

Recent Development in the Theory of Weak Convergence of Vector Measures

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ABSTRACT. The study of vector measures has progressed toward the extensive scrutiny of the interplay between properties of Banach spaces and measures with values in Banach spaces. Recently, the notion of weak convergence of vector measures was introduced by M. Dekiert, and the study of topological properties of spaces of vector measures presents new and interested problems to the field of vector measures. In this survey, we try to explain certain aspects of the recent development in the theory of weak convergence of vector measures.

1. Introduction

According to a splendid book of J. Diestel and J. J. Uhl, Jr., the study of vector measures has progressed toward the extensive scrutiny of the interplay between properties of Banach spaces and measures with values in Banach spaces. Indeed, it has headed for the study of Radon-Nikodým theorem and the martingale convergence theorem and their relation to the topological and geometric structure of Banach spaces, the study of structural properties of operators on spaces of continuous functions, the study of the range of a vector space, the study of the existence of products of vector measures and the Fubini theorem, and so on. These studies are still important and continue to give significant outcomes to the field of vector measures and its related fields. However, most of those studies deal with problems which are involved in not collections of vector measures but a single vector measure.

Recently, the notion of weak convergence of vector measures was introduced by M. Dekiert. It is a natural generalization of the weak convergence of probability measures, which plays an important role in the study of stochastic convergence in probability theory and statistics. Thanks to this weak convergence, the study of topological properties of spaces of vector measures presents new and interested problems to the field of vector measures.

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In this survey, we try to explain certain aspects of the recent development in the theory of weak convergence of vector measures. This will be only a very partial survey, because it is beyond my power to cover adequately all the directions taken by recent research. It will also reflect my personal interests in the area.

Some definitions and basic facts of vector measures are collected in Section 2.

Section 3 deals with compactness and metrizability in the space of vector measures. Included here are Prokhorov-LeCam's compactness criteria and Varadarajan's metrizability criterion for vector measures.

Section 4 devoted to the weak convergence of injective tensor products of vector measures. Presented here are some results concerning the joint continuity of injective tensor products of vector measures with respect to the weak convergence in the following two cases: One is the case that vector measures take values in some nuclear spaces. The other is the case that they take values in the positive cone of Banach lattices.

Strassen's theorem for positive vector measures are dealt with in Section 5. A type of Strassen's theorem is given for positive vector measures with values in the weak dual of a barreled locally convex space which has certain order conditions.

2. Preliminaries

All the topological spaces, uniform spaces, and topological vector spaces are Hausdorff and the scalar fields of topological vector spaces are taken to be the field $\mathbb{R}$ of all real numbers. Denote by $\mathbb{N}$ the set of all natural numbers.

Let X be a locally convex Hausdorff space (for short, lcHs). Denote by X^* the topological dual of X . The weak topology of X means the $\sigma(X, X^*)$ -topology on X . If $x^* \in X^*$ and p is a seminorm on X , we write $x^* \leq p$ whenever $|x^*x| \leq p(x)$ for all $x \in X$.

Let $\mathcal{E}$ be a σ -field of subsets of a non-empty set Ω and $\mu : \mathcal{E} \rightarrow X$ a finitely additive set function. We say that μ is a *vector measure* if it is countably additive, that is, for any sequence $\{E_n\}$ of pairwise disjoint subsets of $\mathcal{E}$, we have $\sum_{n=1}^{\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$ in the original topology of X . Denote by $\mathcal{M}(\Omega, X)$ the set of all vector measures $\mu : \mathcal{E} \rightarrow X$. When $X = \mathbb{R}$, we write $\mathcal{M}(\Omega) := \mathcal{M}(\Omega, \mathbb{R})$. Then, $\mathcal{M}(\Omega)$ is a Banach space with the total variation norm $|m| := |m|(\Omega)$.

If μ is a vector measure, then $x^*\mu$ is a real measure for each $x^* \in X^*$. Conversely, a theorem of Orlicz and Pettis ensures that a finitely additive set function $\mu : \mathcal{E} \rightarrow X$ is countably additive if $x^*\mu$ is countably additive for every $x^* \in X^*$; see, for instance, C. W. McArthur [31, Corollary 1].

Let $\mu : \mathcal{E} \rightarrow X$ be a vector measure and p a seminorm on X . Then the *p-semivariation* of μ is the set function $\|\mu\|_p : \mathcal{E} \rightarrow [0, \infty)$ defined by $\|\mu\|_p(E) := \sup_{x^* \leq p} |x^*\mu|(E)$ for all $E \in \mathcal{E}$, where $|x^*\mu|(\cdot)$ is the total variation of the real measure $x^*\mu$. When X is a Banach space, the *semivariation* of μ is defined by $\|\mu\|(E) := \sup_{\|x^*\| \leq 1} |x^*\mu|(E)$ for all $E \in \mathcal{E}$.

Let $\mu : \mathcal{E} \rightarrow X$ be a vector measure. An $\mathcal{E}$ -measurable, real function f on Ω is said to be *μ -integrable* if (a) f is $x^*\mu$ -integrable for each $x^* \in X^*$, and (b) for each $E \in \mathcal{E}$, there exists an

element of X , denoted by $\int_E f d\mu$, such that

$$x^* \left(\int_E f d\mu \right) = \int_E f d(x^* \mu)$$

for each $x^* \in X^*$. We note here that if X is sequentially complete, then every bounded, $\mathcal{E}$ -measurable real function f is μ -integrable, and

$$p \left(\int_E f d\mu \right) \leq \sup_{x^* \leq p} \int_E |f| d|x^* \mu| \leq \sup_{\omega \in E} |f(\omega)| \cdot \|\mu\|_p(E)$$

for every $E \in \mathcal{S}$ and every continuous seminorm p on X . See R. G. Bartle, N. Dunford and J. T. Schwartz [1], J. Diestel and J. J. Uhl, Jr. [6], D. R. Lewis [30], and I. Kluváněk and G. Knowles [28] for some additional definitions and properties of vector measures.

In what follows, let S be a topological space and $\mathcal{B}(S)$ the σ -field of all Borel subsets of S . Denote by $\mathcal{M}(S, X)$ the set of all vector measures $\mu : \mathcal{B}(S) \rightarrow X$. We define several notions of regularity for vector measures on a topological space. A vector measure $\mu : \mathcal{B}(S) \rightarrow X$ is said to be *Radon* if for each $\varepsilon > 0$, $E \in \mathcal{B}(S)$, and continuous seminorm p on X , there exists a compact subset K of E such that $\|\mu\|_p(E - K) < \varepsilon$, and it is said to be *tight* if the condition is satisfied for $E = S$. We say that μ is *τ -smooth* if for every continuous seminorm p on X and every increasing net $\{G_\alpha\}$ of open subsets of S with $G = \bigcup_\alpha G_\alpha$, we have $\lim_\alpha \|\mu\|_p(G - G_\alpha) = 0$. We say that μ is *scalarly Radon* (respectively, *scalarly tight*, *scalarly τ -smooth*) if for each $x^* \in X^*$ the real measure $x^* \mu$ is Radon (respectively, tight, τ -smooth). It is known that μ is Radon (respectively, tight, τ -smooth) if and only if it is scalarly Radon (respectively, scalarly tight, scalarly τ -smooth). In fact, for Banach space-valued vector measures, this is a consequence of the Rybakov theorem [6, Theorem IX.2.2], which ensures that there exists $x_0^* \in X^*$ for which $x_0^* \mu$ and μ are mutually absolutely continuous. For general lcHs-valued vector measures, see [30, Theorem 1.6] and [23]. Consequently, all of the regularity properties which are valid for positive, finite measures remain true for vector measures. For instance, every vector measure on a topological space with a countable base (in particular, on a separable metric space) is τ -smooth. Further, every vector measure on a complete separable metric space is Radon, so that it is τ -smooth and tight; see N. N. Vakhania, V. I. Tarieladze and S. A. Chobanyan [43, Proposition I.3.1 and Theorem I.3.1].

