

# Moving frames and Eisenstein invariants

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ABSTRACT. We recall combinatorial reconstitution of the periods of Eisenstein series of congruence subgroups of  $\mathrm{SL}_2(\mathbb{Z})$ , and present some consequence of “moving frames” in a free profinite group.

Plan:

1. Moving frames (review)
2. Eisenstein periods
3. Combinatorics in  $\hat{F}_2 = \pi_1^{\text{ét}}(\circlearrowleft)$
4. Some applications

## 1. Review: Moving frames

Suppose we are given a sequence of linear transformations on a vector space  $V$ :

$$V \xleftarrow{f_3} V \xleftarrow{f_2} V \xleftarrow{f_1} V.$$

Fix a basis  $\epsilon_0 = (e_1, \dots, e_n)$  of  $V$ , and let  $A_i$  be the representative matrices of  $f_i$  ( $i = 1, 2, 3$ ) respectively in view of the basis  $\epsilon_0$ . Then, as is well known, the composed transformation  $f_3 \circ f_2 \circ f_1$  is represented by the matrix  $A_3 A_2 A_1$ .

According to the idea of moving frames, we consider not only the initial basis  $\epsilon_0$  but also the moved bases  $\epsilon_1 := f_1(\epsilon_0)$  and  $\epsilon_2 := f_2 f_1(\epsilon_0)$ . Then, letting  $B_i$  denote the representative matrix of  $f_i$  in view of the basis  $\epsilon_{i-1}$  for  $i = 1, 2, 3$ , we derive that

$$B_1 = A_1, \quad B_2 = A_1^{-1} A_2 A_1, \quad B_3 = A_1^{-1} A_2^{-1} A_3 A_2 A_1.$$

Consequently we find that the composition  $f_3 \circ f_2 \circ f_1$  is represented by the reversely multiplied matrix  $B_1 B_2 B_3$  with respect to  $\epsilon_0$ .

We have borrowed from Spivak’s book [Sp99, Chap. 7] the term “moving frames” as an English translation of E. Cartan’s notion “repère mobile”. See loc. cit. for more sophisticated applications. A most typical example of that idea may be what is called the Euler angle representation of the

space rotations  $SO(3) = \{A \in GL_3(\mathbb{R}) \mid {}^tAA = 1, \det(A) = 1\}$ , which was most impressively encountered to the author in his youth 1983 upon an occasion of reading [YS, Chap.II, §2]: Define special matrices

$$\text{Rot}_2(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}, \quad \text{Rot}_3(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

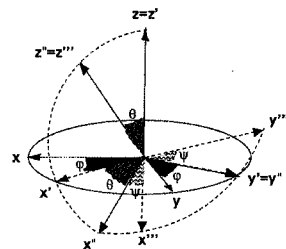
Then, every space rotation in  $SO(3)$  can be written as

$$A_{\varphi, \theta, \psi} = \text{Rot}_3(\varphi) \text{Rot}_2(\theta) \text{Rot}_3(\psi) \quad (0 \leq \varphi, \psi \leq 2\pi, 0 \leq \theta \leq \pi),$$

uniquely with only exceptions  $A_{\varphi, \theta, \psi} = A_{\varphi+\alpha, \theta, \psi-\alpha}$  for  $\theta \in \{0, \pi\}$  and  $\alpha \in \mathbb{R}$ . The above composition of three rotation matrices may be interpreted more naturally if it is read from the left to the right moving  $xyz$ -coordinates

$$(x, y, z) \xrightarrow{\varphi} (x', y', z') \xrightarrow{\theta} (x'', y'', z'') \xrightarrow{\psi} (x''', y''', z''')$$

as illustrated in the picture.



