\omega, \chi) = -4L_F(1-2k, \bar{\chi}^2)(\sigma_{2,\chi_{-4}}(n) - \sigma'_{2,\chi_{-4}}(n)).$$

For example, let $F = \mathbb{Q}(\sqrt{5})$ and $\boldsymbol{\delta} = \omega\sqrt{5}$ where $\omega = (1+\sqrt{5})/2$. Then by applying the theorem on $E_{3/2,1}$ and $E_{5/2,1}$ we may get

$$\begin{aligned}
L_F(0,\chi_{-2-\omega}) &= -\frac{2}{15}\left(\sigma_{2,\chi_{-4}}(2) - \sigma'_{2,\chi_{-4}}(2)\right), \\
2L_F(0,\chi_{-3}) + 2L_F(0,\chi_{-3+\omega}) &= -\frac{2}{15}\left(\sigma_{2,\chi_{-4}}(3) - \sigma'_{2,\chi_{-4}}(3)\right), \\
2L_F(0,\chi_{-4}) &= -\frac{2}{15}\left(\sigma_{2,\chi_{-4}}(4) - \sigma'_{2,\chi_{-4}}(4)\right), \\
2L_F(0,\chi_{-3}) + 2L_F(0,\chi_{-6-\omega}) &= -\frac{2}{15}\left(\sigma_{2,\chi_{-4}}(6) - \sigma'_{2,\chi_{-4}}(6)\right), \\
2\zeta_F(-1) &= \frac{1}{75}\left(\sigma_{4,\chi_{-4}}(1) + \sigma'_{4,\chi_{-4}}(1) + 3s(1)\right), \\
4\zeta_F(-1) + 2L_F(-1,\chi_{5+\omega}) &= \frac{1}{75}\left(\sigma_{4,\chi_{-4}}(5) + \sigma'_{4,\chi_{-4}}(5) + 3s(5)\right), \ldots
\end{aligned}$$

and so on. Here

$$s(n) = \frac{1}{4}\sum_{a^2+b^2=n} (a+b\sqrt{-1})^4.$$

3. Representation theoretic view of the theorem

Let $\psi_0 = \prod_{0,v\le\infty} \psi_{0,v}$ be the additive character of $\mathbb{A}_F/F$ such that for $v \mid \infty$ it satisfies

$$\psi_{0,v}(x) = \exp(2\pi\sqrt{-1}(-1)^k x) \quad (x \in \mathbb{R}).$$

Put $\psi = \psi_0(\boldsymbol{\delta}^{-1}\cdot)$. We denote the metaplectic double covering of SL_2 by Mp_2, which is with respect to the Kubota 2-cocycle. There exists an irreducible representation Ω_ψ of $\prod_{v|2}\mathrm{Mp}_2(\mathfrak{o}_v)$ associated to ψ. This representation is a subquotient of the restricted Weil representation associated to ψ and is of 4-dimension. Note that a modular form of weight $k+1/2$ can be lifted to an automorphic form on $\mathrm{SL}_2(F)\backslash\mathrm{Mp}_2(\mathbb{A}_F)$. Hiraga and Ikeda [?] showed the following theorem

Theorem 4 (Hiraga, Ikeda). *Let $f_0 \in M_{k+1/2}(\Gamma')$ and $W_4 f_0$ be its image under the fricke involution with respect to $z \mapsto -\frac{1}{4z}$. A sufficient necessary condition for f_0 to be in the plus space is*

$$< \rho(\gamma)W_4 f_0 \mid \gamma \in \prod_{v|2}\mathrm{Mp}_2(\mathfrak{o}_v) > \quad \cong \quad \Omega_\psi.$$

Regardless of the choice of F, $\mathrm{SL}_2(\mathbb{Z}_2)$ can be embedded into $\prod_{v|2}\mathrm{Mp}_2(\mathfrak{o}_v)$ and the restriction of Ω_ψ to $\mathrm{SL}_2(\mathbb{Z}_2)$ remains the same. One can show

$$\Omega_\psi\Big|_{\mathrm{SL}_2(\mathbb{Z}_2)} = \pi_k \oplus \sigma_k$$

where π_k and σ_k are irreducible representations of dimension 3 and 1, respectively. They only depend on the parity of k. The following theorem was shown by Kojima [4] (for odd k, but the proof can be easily extended to general positive integer k).

Theorem 5 (Kojima). *Let $h \in M_{2k+1}(\Gamma_0(4), \chi_{-4})$. A sufficient necessary condition for h to be in the plus space is*

$$< \rho(\gamma)W_4 h \mid \gamma \in \mathrm{SL}_2(\mathbb{Z}_2) > \quad \cong \quad \pi_k.$$

By Theorem 4, it is easy to see that if $f \in M^+_{k+1/2}(\Gamma)$ then $W_4\mathcal{R}f$ generates a irreducible representation of $\mathrm{SL}_2(\mathbb{Z}_2)$ equivalent to π_k. Thus we get $\mathcal{R}f \in M^+_{2k+1}(\Gamma_0(4), \chi_{-4})$.

References

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