By $\mathcal{M}_t(S, X)$ we denote the set of all Radon vector measures $\mu : \mathcal{B}(S) \rightarrow X$. As before, we write $\mathcal{M}_t(S) := \mathcal{M}_t(S, \mathbb{R})$. Denote by $C(S)$ the Banach space of all bounded, continuous real functions on S with the norm $\|f\|_\infty := \sup_{s \in S} |f(s)|$.

3. Compactness and metrizability in the space of vector measures

Compactness and metrizability for the weak convergence of measures are important and applicative properties in the space of positive or real measures on topological spaces. In this section, we explain some recent results of the study of compactness and metrizability in the space of vector measures.

3.1. Compactness and metrizability criteria for real measures. Let S be a completely regular space. Let $\{m_\alpha\}$ be a net in $\mathcal{M}(S)$ and $m \in \mathcal{M}(S)$. We say that $\{m_\alpha\}$ *converges*

weakly to m and write $m_\alpha \xrightarrow{w} m$ if for every $f \in C(S)$ we have $\lim_\alpha \int_S f dm_\alpha = \int_S f dm$. In what follows, we always equip $\mathcal{M}(S)$ with the topology determined by this weak convergence and call it the *weak topology of measures*.

A subset M of $\mathcal{M}(S)$ is said to be *uniformly bounded* if $\sup_{m \in M} |m|(S) < \infty$. We say that M is *uniformly tight* if for each $\varepsilon > 0$ there exists a compact subset K of S such that $|m|(S - K) < \varepsilon$ for all $m \in M$.

In 1956, Yu. V. Prokhorov [33, Theorem 1.12] gave a compactness criterion for the weak topology of measures in the space of all positive, finite measures on a complete separable metric space. This criterion was extended by L. LeCam [29, Proposition 1 and Theorem 6] to real Radon measures on an arbitrary completely regular space. These results are called *Prokhorov-LeCam's compactness criteria*, and play an important role in the study of stochastic convergence in probability theory and statistics.

THEOREM 3.1 (Prokhorov-LeCam's compactness criteria). *Let S be a completely regular space. Assume that $M \subset \mathcal{M}_t(S)$ is uniformly bounded and uniformly tight. Then M is relatively compact in $\mathcal{M}_t(S)$. If compact subsets of S are all metrizable, then M is relatively sequentially compact in $\mathcal{M}_t(S)$.*

As to metrizability in the space of measures, it is known that the space of all positive, finite measures on a separable metric space is metrizable; see V. S. Varadarajan [44, Theorem 3.1]. This is not the case for real measures, and in fact it was proved in [45, Theorem 16, Part II] that the set of all real τ -smooth measures on a metric space S is metrizable if and only if S is a finite set. Nevertheless, in [45, Theorem 26, Part II] the following result was actually proved and is called *Varadarajan's metrizability criterion*.

THEOREM 3.2 (Varadarajan's metrizability criterion). *Let S be a locally compact separable metric space. Then, every compact subset M of $\mathcal{M}_t(S)$ is metrizable, so that it is sequentially compact in $\mathcal{M}_t(S)$.*

3.2. Weak convergence of vector measures. Recently, M. Dekiert [5] introduced the notion of weak convergence of Banach space-valued vector measures. Let S be a completely regular space. Let X be a sequentially complete lchS with locally convex topology τ . Let $\{\mu_\alpha\}$ be a net in $\mathcal{M}_t(S, X)$ and $\mu \in \mathcal{M}_t(S, X)$. We say that $\{\mu_\alpha\}$ *converges weakly* to μ for τ if for every $f \in C(S)$ we have $\int_S f d\mu_\alpha \rightarrow \int_S f d\mu$ for the topology τ of X .

This is a natural analogy of the convergence studied by [5, Sections 2 and 3, Chapter IV] for Banach space-valued vector measures, and coincides with the usual weak convergence of measures in the case that $X = \mathbb{R}$; see [33], [29], [45], and [43]. The topology determined by this weak convergence is called the *weak topology of vector measures for τ* (for short, WTVM for τ).

In 1994, M. März and R. M. Shortt [32, Theorem 1.5 and Corollary 1.6] gave a sequential compactness criterion for Banach space-valued vector measures on a metric space, which is the starting point of our studies of weak convergence of vector measures. Let S be a topological space and X a Banach space. Let $\mathcal{V} \subset \mathcal{M}_t(S, X)$. We say that $\mathcal{V}$ is *uniformly bounded* if

$\sup_{\mu \in \mathcal{V}} \|\mu\|(S) < \infty$ and that it is *uniformly tight* if for each $\varepsilon > 0$ there exists a compact subset K of S such that $\sup_{\mu \in \mathcal{V}} \|\mu\|(S - K) < \varepsilon$.

THEOREM 3.3 (März-Shortt's sequential compactness criterion). *Let S be a metric space and X a Banach space. Assume that $\mathcal{V} \subset \mathcal{M}_t(S, X)$ satisfies the following conditions:*

- (i) $\mathcal{V}$ is *uniformly bounded*.
- (ii) $\mathcal{V}$ is *uniformly tight*.
- (iii) For each compact subset K of S , $\{\int_K f d\mu : f \in C(S), \|f\|_\infty \leq 1, \mu \in \mathcal{V}\}$ is a *relatively weakly compact subset of X* .

Then $\mathcal{V}$ is relatively sequentially compact in $\mathcal{M}_t(S, X)$ with respect to the WTVM for $\sigma(X, X^)$. Further, if X is reflexive, (iii) follows from (i).*

3.3. Uniform tightness for vector measures with values in a lcHs. The notion of uniform boundedness and uniform tightness can be naturally extended to vector measures with values in a lcHs. Let S be a completely regular space and X a lcHs. Let $\mathcal{V} \subset \mathcal{M}(S, X)$. We say that $\mathcal{V}$ is *uniformly bounded* if $\sup_{\mu \in \mathcal{V}} \|\mu\|_p(S) < \infty$ for every continuous seminorm p on X and that $\mathcal{V}$ is *scalarly uniformly bounded* if for each $x^* \in X^*$ the set $x^*(\mathcal{V}) := \{x^*\mu : \mu \in \mathcal{V}\}$ of real measures is uniformly bounded. Since every weakly bounded subset of X is bounded, $\mathcal{V}$ is uniformly bounded if and only if it is scalarly uniformly bounded. Further, the principle of uniform boundedness (see H. H. Schaefer [34, Corollary to III.4.2]) ensures that if every element of $x^*(\mathcal{V})$ is Radon, then the scalarly uniform boundedness follows from the condition that $\sup_{\mu \in \mathcal{V}} |\int_S f d(x^*\mu)| < \infty$ for every $x^* \in X^*$ and $f \in C(S)$.

We say that $\mathcal{V}$ is *uniformly tight* if for each $\varepsilon > 0$ and continuous seminorm p on X there exists a compact subset K of S such that $\sup_{\mu \in \mathcal{V}} \|\mu\|_p(S - K) < \varepsilon$ and that $\mathcal{V}$ is *scalarly uniformly tight* if for each $x^* \in X^*$ the set $x^*(\mathcal{V})$ is uniformly tight.

As is stated above, the notions of countable additivity, Radonness, and uniform boundedness for vector measures are equivalent to the corresponding scalarly notions. However, the following example shows that the notion of uniform tightness is not the case even for Hilbert space-valued vector measures.

EXAMPLE 3.4 ([19, Example]). We give a set of Radon vector measures, which is scalarly uniformly bounded and scalarly uniformly tight, but which is not uniformly tight.

Let H be a separable Hilbert space with inner product $(\cdot, \cdot)$, and $\{e_n\}$ a complete orthonormal basis in H . Let $\{m_n\}$ be a sequence of Gaussian measures on $\mathbb{R}$ with zero mean and variance n .

For each $n \in \mathbb{N}$, define a vector measure $\mu_n : \mathcal{B}(\mathbb{R}) \rightarrow H$ by $\mu_n(E) := m_n(E)e_n$ for all $E \in \mathcal{B}(\mathbb{R})$. Then it is easy to see that $\mu_n \in \mathcal{M}_t(\mathbb{R}, H)$ for all $n \in \mathbb{N}$.