## 2. Eisenstein periods

Let  $\mathfrak{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$  be the complex upper half plane on which  $SL_2(\mathbb{Z})$  acts in the usual way. For each  $\mathbf{x} = \frac{\mathbf{u}}{N} = (\frac{u}{N}, \frac{v}{N}) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ , we have the holomorphic Eisenstein series of weight 2 and 'label'  $\mathbf{x}$  on  $\mathfrak{H}$  defined by

$$E_2^{(\mathbf{x})}(\tau) := \sum_{\mathbf{a} \in (\mathbb{Z}/N\mathbb{Z})^2} \frac{e^{2\pi i \det(\mathbf{a})}}{(2\pi i)^2} \left( \sum'_{\substack{(m_1, m_2) \equiv \mathbf{a} \\ \text{mod } N}} \frac{1}{(m_1\tau + m_2)^2} \cdot \frac{1}{|m_1\tau + m_2|^s} \right)_{s \rightarrow 0}.$$

The classical Eisenstein periods of  $E_2^{(\mathbf{x})}$  for those  $\mathbf{x} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$  are well known to be encoded in what are called the (generalized) Rademacher functions  $\Phi_{\mathbf{x}} : SL_2(\mathbb{Z}) \rightarrow \mathbb{Q}$ , which are good extensions of the period mapping  $A \mapsto \int_z^{Az} E_2^{(\mathbf{x})}(\tau) d\tau$  for  $A \in \Gamma(N)$  with  $N\mathbf{x} \in \mathbb{Z}^2$ . The value of  $\Phi_{\mathbf{x}}(A) \in \mathbb{Q}$  for every  $\mathbf{x} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$  and  $A \in SL_2(\mathbb{Z})$  is explicitly calculated in terms of Bernoulli polynomials and Dedekind sums (B.Schoeneberg [Sc74]).

Based on our recent work [N13], we can introduce a (profinite) *combinatorial avatar* “ $\mathbb{E}_x$ ” of  $\Phi_x : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathbb{Q}$ . Here, we consider the label  $x$  to lie in  $\mathbb{Q}_f^2 := (\mathbb{Q} \otimes \hat{\mathbb{Z}})^2$  (adelic row vectors) and replace  $\mathrm{SL}_2(\mathbb{Z})$  by a certain profinite group  $\pi_1^{\text{ét}}(\mathfrak{M})$  which is:

(1) in the form of a semi-direct product  $G_{\mathbb{Q}} \ltimes \hat{B}_3$  of two profinite groups, where  $\hat{B}_3$  is a central extension of  $\widehat{\mathrm{SL}_2(\mathbb{Z})}$  and  $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ ;

(2) equipped with a standard representation  $\rho : \pi_1^{\text{ét}}(\mathfrak{M}) \rightarrow \mathrm{GL}_2(\hat{\mathbb{Z}})$ ; as explained soon in more details. Throughout below, we write  $A_{\sigma} \in \mathrm{GL}_2(\hat{\mathbb{Z}})$  for the transposed matrix of  $\rho(\sigma)$ :

$$A_{\sigma} = {}^t\rho(\sigma) \quad (\sigma \in \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3).$$

The main aim of the present article is to illustrate roughly a use of “moving frames” idea to get the following composition law for our invariant  $\mathbb{E}_x$ :

**Theorem 2.1** (Composition law [N16b]). *Let  $x \in \mathbb{Q}_f^2 := (\mathbb{Q} \otimes \hat{\mathbb{Z}})^2$ . Then,*

$$\mathbb{E}_x(\sigma_1\sigma_2) = \mathbb{E}_{xA_{\sigma_2}}(\sigma_1) + \det(A_{\sigma_1})\mathbb{E}_x(\sigma_2)$$

holds for  $\sigma_1, \sigma_2 \in \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3$ . □

Before going further, we quickly introduce a relation between the classical period  $\Phi_x$  and our avatar  $\mathbb{E}_x$ . Just for now, we recall that the discrete Artin braid group  $B_3$  with three strands fits in a central extension

$$(2.2) \quad \begin{array}{ccccccc} 1 & \rightarrow & \mathbb{Z} & \rightarrow & \widehat{\mathrm{SL}_2(\mathbb{Z})} & \cong & B_3 & \xrightarrow{\rho} & \mathrm{SL}_2(\mathbb{Z}) & \rightarrow & 1. \\ & & & & \psi & & \psi & & & & \\ & & & & \sigma & \mapsto & \rho(\sigma) & & & & \end{array}$$

As seen later in §3, the above  $\rho$  extends to a continuous homomorphism

$$\rho : \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3 \longrightarrow \mathrm{GL}_2(\hat{\mathbb{Z}})$$

representing the monodromy actions on the torsion points of an elliptic curve.