For each $x \in H$ and $\mu \in \mathcal{M}_t(\mathbb{R}, H)$, define a real measure $x\mu$ on $\mathbb{R}$ by $(x\mu)(E) := (x, \mu(E))$ for all $E \in \mathcal{B}(\mathbb{R})$. Then we have $|x\mu_n| = |(x, e_n)|m_n$ and $\|\mu_n\| = m_n$ for all $n \in \mathbb{N}$.

Put $\mathcal{V} = \{\mu_n\}$ and fix $x \in H$. Then we have $|x\mu_n|(\mathbb{R}) = |(x, e_n)|m_n(\mathbb{R}) \leq \|x\|$ for all $n \in \mathbb{N}$, so that $x(\mathcal{V}) := \{x\mu : \mu \in \mathcal{V}\}$ is uniformly bounded.

Let $\varepsilon > 0$. Since (x, e_n) converges to 0, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|(x, e_n)| < \varepsilon$. Hence we have $\sup_{n \geq n_0} |x\mu_n|(\mathbb{R}) = \sup_{n \geq n_0} |(x, e_n)| \leq \varepsilon$.

On the other hand, since each $x\mu_n$ is Radon, the finite set $\{x\mu_n; 1 \leq n < n_0\}$ is uniformly tight, so that there exists a compact subset K of $\mathbb{R}$ such that $\sup_{1 \leq n < n_0} |x\mu_n|(\mathbb{R} - K) < \varepsilon$. Consequently, we have

$$\sup_{n \geq 1} |x\mu_n|(\mathbb{R} - K) \leq \max \left(\sup_{1 \leq n < n_0} |x\mu_n|(\mathbb{R} - K), \sup_{n \geq n_0} |x\mu_n|(\mathbb{R}) \right) = \varepsilon,$$

which implies that $x(\mathcal{V})$ is uniformly tight.

However, $\mathcal{V}$ is not uniformly tight, which will be proved below: Put

$$\varepsilon_0 = 2 \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt > 0.$$

Since any compact subset K of $\mathbb{R}$ is contained in some bounded interval $[-N_0, N_0]$ ($N_0 \in \mathbb{N}$), we have

$$\begin{aligned} \|\mu_{N_0^2}\|(\mathbb{R} - K) &\geq m_{N_0^2}(\mathbb{R} - [-N_0, N_0]) \\ &= 2 \int_{N_0}^\infty \frac{1}{\sqrt{2\pi N_0^2}} e^{-t^2/(2N_0^2)} dt \\ &= 2 \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = \varepsilon_0, \end{aligned}$$

so that $\mathcal{V}$ is not uniformly tight.

Thanks to the above example, it is an interested problem to study the relation between the scalarly uniform tightness and the uniform tightness. In addition, the above example suggests that we need to study vector measures with values in not only normable spaces but locally convex spaces such as nuclear spaces.

3.4. Compactness and metrizability – Fréchet space-valued case. Now we shall explain some recent results of the study of compactness and metrizability in the space of vector measures. Let us begin with extending Prokhorov-LeCam's compactness criteria and Varadarajan's metrizability criterion to vector measures with values in a Fréchet space. The following theorem contains those criteria for real measures and a sequential compactness criterion given by [32, Theorem 1.5 and Corollary 1.6]; see also [20, Theorem 2].

THEOREM 3.5 ([20, Theorem 3]). *Let S be a completely regular space whose compact subsets are all metrizable. Let X be a Fréchet space whose topological dual X^* has a countable set which separates points of X (this is equivalent to X^* being separable for the weak topology $\sigma(X^*, X)$). Assume that $\mathcal{V} \subset \mathcal{M}_t(S, X)$ satisfies the following three conditions:*

- (i) $\mathcal{V}$ is uniformly bounded.
- (ii) $\mathcal{V}$ is uniformly tight.
- (iii) The set $\{\int_S f d\mu : f \in C(S), \|f\|_\infty \leq 1, \mu \in \mathcal{V}\}$ is relatively weakly compact in X .

Then, the closure of $\mathcal{V}$ with respect to the WTVM for $\sigma(X, X^)$ is compact and metrizable, so that it is sequentially compact in $\mathcal{M}_t(S, X)$ with respect to the WTVM for $\sigma(X, X^*)$. Further, if X is reflexive, (iii) follows from (i).*

REMARK 3.6. (1) Let S be a metric space and X a Banach space. Then the condition (iii) of Theorem 3.3 follows from the condition (iii) of Theorem 3.5. Indeed, we have only to

observe that for each compact subset K of S the set $\{\int_K f d\mu : f \in C(S), \|f\|_\infty \leq 1, \mu \in \mathcal{V}\}$ is contained in the weak closure of the set $\{\int_S f d\mu : f \in C(S), \|f\|_\infty \leq 1, \mu \in \mathcal{V}\}$. On the other hand, using Grothendieck's lemma [7, Lemma XIII.2], it is proved in [20, Remark] that for a uniformly tight subset $\mathcal{V}$ of $\mathcal{M}_t(S, X)$ the condition (iii) of Theorem 3.3 implies the condition (iii) of Theorem 3.5.

(2) Every locally compact separable metric space S is a Polish space (see L. Schwartz [37, Theorem 6, Chapter II]), so that by [45, Theorem 30, Part II] relative compactness coincides with the combination of uniform boundedness and uniform tightness for subsets of $\mathcal{M}_t(S)$. Therefore, Theorem 3.5 also extends Varadarajan's metrizability criterion to vector measures that take their values in a Fréchet space with a certain separability condition.

3.5. Compactness and metrizability – semi-reflexive or semi-Montel space-valued case. We turn our attention to vector measures with values in a semi-reflexive or a semi-Montel space. In this case, we have only to assume the *scalarly* uniform tightness for a bounded subset of $\mathcal{M}_t(S, X)$ to obtain its metrizability and sequential compactness. The following theorem contains Prokhorov-LeCam's sequential compactness criteria and Varadarajan's metrizability criterion for real measures. Further, it applies to the cases that vector measures take values in reflexive Banach spaces L^p and ℓ^p ($1 < p < \infty$) and in semi-Montel spaces such as the space $\mathcal{S}$ of all rapidly decreasing, infinitely differentiable functions, the space $\mathcal{D}$ of all test functions, and the strong duals of those spaces.

THEOREM 3.7 ([21, Theorem 2]). *Let S be a completely regular space whose compact subsets are all metrizable. Let X be a semi-reflexive space whose topological dual X^* has a countable set which separates points of X (this is equivalent to X^* being separable for the weak topology $\sigma(X^*, X)$). Assume that $\mathcal{V} \subset \mathcal{M}_t(S, X)$ is scalarly uniformly bounded and scalarly uniformly tight. Then, the closure of $\mathcal{V}$ with respect to the WTVM for $\sigma(X, X^*)$ is compact and metrizable, so that it is sequentially compact in $\mathcal{M}_t(S, X)$ with respect to the WTVM for $\sigma(X, X^*)$. When X is a semi-Montel space, the same conclusion holds with respect to the WTVM for the original topology of X .*

REMARK 3.8. It is readily seen that the above results characterize locally convex spaces which are semi-reflexive and semi-Montel.

3.6. A converse to Prokhorov-LeCam's compactness criteria. Let S be a complete separable metric space. It is known that a subset M of $\mathcal{M}_t(S)$ is uniformly bounded and uniformly tight if and only if it is relatively sequentially compact in $\mathcal{M}_t(S)$; see [45, Theorem 30, Part II]. This contains a converse to Theorem 3.1 and does not hold in general (not even for standard spaces; see, for instance, X. Fernique [11, Example 1.6.4]). The following theorem asserts that the same result stated above holds for vector measures that take their values in a semi-Montel space with a certain separability condition.

THEOREM 3.9 ([24]). *Let S be a complete separable metric space. Let X be a semi-Montel space whose topological dual X^* has a countable set which separates points of X . We equip $\mathcal{M}_t(S, X)$ with the WTVM for the original topology of X . Let $\mathcal{V} \subset \mathcal{M}(S, X)$. Then the following six conditions are equivalent:*

- (i) $\mathcal{V}$ is scalarly uniformly bounded and scalarly uniformly tight.
- (ii) For each $x^* \in X^*$, the closure of the set $x^*(\mathcal{V})$ is compact and metrizable in $\mathcal{M}_t(S)$.
- (iii) For each $x^* \in X^*$, the set $x^*(\mathcal{V})$ is relatively sequentially compact in $\mathcal{M}_t(S)$.
- (iv) $\mathcal{V}$ is uniformly bounded and uniformly tight.
- (v) The closure of $\mathcal{V}$ is compact and metrizable in $\mathcal{M}_t(S, X)$.
- (vi) $\mathcal{V}$ is relatively sequentially compact in $\mathcal{M}_t(S, X)$.

4. Weak convergence of injective tensor products of vector measures

In this section, we explain some results concerning the joint continuity of injective tensor product of vector measures with respect to the weak convergence in the following two cases: One is the case that the vector measures take values in some nuclear spaces such as the space $\mathcal{S}$, the space $\mathcal{D}$, and the strong duals of those spaces. The other is the case that they take values in the positive cone of Banach lattices.

4.1. Product measures of two vector measures. The notion of injective tensor product of vector measures was introduced by M. Duchon and I. Kluváněk [8] in 1967: Let X and Y be lcHs. Let $(\Omega, \mathcal{E})$ and $(\Gamma, \mathcal{F})$ be measurable spaces. Denote by $X \widehat{\otimes} Y$ and $X \widehat{\otimes}_\pi Y$ the injective and projective tensor products of X and Y , respectively; see H. Jarchow [16, 15.1 and 16.1]. Let $\mu \in \mathcal{M}(\Omega, X)$ and $\nu \in \mathcal{M}(\Gamma, Y)$. If a set C is of the form $C = \bigcup_{k=1}^n (E_k \times F_k)$, where the union is disjoint and $E_k \in \mathcal{E}$, $F_k \in \mathcal{F}$, then the set function $\lambda(C) = \sum_{k=1}^n \mu(E_k) \otimes \nu(F_k)$ is unambiguously defined on the field of sets of the above form C and is finitely additive. Then, it was proved in [8, Theorem] that λ is countably additive and can be uniquely extended to a countably additive set function, which is denoted by $\mu \widehat{\otimes} \nu$, on the σ -field $\mathcal{E} \times \mathcal{F}$ generated by all sets of the above form C with values in $X \widehat{\otimes} Y$. This vector measure is called the *injective tensor product* of μ and ν ; see also [27, Theorem]. This fact is not true in the case of the projective tensor product of X and Y , as it was shown in [26, Remarks]. However, if X is nuclear, then the projective tensor product $X \widehat{\otimes}_\pi Y$ coincides with the injective tensor product $X \widehat{\otimes} Y$, so that the projective tensor product of μ and ν exists.

The injective tensor product of two probability measures is just the usual product measure, so that its joint continuity is well-known in the case that the underlying topological spaces, on which measures are defined, are separable metric spaces (see P. Billingsley [3, Theorem 3.2]), and more generally completely regular spaces (see [43, Proposition I.4.1]). It was also shown in I. Csiszár [4, Corollary] that the convolution of probability measures on an arbitrary topological group is jointly continuous. These results are important and applicative in probability theory.

4.2. Joint continuity problem – nuclear space-valued case. We consider a joint continuity problem of vector measures with values in certain nuclear spaces. Let X be a lcHs. Denote by X_σ^* the weak dual of X , that is, the dual of X with the weak topology $\sigma(X^*, X)$. We also denote by X_β^* the strong dual of X , that is, the dual of X with the strong topology $\beta(X^*, X)$.

Throughout this subsection, let X be a strict inductive limit of an increasing sequence $\{X_n\}$ of nuclear Fréchet spaces and Y a strict inductive limit of an increasing sequence $\{Y_n\}$ of Fréchet spaces; see [16, 4.6]. Denote by $X \widehat{\otimes} Y$ a strict inductive limit of the increasing sequence

$\{X_n \widehat{\otimes}_\pi Y_n\}$ of the projective tensor products of X_n and Y_n . In this case, for $\mu \in \mathcal{M}(\Omega, X)$ and $\nu \in \mathcal{M}(\Gamma, Y)$ there exists a unique product measure $\mu \bar{\otimes} \nu : \mathcal{E} \times \mathcal{F} \rightarrow X \bar{\otimes} Y$ such that $(\mu \bar{\otimes} \nu)(E \times F) = \mu(E) \otimes \nu(F)$ for all $E \in \mathcal{E}$ and $F \in \mathcal{F}$. For, since X and Y are strict inductive limits of increasing sequences $\{X_n\}$ and $\{Y_n\}$, there exists an $n_0 \in \mathbb{N}$ such that $\mu \in \mathcal{M}(\Omega, X_{n_0})$ and $\nu \in \mathcal{M}(\Gamma, Y_{n_0})$. Since X_{n_0} is nuclear, the projective tensor product of X_{n_0} and Y_{n_0} coincides with the injective tensor product $X_{n_0} \widehat{\otimes} Y_{n_0}$, so that there exists a vector measure $\mu \widehat{\otimes} \nu : \mathcal{E} \times \mathcal{F} \rightarrow X_{n_0} \widehat{\otimes} Y_{n_0}$. It is obvious that $\mu \widehat{\otimes} \nu$ can be considered as a vector measure with values in $X \bar{\otimes} Y$, which we denote by $\mu \bar{\otimes} \nu$.

We also obtain a product of two vector measures with values in dual spaces. Since X_β^* is nuclear, for any $\mu \in \mathcal{M}(\Omega, X_\beta^*)$ and $\nu \in \mathcal{M}(\Gamma, Y_\beta^*)$, there exists a unique vector measure $\mu \widehat{\otimes} \nu \in \mathcal{M}(\Omega \times \Gamma, X_\beta^* \widehat{\otimes}_\pi Y_\beta^*)$ such that $(\mu \widehat{\otimes} \nu)(E \times F) = \mu(E) \otimes \nu(F)$ for all $E \in \mathcal{E}$ and $F \in \mathcal{F}$. Since $X_\beta^* \widehat{\otimes}_\pi Y_\beta^* = (X \bar{\otimes} Y)_\beta^*$, we may view the product as a vector measure with values in $(X \bar{\otimes} Y)^*$, and we still denote it by $\mu \bar{\otimes} \nu$ again.

EXAMPLE 4.1. (1) Let $\mathcal{S}(\mathbb{R}^m)$ and $\mathcal{S}(\mathbb{R}^n)$ be the spaces of all rapidly decreasing, infinitely differentiable functions on Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. These are examples of nuclear Fréchet spaces. The strong dual spaces $\mathcal{S}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^n)$ are called the spaces of all *slowly increasing distributions*. Then, we have the canonical isomorphisms (see F. Trèves [42, Theorem 51.6 and its Corollary]):

$$\mathcal{S}(\mathbb{R}^m) \widehat{\otimes}_\pi \mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{m+n}) \quad \text{and} \quad \mathcal{S}'(\mathbb{R}^m) \widehat{\otimes}_\pi \mathcal{S}'(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^{m+n}).$$

Consequently, for $\mu \in \mathcal{M}(\Omega, \mathcal{S}(\mathbb{R}^m))$ and $\nu \in \mathcal{M}(\Gamma, \mathcal{S}(\mathbb{R}^n))$, the tensor product $\mu \bar{\otimes} \nu$ exists and takes values in $\mathcal{S}(\mathbb{R}^{m+n})$. When $\mu \in \mathcal{M}(\Omega, \mathcal{S}'(\mathbb{R}^m))$ and $\nu \in \mathcal{M}(\Gamma, \mathcal{S}'(\mathbb{R}^n))$, then $\mu \bar{\otimes} \nu$ also exists and takes values in $\mathcal{S}'(\mathbb{R}^{m+n})$.