If  $\sigma$  lies in the discrete part  $B_3$  of  $\hat{B}_3 \subset \pi_1^{\text{ét}}(\mathfrak{M})$ , then  $\rho(\sigma)$  and  $A_{\sigma}$  lie in  $\mathrm{SL}_2(\mathbb{Z})$ .

The following theorem is based on our work [N13].

**Theorem 2.3.** *One can introduce  $\mathbb{E}_x(\sigma) \in \hat{\mathbb{Z}}$  for  $\sigma \in \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3$  and  $x \in \mathbb{Q}_f^2$  in a purely combinatorial way (Fox calculus) so that when  $x \in \mathbb{Q}^2$  and  $\sigma \in B_3$  with  $A_\sigma \in \text{SL}_2(\mathbb{Z})$ ,*

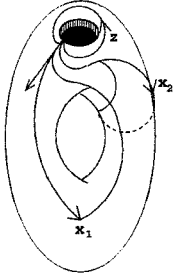
$$\mathbb{E}_x(\sigma) = -\Phi_x(A_\sigma) + (\text{explicit error term}).$$

$$\begin{array}{ccc} \cap & \cap & \cap \\ \mathbb{Z} & \mathbb{Q} & \mathbb{Q} \end{array} \quad \square$$

**Remark 2.4.** It is noteworthy to observe that the above error term sweeps out the denominator of  $\Phi_x(A_\sigma) \in \mathbb{Q}$  to obtain an integer value  $\mathbb{E}_x(\sigma) \in \mathbb{Z}$ . The explicit form of the error term ' $K_x(A_\sigma) - \frac{1}{12}\rho_\Delta(\sigma)$ ' is calculated in [N13, Th.7.2.3]. As a consequence, it follows, e.g., that the denominator of  $\Phi_{(\frac{y}{N}, \frac{v}{N})}(A)$  for  $A \in \text{SL}_2(\mathbb{Z})$  is bounded by  $12N^2$ .

### 3. Combinatorics in $\hat{F}_2 = \hat{\pi}_{1,1}$

In order to introduce our combinatorial avatar of Eisenstein periods, we shall set up the universal elliptic curves  $E \setminus \{O\} := \{y^2 = 4x^3 - g_2x - g_3\}$  over the parameter space  $\mathfrak{M} := \{(g_2, g_3) \mid \Delta := g_2^3 - 27g_3^2 \neq 0\}$ . We consider both  $E \setminus \{O\}$  and  $\mathfrak{M}$  as affine varieties over  $\mathbb{Q}$ . The natural projection  $E \setminus \{O\} \rightarrow \mathfrak{M}$  is the Weierstrass family of elliptic curves whose structured chart from a viewpoint of anabelian geometry was discussed in [N13, §5]. In summary, we have a tangential section  $\tilde{w} : \mathfrak{M} \dashrightarrow E \setminus \{O\}$  (normalized with  $t := -2x/y$ ) and a tangential fiber  $\text{Tate}(q) \hookrightarrow E \setminus \{O\}$ . Using the van-Kampen construction of the Tate curve, we also introduced standard loops  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{z}$  of  $\hat{\pi}_{1,1} := \pi_1^{\text{ét}}(\text{Tate}(q) \otimes \overline{\mathbb{Q}})$  based at  $\text{Im}(\tilde{w}) \cap \text{Tate}(q)$  on  $E(\mathbb{C}) \setminus \{O\}$  with  $[\mathbf{x}_1, \mathbf{x}_2]\mathbf{z} = 1$  ( $[\mathbf{x}_1, \mathbf{x}_2] := \mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}$ ). Note that  $\hat{\pi}_{1,1}$  is isomorphic to a free profinite group  $\hat{F}_2$  freely generated by  $\mathbf{x}_1, \mathbf{x}_2$ .

$$\begin{array}{ccc} E \setminus \{O\} := \{y^2 = 4x^3 - g_2x - g_3\} & \dashrightarrow & \text{Tate}(q) \\ \Downarrow \tilde{w} & & \Downarrow \\ \mathfrak{M} := \{(g_2, g_3) \mid \Delta := g_2^3 - 27g_3^2 \neq 0\} & \dashrightarrow & \text{Spec } \mathbb{Q}((q)) \end{array}$$


It is natural to employ the images of  $\text{Spec } \mathbb{Q}((q))$  as base points of those étale fundamental groups of individual spaces in the above diagram. Then, we obtain the basic identification :

$$\pi_1^{\text{ét}}(E \setminus \{O\}) = \pi_1^{\text{ét}}(\mathfrak{M}) \ltimes \hat{\pi}_{1,1}, \quad \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3.$$

In fact, the moduli space  $\mathfrak{M}$  is naturally interpreted as the space of (normalized) cubics, and a topological loop in  $\pi_1(\mathfrak{M}(\mathbb{C}))$  is a motion of three points on the plane: we may identify  $\pi_1(\mathfrak{M}(\mathbb{C}))$  with the Artin braid group  $B_3$  of three strands, consequently,  $\pi_1^{\text{ét}}(\mathfrak{M})$  as the semidirect product of  $G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  with the profinite completion  $\hat{B}_3$ .

The conjugate action in the above splitting  $\pi_1^{\text{ét}}(E \setminus \{O\}) = \pi_1^{\text{ét}}(\mathfrak{M}) \ltimes \hat{\pi}_{1,1}$  induces the monodromy action of  $\pi_1^{\text{ét}}(\mathfrak{M})$  on  $\hat{\pi}_{1,1} = \hat{F}_2$ :

$$\begin{array}{ccccc} \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3 & \xrightarrow{\varphi} & \text{Aut}^*(\hat{F}_2) & \xrightarrow{\text{mod } \hat{F}_2'} & \text{GL}(\hat{\mathbb{Z}}^2) \\ \downarrow \Psi & & \downarrow \Psi & & \downarrow \Psi \\ \sigma & \longmapsto & \varphi(\sigma) & \longmapsto & \rho(\sigma) \end{array}$$

where  $\text{Aut}^*(\hat{F}_2)$  denotes the group of *special automorphisms* defined by

$$\text{Aut}^*(\hat{F}_2) = \{\sigma \in \text{Aut}(\hat{F}_2) \mid \sigma(\langle z \rangle) = \langle z \rangle\}.$$

Given  $m \geq 1$ ,  $\sigma \in \pi_1^{\text{ét}}(\mathfrak{M})$  and  $(u, v) \in \hat{\mathbb{Z}}^2$ , let  $\rho(\sigma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and set

$$\mathcal{S}_{uv}(\sigma) := \sigma(\mathbf{x}_2^{-v} \mathbf{x}_1^{-u}) \mathbf{x}_1^{au+bv} \mathbf{x}_2^{cu+dv} \in \hat{F}_2' := [\hat{F}_2, \hat{F}_2].$$

By Ihara's theory (cf. [I99]), with the class of  $\mathcal{S}_{uv}(\sigma)$  in the 2nd derived quotient  $\hat{F}_2'/\hat{F}_2''$ , we may associate a unique element of the complete group algebra  $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]] = \varprojlim_m \frac{\hat{\mathbb{Z}}[\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2]}{(\bar{\mathbf{x}}_1^m - 1, \bar{\mathbf{x}}_2^m - 1)}$ , where  $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2$  designate the abelianization images of  $\mathbf{x}_1, \mathbf{x}_2 \in \hat{F}_2$  respectively. In order to explain this procedure in a more fitting form with the moving frame idea, it is useful to introduce a sequence of maps composed of the Fox derivative  $\partial_{x_1}$  with projections

$$\hat{\mathbb{Z}}[[\hat{F}_2]] \xrightarrow{\partial_{x_1}} \hat{\mathbb{Z}}[[\hat{F}_2]] \xrightarrow{\text{ab}} \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]] \xrightarrow{\text{mod } m} \frac{\hat{\mathbb{Z}}[\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2]}{(\bar{\mathbf{x}}_1^m - 1, \bar{\mathbf{x}}_2^m - 1)}$$