(2) Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets. Denote by $\mathcal{D}(U)$, $\mathcal{D}(V)$ and $\mathcal{D}(U \times V)$ the spaces of all test functions on U , V and $U \times V$, respectively. These are examples of lcHs whose type is a strict inductive limit of an increasing sequence of nuclear Fréchet spaces. The strong dual spaces $\mathcal{D}'(U)$, $\mathcal{D}'(V)$, and $\mathcal{D}'(U \times V)$ are called the spaces of all *distributions*. Then, we have the canonical isomorphisms (see A. Grothendieck [13, page 84, Chapter II] and [42, Theorem 51.7]):

$$\mathcal{D}(U) \bar{\otimes} \mathcal{D}(V) = \mathcal{D}(U \times V) \quad \text{and} \quad \mathcal{D}'(U \times V) = \mathcal{D}'(U) \widehat{\otimes}_\pi \mathcal{D}'(V).$$

Consequently, for $\mu \in \mathcal{M}(\Omega, \mathcal{D}(U))$ and $\nu \in \mathcal{M}(\Gamma, \mathcal{D}(V))$, the tensor product $\mu \bar{\otimes} \nu$ exists and takes values in $\mathcal{D}(U \times V)$. When $\mu \in \mathcal{M}(\Omega, \mathcal{D}'(U))$ and $\nu \in \mathcal{M}(\Gamma, \mathcal{D}'(V))$, then $\mu \bar{\otimes} \nu$ also exists and takes values in $\mathcal{D}'(U \times V)$.

In what follows, let S and T be completely regular spaces which satisfy $\mathcal{B}(S \times T) = \mathcal{B}(S) \times \mathcal{B}(T)$ (it is routine to check that this condition is satisfied, for instance, either S or T has a countable base of open sets). Then, we have an affirmative answer for a problem of joint continuity of product of vector measures with values in above nuclear spaces.

The following two theorems insist that the weak convergence of a net of tensor products of uniformly bounded vector measures follows from that of the corresponding net of real product measures. We recall that for $\mu \in \mathcal{M}(S, X)$ and $\nu \in \mathcal{M}(T, Y)$, the tensor product $\mu \bar{\otimes} \nu$ exists

and takes values in $Z := X \bar{\otimes} Y$, and Z_β^* can be identified with $X_\beta^* \widehat{\otimes}_\pi Y_\beta^*$ as a topological vector space.

THEOREM 4.2 ([17, Theorem 5]). *Let $\{\mu_\alpha\} \subset \mathcal{M}(S, X)$ and $\{\nu_\alpha\} \subset \mathcal{M}(T, Y)$ be uniformly bounded nets. Let $\mu \in \mathcal{M}(S, X)$ and $\nu \in \mathcal{M}(T, Y)$. Assume that for each $x^* \in X^*$ and $y^* \in Y^*$ the net $\{x^* \mu_\alpha \times y^* \nu_\alpha\}$ of real product measures converges weakly to the real product measure $x^* \mu \times y^* \nu$. Then $\{\mu_\alpha \bar{\otimes} \nu_\alpha\} \subset \mathcal{M}(S \times T, Z)$ converges weakly to $\mu \bar{\otimes} \nu \in \mathcal{M}(S \times T, Z)$ for $\sigma(Z, Z^*)$. Further, if Y is nuclear, it also converges weakly for the inductive limit topology on Z .*

In the case of vector measures with values in dual spaces, we have

THEOREM 4.3 ([17, Theorem 7]). *Let $\{\mu_\alpha\} \subset \mathcal{M}(S, X_\beta^*)$ and $\{\nu_\alpha\} \subset \mathcal{M}(T, Y_\beta^*)$ be uniformly bounded nets. Let $\mu \in \mathcal{M}(S, X_\beta^*)$ and $\nu \in \mathcal{M}(T, Y_\beta^*)$. Assume that for each $x \in X$ and $y \in Y$ the net $\{x \mu_\alpha \times y \nu_\alpha\}$ converges weakly to $x \mu \times y \nu$. Then $\{\mu_\alpha \bar{\otimes} \nu_\alpha\} \subset \mathcal{M}(S \times T, Z_\beta^*)$ converges weakly to $\mu \bar{\otimes} \nu \in \mathcal{M}(S \times T, Z_\beta^*)$ for $\sigma(Z^*, Z)$. Further, if Y is nuclear, it also converges weakly for $\beta(Z^*, Z)$.*

4.3. Banach lattice-valued measures. Let $(\Omega, \mathcal{E})$ be a measurable space. Let $(X, \leq)$ be a Banach lattice. When a Banach space X is equipped with the additional structure of a Banach lattice, we may introduce the notion of positivity for vector measures. We say that a vector measure $\mu : \mathcal{E} \rightarrow X$ is *positive* if $\mu(E) \geq 0$ for every $E \in \mathcal{E}$. By [38, Lemma 1.1], for every positive vector measure μ we have $\|\mu\|(E) = \|\mu(E)\|$ for all $E \in \mathcal{E}$. Further, it is easy to verify that for any μ -integrable, $\mathcal{E}$ -measurable real functions f and g with $|f| \leq g$ almost everywhere, we have

$$\left| \int_\Omega f d\mu \right| \leq \int_\Omega |f| d\mu \leq \int_\Omega g d\mu \quad \text{and} \quad \left\| \int_\Omega f d\mu \right\| \leq \left\| \int_\Omega g d\mu \right\|.$$

These facts greatly facilitate the analysis of positive vector measures. For further properties of positive vector measures on metric spaces see [32] and [38]. We refer the reader to the book of [35] for the basic theory of Banach lattices.

Let S be a uniform space. Denote by $U(S)$ the space of all uniformly continuous real functions on S . Let $(X, \leq)$ be a Banach lattice. Denote by $\mathcal{M}^+(S, X)$ the space of all positive vector measures $\mu : \mathcal{B}(S) \rightarrow X$.

Let $\{\mu_\alpha\}$ be a net in $\mathcal{M}(S, X)$ and $\mu \in \mathcal{M}(S, X)$. Recall that $\{\mu_\alpha\}$ converges weakly to μ , and we write $\mu_\alpha \xrightarrow{w} \mu$, if for every $f \in C(S)$ we have $\lim_\alpha \int_S f d\mu_\alpha = \int_S f d\mu$ in the norm of X . The following proposition asserts that the weak convergence of *positive* vector measures follows from the validity of the above convergence only for bounded *uniformly* continuous functions f on S ; see F. Topsøe [41, Theorem 8.1 (the Portmanteau Theorem)] for positive scalar measures.

PROPOSITION 4.4 ([22, Proposition 5.1]). *Let S be a uniform space and X a Banach lattice. Let $\{\mu_\alpha\}$ be a net in $\mathcal{M}^+(S, X)$ and μ a tight measure in $\mathcal{M}^+(S, X)$. Then the following two conditions are equivalent:*

- (i) *For every $f \in U(S)$, we have $\int_S f d\mu_\alpha \rightarrow \int_S f d\mu$.*
- (ii) *For every $f \in C(S)$, we have $\int_S f d\mu_\alpha \rightarrow \int_S f d\mu$.*

4.4. Injective tensor integral. We define the Bartle bilinear integration in our setting; see R. G. Bartle [2]. Let X and Y be Banach spaces. Denote by $X \widehat{\otimes} Y$ the injective tensor product of X and Y ; see [6, Chapter VIII]. Denote by χ_E the indicator function of a set E . Let $(\Gamma, \mathcal{F})$ be a measurable space and $\nu : \mathcal{F} \rightarrow Y$ a vector measure. A ν -null set is a set $F \in \mathcal{F}$ for which $\|\nu\|(F) = 0$; the term ν -almost everywhere refers to the complement of a ν -null set.