and, writing any element of  $\frac{\hat{\mathbb{Z}}[\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2]}{(\bar{\mathbf{x}}_1^m - 1, \bar{\mathbf{x}}_2^m - 1)}$  as  $\sum_{i,j=0}^{m-1} c_{ij} \bar{\mathbf{x}}_1^i \bar{\mathbf{x}}_2^j$ , define

$$\mathbb{E}_m(\sigma; u, v) := \text{constant term } c_{00} \text{ of } \left[ \frac{[\partial_{\mathbf{x}_1}(\mathcal{S}_{uv}(\sigma))]^{\text{ab}}}{\bar{\mathbf{x}}_2 - 1} \right]_{\bar{\mathbf{x}}_1^m = \bar{\mathbf{x}}_2^m = 1} \quad \left( \in \hat{\mathbb{Z}} \right).$$

Cf. [N13, (3.2.3)].

**Proposition 3.1** ([N16b], Theorem A). *It holds that*

$$\mathbb{E}_m(\sigma_1\sigma_2; \mathbf{u}) = \mathbb{E}_m(\sigma_1; \mathbf{u}A_{\sigma_2}) + (\det \rho(\sigma_1)) \cdot \mathbb{E}_m(\sigma_2; \mathbf{u})$$

for  $\sigma_1, \sigma_2 \in \text{Aut}^*(\hat{F}_2)$  and  $\mathbf{u} \in \hat{\mathbb{Z}}^2$ .

*Proof motivation* of the above composition law: Given any  $\sigma \in \pi_1^{\text{ét}}(\mathfrak{M})$ , view the data  $\mathbb{E}_m(\sigma) := [\mathbb{E}_m(\sigma; u, v)]_{(u,v) \in \hat{\mathbb{Z}}^2}$  as a profinite tableau on the plane  $\hat{\mathbb{Z}}^2$  with entries  $\hat{\mathbb{Z}}$ . Let us consider traveling in  $\hat{F}_2$  (with portable  $\mathbb{E}_m$ -board in one hand) along the composition of two automorphisms  $\sigma \circ \tau \in \text{Aut}^*(\hat{F}_2)$  and observe effects on the  $\mathbb{E}_m$ . Noting that the definition of  $\mathbb{E}_m$  depends entirely on the choice of free generator system  $\underline{\mathbf{x}} = (\mathbf{x}_1, \mathbf{x}_2)$  of  $\hat{F}_2$ , we are urged to look closely at the diagram

$$(3.2) \quad \begin{array}{ccccc} & & \sigma\tau & & \\ & \swarrow & & \searrow & \\ \hat{F}_2 & \xleftarrow[\tau(\underline{\mathbf{x}})]{\sigma} & \hat{F}_2 & \xleftarrow[\underline{\mathbf{x}}]{\tau} & \hat{F}_2 \\ \uparrow \tau & & & & \downarrow \tau^{-1} \\ \hat{F}_2 & \xleftarrow[\underline{\mathbf{x}}]{\sigma'} & \hat{F}_2 & & \end{array}$$

and especially at the effect of  $\sigma$  with regard to the moved frame  $\tau(\underline{\mathbf{x}}) = (\tau(\mathbf{x}_1), \tau(\mathbf{x}_2))$ . In fact, one symbolically finds

$$\mathcal{S}_{\mathbf{u}}(\sigma\tau) = \mathcal{S}_{\mathbf{u}}(\sigma; \text{"rel.}\tau(\underline{\mathbf{x}})\text{"}) \cdot \mathcal{S}_{\sigma'\mathbf{u}}(\tau)$$

which approximately leads to

$$\mathbb{E}_m(\sigma\tau, \mathbf{u}) \approx (\det \rho(\tau)) \cdot \mathbb{E}_m(\sigma', \mathbf{u}) + \mathbb{E}_m(\tau, \mathbf{u}A_{\sigma'}).$$

Proposition 3.1 follows then by rewriting:  $\sigma_2 = \sigma' = \tau^{-1}\sigma\tau$ ,  $\sigma_1 = \tau$  so that  $\sigma_1\sigma_2 = \sigma\tau$ .  $\square$

**Remark 3.3.** In [N13], it is shown that the adelic tableau  $\mathbb{E}_m(\sigma) \in \hat{\mathbb{Z}}^{\hat{\mathbb{Z}}^2}$  encodes the image of  $\sigma$  by  $\pi_1^{\text{ét}}(\mathfrak{M}) \rightarrow \text{Aut}^*(F_2/F_2'')$  (the meta-abelian monodromy).