Given an X -valued simple function $\varphi = \sum_{k=1}^m x_k \chi_{F_k}$ with $x_1, \dots, x_m \in X$, $F_1, \dots, F_m \in \mathcal{F}$, $m \in \mathbb{N}$, define its integral $\int_F \varphi \widehat{\otimes} d\nu$ over a set $F \in \mathcal{F}$ by $\int_F \varphi \widehat{\otimes} d\nu = \sum_{k=1}^m x_k \otimes \nu(F_k \cap F)$. We say that a vector function $\varphi : \Gamma \rightarrow X$ is ν -measurable if there exists a sequence $\{\varphi_n\}$ of X -valued simple functions converging ν -almost everywhere to φ . The function φ is said to be ν -integrable in the sense of Bartle if there exists a sequence $\{\varphi_n\}$ of X -valued simple functions converging ν -almost everywhere to φ such that the sequence $\{\int_F \varphi_n \widehat{\otimes} d\nu\}$ converges in the norm of $X \widehat{\otimes} Y$ for each $F \in \mathcal{F}$. This limit $\int_F \varphi \widehat{\otimes} d\nu$ does not depend on the choice of such X -valued simple functions φ_n , $n \in \mathbb{N}$, and the indefinite integral $F \rightarrow \int_F \varphi \widehat{\otimes} d\nu$ is an $X \widehat{\otimes} Y$ -valued vector measure on $\mathcal{F}$.

For simplicity, we say that the φ is ν -integrable if it is ν -integrable in the sense of Bartle. The integral $\int_F \varphi \widehat{\otimes} d\nu$ is called the *injective tensor integral of φ over F with respect to ν* . See a recent paper of F. J. Freniche and J. C. García-Vázquez [12] for further properties of injective tensor integrals such as some characterizations of integrable functions and the general Fubini theorem.

Let T be a topological space. Here and in what follows, $C(T, X)$ denotes the Banach space of all bounded continuous functions $\varphi : T \rightarrow X$ with the norm $\|\varphi\|_\infty := \sup_{t \in T} \|\varphi(t)\|$. When $X = \mathbb{R}$, we write $C(T) := C(T, \mathbb{R})$. By the following proposition, every $\varphi \in C(T, X)$ is integrable with respect to any *tight* vector measure $\nu : \mathcal{B}(T) \rightarrow Y$.

PROPOSITION 4.5 ([22, Proposition 3.3]). *Let T be a topological space. Let X and Y be Banach spaces. Let $\nu : \mathcal{B}(T) \rightarrow Y$ be a tight vector measure and $\varphi \in C(T, X)$. Then, φ is ν -integrable, and $\|\int_F \varphi \widehat{\otimes} d\nu\| \leq \sup_{t \in F} \|\varphi(t)\| \cdot \|\nu\|(F)$ for all $F \in \mathcal{B}(T)$.*

4.5. A diagonal convergence theorem. Let T be a uniform space and X a Banach space. Denote by $U(T, X)$ the Banach space of all bounded uniformly continuous functions $\varphi : T \rightarrow X$ with the norm $\|\varphi\|_\infty := \sup_{t \in T} \|\varphi(t)\|$. When $X = \mathbb{R}$, we write $U(T) := U(T, \mathbb{R})$.

We give a diagonal convergence theorem for injective tensor integrals with respect to positive vector measures. The following theorem is not only crucial to prove our results, that is Theorems 4.7 and 4.8, but seems to be of some interest.

THEOREM 4.6 ([22, Theorem 4.1]). *Let T be a uniform space with the uniformity $\mathcal{U}_T$. Let X be a Banach space and Y a Banach lattice. Consider a net $\{\varphi_\alpha\} \subset U(T, X)$ and $\varphi \in U(T, X)$ satisfying the following conditions:*

- (i) $\varphi_\alpha(t) \rightarrow \varphi(t)$ for every $t \in T$;
- (ii) $\{\varphi_\alpha\}$ is uniformly bounded, that is, $\sup_\alpha \|\varphi_\alpha\|_\infty < \infty$; and
- (iii) $\{\varphi_\alpha\}$ is uniformly equicontinuous on T , that is, for any $\varepsilon > 0$, there exists a set $V \in \mathcal{U}_T$ such that $\sup_\alpha \|\varphi_\alpha(t) - \varphi_\alpha(t')\| < \varepsilon$ whenever $(t, t') \in V$.

Given a net $\{\nu_\alpha\}$ of tight measures in $\mathcal{M}^+(T, Y)$ and a tight and τ -smooth measure ν in $\mathcal{M}^+(T, Y)$, if $\lim_\alpha \int_T g d\nu_\alpha = \int_T g d\nu$ for every $g \in U(T)$, then $\lim_\alpha \int_T \varphi_\alpha \widehat{\otimes} d\nu_\alpha = \int_T \varphi \widehat{\otimes} d\nu$.

4.6. Joint continuity problem – Banach lattice-valued case. In 4.2, we have already studied a joint continuity problem for vector measures with values in certain nuclear spaces, such as the space $\mathcal{S}$, the space $\mathcal{D}$, and the strong duals of those spaces. The way of proving the joint continuity of product of nuclear space-valued measures is essentially based on a finite dimensional aspect of nuclear spaces, that is, the weak topology coincides with the original topology on every bounded subset of any barreled, quasi-complete nuclear space. Therefore, the same method may not apply to the case of vector measures with values in Banach spaces. We state here that the joint continuity of product measures remains true for the injective tensor products of positive vector measures in certain Banach lattices. Our approach to this problem is based on the Bartle bilinear vector integration [2].

Let S and T be uniform spaces. Let X and Y be Banach lattices. Let us recall that for any vector measures $\mu \in \mathcal{M}(S, X)$ and $\nu \in \mathcal{M}(T, Y)$ there exists a unique vector measure $\mu \widehat{\otimes} \nu : \mathcal{B}(S) \times \mathcal{B}(T) \rightarrow X \widehat{\otimes} Y$, which is called an *injective tensor product* of μ and ν , such that $(\mu \widehat{\otimes} \nu)(E \times F) = \mu(E) \otimes \nu(F)$ for all $E \in \mathcal{B}(S)$ and $F \in \mathcal{B}(T)$.

In the rest of this section, we assume that S and T satisfy $\mathcal{B}(S \times T) = \mathcal{B}(S) \times \mathcal{B}(T)$. This restriction, however, may be dropped if, for instance, both μ and ν are τ -smooth positive vector measures, and either of the ranges of μ and ν is separable, since in this case the injective tensor product measure $\mu \widehat{\otimes} \nu$ can be uniquely extended to a τ -smooth positive vector measure on $\mathcal{B}(S \times T)$, which contains $\mathcal{B}(S) \times \mathcal{B}(T)$ in general; see [23]. We can also obtain the same form of the general Fubini theorem [12, Theorem 13] for this extended injective tensor product measure.

Anyway, under our assumption, we can view the injective tensor product $\mu \widehat{\otimes} \nu$ as a vector measure defined on $\mathcal{B}(S \times T)$, and integrate every (uniformly) continuous real functions with respect to $\mu \widehat{\otimes} \nu$.

As an application of Theorem 4.6, we obtain the following result which seems to be of some interest.

THEOREM 4.7 ([22, Theorem 5.3]). *Let X and Y be Banach lattices. Let $\{\mu_\alpha\}$ be a net in $\mathcal{M}^+(S, X)$ and $\mu \in \mathcal{M}^+(S, X)$. Let $\{\nu_\alpha\}$ be a net of tight measures in $\mathcal{M}^+(T, Y)$ and ν a tight and τ -smooth measure in $\mathcal{M}^+(T, Y)$. If $\int_S f d\mu_\alpha \rightarrow \int_S f d\mu$ and $\int_T g d\nu_\alpha \rightarrow \int_T g d\nu$ for every $f \in U(S)$ and $g \in U(T)$, then $\int_{S \times T} h d(\mu_\alpha \widehat{\otimes} \nu_\alpha) \rightarrow \int_{S \times T} h d(\mu \widehat{\otimes} \nu)$ for every $h \in U(S \times T)$.*

Let X and Y be Banach lattices. Then, in general, the injective tensor product $X \widehat{\otimes} Y$ or the projective tensor product may not be a vector lattice for the natural ordering. However, the injective tensor products of some important examples of Banach lattices are also Banach lattices; see Example 4.10.

Let X and Y be Banach lattices such that the injective tensor product $X \widehat{\otimes} Y$ is also a Banach lattice satisfying the condition $x \otimes y \geq 0$ for every $x \geq 0$ and $y \geq 0$. Let $(\Omega, \mathcal{E})$ and $(\Gamma, \mathcal{F})$ be measurable spaces. Let $\mu : \mathcal{E} \rightarrow X$ and $\nu : \mathcal{F} \rightarrow Y$ be vector measures. Then it is easy to verify that if μ and ν are positive, so is the injective tensor product $\mu \widehat{\otimes} \nu$. In this case,

we have an affirmative answer for a problem of joint continuity of the injective tensor products with respect to the weak convergence of vector measures.

THEOREM 4.8 ([22, Theorem 5.4]). *Let X and Y be Banach lattices such that the injective tensor product $X \widehat{\otimes} Y$ is also a Banach lattice satisfying the condition $x \otimes y \geq 0$ for every $x \geq 0$ and $y \geq 0$. Let $\{\mu_\alpha\}$ be a net in $\mathcal{M}^+(S, X)$ and μ a tight measure in $\mathcal{M}^+(S, X)$. Let $\{\nu_\alpha\}$ be a net of tight measures in $\mathcal{M}^+(T, Y)$ and ν a tight and τ -smooth measure in $\mathcal{M}^+(T, Y)$. If $\mu_\alpha \xrightarrow{w} \mu$ and $\nu_\alpha \xrightarrow{w} \nu$, then $\mu_\alpha \widehat{\otimes} \nu_\alpha \xrightarrow{w} \mu \widehat{\otimes} \nu$.*

REMARK 4.9. In the special case that $X = Y = \mathbb{R}$, an alternative proof of Theorem 4.8 is executed by a well-known criterion that one can prove the weak convergence of μ_α to μ by showing that $\mu_\alpha(E) \rightarrow \mu(E)$ for some special class of sets E (see, for instance, [43, Corollary 1 to Theorem I.3.5 and Proposition I.4.1]). However, it seems that the usual proof of the above criterion does not work well for positive vector measures, since the notions of limit infimum and limit supremum cannot be extended to general Banach lattices.

We finish this section with examples of Banach lattices X and Y such that the injective tensor product $X \widehat{\otimes} Y$ is also a Banach lattice satisfying the condition $x \otimes y \geq 0$ for every $x \geq 0$ and $y \geq 0$; see examples in [35, pages 274–276] and [13, page 90, Chapter I].

EXAMPLE 4.10. (1) Let K be a compact space and Y be any Banach lattice. Then $C(K) \widehat{\otimes} Y$ is isometrically lattice isomorphic to the Banach lattice $C(K, Y)$. Especially, when $Y = C(L)$ for some compact space L , $C(K) \widehat{\otimes} C(L)$ is isometrically lattice isomorphic to $C(K \times L)$.

(2) Let P be a locally compact space and Y be any Banach lattice. Denote by $C_0(P, Y)$ the Banach lattice with its canonical ordering of all continuous functions $\varphi : P \rightarrow Y$ such that for every $\varepsilon > 0$ the set $\{s \in P : \|\varphi(s)\| \geq \varepsilon\}$ is compact. We write $C_0(P) := C_0(P, \mathbb{R})$. Then $C_0(P) \widehat{\otimes} Y$ is isometrically lattice isomorphic to $C_0(P, Y)$. Especially, when $Y = C_0(Q)$ for some locally compact space Q , $C_0(P) \widehat{\otimes} C_0(Q)$ is isometrically lattice isomorphic to $C_0(P \times Q)$.

(3) Let $(\Omega, \mathcal{E}, m)$ be a measure space and Y be any Banach lattice. Denote by $L^\infty(\Omega, Y)$ the Banach lattice of all (equivalence classes of) m -essentially bounded measurable functions $\varphi : \Omega \rightarrow Y$ with its canonical ordering. We write $L^\infty(\Omega) := L^\infty(\Omega, \mathbb{R})$. Then, $L^\infty(\Omega) \widehat{\otimes} Y$ is a Banach lattice. However, in general, $L^\infty(\Omega) \widehat{\otimes} Y$ is a proper closed subset of $L^\infty(\Omega, Y)$.

5. Strassen's theorem for positive vector measures

In a celebrated paper, V. Strassen [40] gave necessary and sufficient conditions for the existence of probability measures with given marginals. His results have been extended by many authors in more general settings; see, D. A. Edwards [10], G. Hansel and J. P. Troallic [14], H. G. Kellerer [25], H. J. Skala [39] and so on. In this section, we explain two types of Strassen's conditions for the existence of positive vector measures with given marginals.

5.1. Two theorems of V. Strassen. Let S and T be topological spaces. Denote by $\mathcal{M}_1^+(S)$ the space of all Radon probability measures on S with the weak topology of measures. Let us recall that a $r \in \mathcal{M}_1^+(S \times T)$ is called a measure *with marginals* $p \in \mathcal{M}_1^+(S)$ and $q \in \mathcal{M}_1^+(T)$ if $r(E \times T) = p(E)$ and $r(S \times F) = q(F)$ for all $E \in \mathcal{B}(S)$ and $F \in \mathcal{B}(T)$.

The following two types of Strassen's conditions for the existence of probability measures with given marginals are well-known and have many applications in the theory of probability and statistics.

THEOREM 5.1 ([39, Theorem 1]). *Let S and T be topological spaces. Let R be a non-empty closed convex subset of $\mathcal{M}_1^+(S)$. In order that there exists a $r \in R$ with given marginals $p \in \mathcal{M}_1^+(S)$ and $q \in \mathcal{M}_1^+(T)$, it is necessary and sufficient that*

$$\int_S f dp + \int_T g dq \leq \sup \left\{ \int_{S \times T} (f \oplus g) dr : r \in R \right\}$$

for all bounded Borel measurable functions $f : S \rightarrow \mathbb{R}$ and $g : T \rightarrow \mathbb{R}$, where $(f \oplus g)(s, t) := f(s) + g(t)$ for all $(s, t) \in S \times T$.

THEOREM 5.2 ([39, Corollary 6]). *Let S and T be topological spaces. Let D be a non-empty closed subset of $S \times T$. Let $\varepsilon > 0$. Then there exists a $r \in \mathcal{M}_1^+(S \times T)$ with given marginals $p \in \mathcal{M}_1^+(S)$ and $q \in \mathcal{M}_1^+(T)$ such that $r(D) \geq 1 - \varepsilon$ if and only if $p(E) + q(F) \leq 1 + \varepsilon$ whenever $E \times F \subset D^c$.*

An attempt to extend Strassen's results to vector measures has been made by I. März, R. M. Shortt and A. Hirshberg, and they deal with vector measures with values in the positive cone of a reflexive Banach lattice or a Banach lattice of a certain type: the so-called KB-spaces. A Banach lattice $(X, \leq)$ is called a *KB-space* if each norm bounded increasing sequence in X is convergent. The following extends Theorem 5.2 to positive vector measures with values in a KB-space.

THEOREM 5.3 ([15, Theorem 2]). *Let $\mathcal{E}$ and $\mathcal{F}$ be σ -fields of subsets of non-empty sets Ω and Γ , respectively. Let X be a KB-space. Let $\mu \in \mathcal{M}^+(\Omega, X)$ and $\nu \in \mathcal{M}^+(\Gamma, X)$ satisfy $\mu(\Omega) = \nu(\Gamma) = u$. Suppose that μ is perfect (see [38]) and that $D \in \mathcal{E} \times \mathcal{F}$ is a countable intersection of sets in the field on $\Omega \times \Gamma$ generated by all rectangles $E \times F$ for $E \in \mathcal{E}$ and $F \in \mathcal{F}$. For every positive element $v \in X$, the following are equivalent:*

- (i) *There exists a vector measure $\lambda \in \mathcal{M}^+(\Omega \times \Gamma, X)$ with marginals μ and ν such that $\lambda(D) \geq v$.*
- (ii) *For all $E \in \mathcal{E}$ and $F \in \mathcal{F}$, we have $\mu(E) + \nu(F) \leq 2u - v$ whenever $E \times F \subset D^c$.*

5.2. Another type of Strassen's theorem for vector measures. We extend Theorem 5.1 to positive vector measures with values in the weak dual of a barreled lcHs which has certain order conditions.