#### 4. Some applications

Let us briefly pick up a few topics from [N16b].

**4.1. Homogeneity.** The above composition law Proposition 3.1 leads us to the following basic property:

**Corollary 4.1** (Homogeneity [N16b] Theorem C). *Let  $\mathbf{u} \in \hat{\mathbb{Z}}^2$ ,  $\sigma \in \pi_1^{\text{ét}}(\mathfrak{M})$ . Then, for each positive integer  $k \in \mathbb{N}$ , it holds that*

$$\mathbb{E}_m(\sigma, \mathbf{u}) = \mathbb{E}_{mk}(\sigma, k\mathbf{u}).$$

In fact, by virtue of Proposition 3.1, expressing  $\sigma$  as a product of  $\sigma_1 \in G_{\mathbb{Q}}$  and  $\sigma_2 \in \hat{B}_3$ , we may reduce the proof of Corollary to individual cases where  $\sigma \in G_{\mathbb{Q}}$  or  $\sigma \in \hat{B}_3$ . In the latter case, since  $B_3 \times \mathbb{Z}^2$  is dense in  $\hat{B}_3 \times \hat{\mathbb{Z}}^2$ , the result follows from the explicit formula of  $\mathbb{E}_{km}(\sigma, k\mathbf{u})$  for  $\sigma \in B_3$ ,  $\mathbf{u} \in \mathbb{Z}^2$  given in Theorem 2.3 (cf. [N13, Th.7.2.3]). In the former case where  $\sigma \in G_{\mathbb{Q}}$ , the result follows from an explicit calculation of  $\mathbb{E}_m(\sigma, \mathbf{u})$  which is based on the Grothendieck-Teichmüller theory on  $\pi_1^{\text{ét}}(\text{Tate}(q) \setminus O)$  (see [N16b]).

The above corollary allows us to define the “*adelic Eisenstein function*”  $\mathbb{E}_{\mathbf{x}}(\sigma)$ :

$$\pi_1^{\text{ét}}(\mathfrak{M}) \times \mathbb{Q}_f^2 \ni (\sigma, \mathbf{x}) \longmapsto \mathbb{E}_{\mathbf{x}}(\sigma) \in \hat{\mathbb{Z}}$$

by assigning  $\mathbb{E}_m(\sigma, \mathbf{u})$  for any choice of  $m \in \mathbb{N}$  and  $\mathbf{u} \in \hat{\mathbb{Z}}^2$  so that  $\mathbf{x} = \frac{\mathbf{u}}{m} \in \mathbb{Q}_f^2$ . Then, Theorem 2.1 is only the reload of Proposition 3.1.

**4.2. Level splitter homomorphism** ([N16b, §7]). Let  $m, M$  be positive integers and set  $N = \gcd(2, M) \cdot M$ . We define the principal congruence subgroup of level  $N$  by  $\pi_1^{\text{ét}}(\mathfrak{M})[N] := \{\sigma \mid A_{\sigma} \equiv \mathbf{1} \bmod N\}$ . Then, combining results of [N12], [N13] and [N16b], we see that  $\mathbb{E}_m(\sigma, \mathbf{u}) \bmod M$  has  $m \times m$ -periodicity in  $\mathbf{u} \in \hat{\mathbb{Z}}^2$ , hence that it induces a homomorphism

$$\mathbb{E}_{m \rfloor M} : \pi_1^{\text{ét}}(\mathfrak{M})[mN] \rightarrow (\mathbb{Z}/M\mathbb{Z})[(\mathbb{Z}/m\mathbb{Z})^2].$$

Generally, the above level splitter  $\mathbb{E}_{m \rfloor M}$  affords a non-trivial abelian quotient of  $\pi_1^{\text{ét}}(\mathfrak{M})[N]$  and should involve highly arithmetic information about “Eisenstein quotient”. We hope to discuss it in more details on some other occasion.

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