We recall that a vector space X with a partial ordering $\leq$ is an *ordered vector space* if

- (1) $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in X$;
- (2) $x \leq y$ implies $cx \leq cy$ for all $x, y \in X$ and $c > 0$.

A *Riesz space* is defined to be an ordered vector space such that every pair of elements x, y of X has a supremum $x \vee y$ and an infimum $x \wedge y$. An element $x \in X$ is said to be *positive* if $x \geq 0$. We say that an ordered vector space is *of type (R)* if for each $x \in X$, there exist two positive elements x^+ and x^- of X with $x = x^+ - x^-$. Riesz spaces are of type (R). See Example 5.6

for other ordered vector spaces of type (R) . We refer the reader to the book of [35] for further information on ordered vector spaces and Riesz spaces.

Let X be a lcHs and X_σ^* the weak dual of X , that is, the topological dual of X with the weak topology $\sigma(X^*, X)$. Denote by $\langle x, x^* \rangle$ the natural duality between X and X^* .

An element $x^* \in X^*$ is said to be *positive* if $\langle x, x^* \rangle \geq 0$ for any positive element $x \in X$. We say that a vector measure $\mu : \mathcal{B}(S) \rightarrow X_\sigma^*$ is *positive* if $\mu(E)$ is a positive element in X^* for all $E \in \mathcal{B}(S)$. Then it is easy to prove that $\mu \in \mathcal{M}_t(S, X_\sigma^*)$ is positive if and only if $\int_S f d(x\mu) \geq 0$ for every positive $x \in X$ and every $f \in C(S)$ with $f \geq 0$. Denote by $\mathcal{M}_t^+(S, X_\sigma^*)$ the set of all positive vector measures in $\mathcal{M}_t(S, X_\sigma^*)$ and we write $\mathcal{M}_t^+(S) := \mathcal{M}_t^+(S, \mathbb{R})$.

The following extends Theorem 5.1 to positive vector measures with values in the weak dual of a barreled lcHs which is an ordered vector space of type (R) .

THEOREM 5.4 ([18, Theorem 1]). *Let S and T be completely regular spaces. Let X be a barreled lcHs which is an ordered vector space of type (R) . Assume that Γ is a uniformly bounded, non-empty convex subset of $\mathcal{M}_t^+(S \times T, X_\sigma^*)$ which is closed for the WTVM for $\sigma(X^*, X)$. In order that there exists a $\gamma \in \Gamma$ with given marginals $\mu \in \mathcal{M}_t^+(S, X_\sigma^*)$ and $\nu \in \mathcal{M}_t^+(T, X_\sigma^*)$, it is necessary and sufficient that for every $\{f_i\}_{i=1}^n \subset C(S)$, $\{g_i\}_{i=1}^n \subset C(T)$ and $\{x_i\}_{i=1}^n \subset X$, we have*

$$\sum_{i=1}^n \left\langle x_i, \int_S f_i d\mu + \int_T g_i d\nu \right\rangle \leq \sup \left\{ \sum_{i=1}^n \left\langle x_i, \int_{S \times T} (f_i \oplus g_i) d\lambda \right\rangle : \lambda \in \Gamma \right\}.$$

REMARK 5.5. When X is reflexive, the existing measure $\gamma \in \Gamma$ in Theorem 5.4 is countably additive and Radon for the strong topology $\beta(X^*, X)$ since in this case $\mathcal{M}_t(S \times T, X_\sigma^*) = \mathcal{M}_t(S \times T, X_\beta^*)$; see [18, Remark 2].

EXAMPLE 5.6. (1) The following (a)–(g) are barreled lcHs which are Riesz spaces, and hence of type (R) :

(a) The Banach lattice $L^p(\Omega, \mathcal{E}, m)$ with a measure space $(\Omega, \mathcal{E}, m)$ and the Banach lattice ℓ^p ($1 \leq p \leq \infty$). Then $L^p(\Omega, \mathcal{E}, m)^* = L^q(\Omega, \mathcal{E}, m)$ and $(\ell^p)^* = \ell^q$ ($1 \leq p < \infty, 1/p + 1/q = 1$).

(b) The Banach lattice $C(S)$ with a topological space S . See N. Dunford and J. T. Schwartz [9, Theorems IV.6.2 and 6.3] for the topological dual of $C(S)$.

(c) The Banach lattice $\mathcal{M}(\Omega)$ of all real measures on a measurable space $(\Omega, \mathcal{E})$.

(d) Let S be a σ -compact and locally compact space. Denote by $\mathcal{C}(S)$ the space of all continuous real functions on S . We endow $\mathcal{C}(S)$ with the topology generated by the family of seminorms p_K given by $f \mapsto p_K(f) := \sup_{s \in K} |f(s)|$ (K varies in the family of all compact subsets of S). Then $\mathcal{C}(S)$ is a Fréchet space which is a Riesz space.

(e) Let S be a locally compact space. Denote by $\mathcal{C}_{00}(S)$ the space of all continuous real functions on S with compact support. For any fixed compact subset K of S , denote by C_K the Banach space of functions in $\mathcal{C}_{00}(S)$ that are supported by K , with the uniform norm. We endow $\mathcal{C}_{00}(S)$ with the inductive topology generated by the family of Banach spaces C_K . Then $\mathcal{C}_{00}(S)$ is a barreled lcHs which is a Riesz space, and the dual $\mathcal{C}_{00}(S)^*$ is the space of all real Radon measures on S ; see [34, pages 57 and 58].

(f) Let $\mathbb{R}^\infty$ be the Fréchet-Montel space of all real sequences with the topology of simple convergence. Let $\mathbb{R}_0^\infty$ be the Montel space of all real sequences which have only a finite number

of non-zero coordinates with the topology of uniform convergence on compact sets. We endow those spaces with the canonical coordinatewise order. Then they are Riesz spaces and we have that $(\mathbb{R}^\infty)^* = \mathbb{R}_0^\infty$ and $(\mathbb{R}_0^\infty)^* = \mathbb{R}^\infty$.

(g) Let $\Lambda(P)$ be the Köthe sequence space with a Köthe set P . Then it is a Fréchet space, provided that P is countable, and a Riesz space under the canonical coordinatewise order; see [16, pages 27, 50, 69 and 497] for definition and properties. Especially, the Fréchet-Montel space (s) of all rapidly decreasing sequences is a Riesz space and the dual $(s)^*$ is the space of all slowly increasing sequences.

(2) We present here some examples which are not Riesz spaces but of type (R) . Let H be a real Hilbert space with inner product $(\cdot, \cdot)$. Denote by $\mathcal{L}_s(H)$ and $\mathcal{C}_s(H)$ the Banach spaces of all bounded self-adjoint operators on H and of all completely continuous self-adjoint operators on H with the usual operator norm. We also denote by $\mathcal{T}_s(H)$ and $\mathcal{S}_s(H)$ the Banach space of all trace class self-adjoint operators on H with the trace norm and the Hilbert space of all Hilbert-Schmidt class self-adjoint operators on H with the Hilbert-Schmidt norm. We endow those spaces with the order defined by the relation " $A \leq B \Leftrightarrow (Ax, x) \leq (Bx, x)$ for all $x \in H$ ". For any $A \in \mathcal{L}_s(H)$, put $|A| = (A^2)^{1/2}$, $A^+ = (|A| + A)/2$ and $A^- = (|A| - A)/2$. Then they are positive operators on H . If A belongs to $\mathcal{L}_s(H)$, $\mathcal{C}_s(H)$, $\mathcal{T}_s(H)$ and $\mathcal{S}_s(H)$, then so do $|A|$, A^+ and A^- , and we have $A = A^+ - A^-$. Consequently, the above spaces are ordered vector spaces of type (R) and we have $\mathcal{C}_s(H)^* = \mathcal{T}_s(H)$, $\mathcal{T}_s(H)^* = \mathcal{L}_s(H)$ and $\mathcal{S}_s(H)^* = \mathcal{S}_s(H)$. See R. Schatten [36] for details.

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