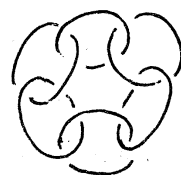


# 3 次元多様体の基本群の表現 I I

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右図の様な link  $L$  を考える。  
 $L$  の component を順に、 $L_1, L_2,$   
 $L_3, L_4, L_5, L_6$  とする。  $L$  の link 群を  
 $\pi(L)$  とすると  $\pi(L)$  の Wirtinger  
 表示は次の様になる。



$$\pi(L) \cong \langle x_1, x_2, x_3, x_4, x_5, x_6 : [x_i, x_{i-1}x_{i+1}^{-1}] = 1, i=1, \dots, 6(\text{mod } 6) \rangle$$

$$\cong \langle x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6 :$$

$$[x_i, y_i] = 1, x_{i-1}x_{i+1}^{-1}y_i^{-1} = 1, i=1, \dots, 6(\text{mod } 6) \rangle$$

第 2 の表示で  $y_1y_3y_5=1, y_2y_4y_6=1$  も成り立つ。

$L$  の各 component  $L_i$  を  $p_i/r_i$  Dehn surgery して得られる閉多様体を

$$M = L(p_1/r_1, p_2/r_2, p_3/r_3, p_4/r_4, p_5/r_5, p_6/r_6)$$

とし、 $M$  の基本群を  $\pi_1(M)$  とすれば

$$\pi_1(M) \cong \langle x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6 :$$

$$[x_i, y_i] = 1, x_{i-1} x_{i+1}^{-1} y_i^{-1} = 1, x_i^p = y_i^q, i=1, \dots, 6 \pmod{6} >$$

$$\cong \langle a_1, a_2, a_3, a_4, a_5, a_6 : a_{i-1}^{-1} a_{i+1}^{-1} a_i^{-p} = 1, i=1, \dots, 6 \pmod{6} \rangle$$

となる。

$\pi(L)$  の  $PSL(2, C)$  への表現が次の定理により構成される。

$z \in C$  に対し  $c(z) = (z+1/z)/2$ ,  $s(z) = (z-1/z)/2$  とおく。

定理  $\lambda_1, \lambda_3, \lambda_5, \mu_1, \mu_3, \mu_5$  を  $0, \pm 1$  と異なる 6 つの複素数とする。

これに対し複素数  $\lambda_2, \lambda_4, \lambda_6, \mu_2, \mu_4, \mu_6 = 0, \pm 1$  を、次の条件を満たす様にとったとする ( $i=2, 4, 6 \pmod{6}$ )。

$$c(\lambda_i) = c(\mu_{i-1})c(\mu_{i+1})$$

$$+(c(\lambda_{i-1})c(\lambda_{i+1}) - c(\lambda_{i+3}))s(\mu_{i-1})s(\mu_{i+1})/s(\lambda_{i-1})s(\lambda_{i+1}),$$

$$s(\lambda_i)c(\mu_i)/s(\mu_i) = -c(\lambda_{i-1})c(\mu_{i+1})s(\mu_{i-1})/s(\lambda_{i-1})$$

$$-c(\mu_{i-1})c(\lambda_{i+1})s(\mu_{i+1})/s(\lambda_{i+1})$$

$$-c(\mu_{i+3})s(\lambda_{i+3})s(\mu_{i-1})s(\mu_{i+1})/(s(\mu_{i+3})s(\lambda_{i-1})s(\lambda_{i+1})).$$

このとき  $SL(2, C)$  の元  $A_i$  ( $i=1, \dots, 6$ ) を適当にとると上の表示で

$$y_i \mapsto A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1}$$

$$x_i \longmapsto A_i \begin{pmatrix} \mu_i & 0 \\ 0 & \frac{1}{\mu_i} \end{pmatrix} A_i^{-1}$$

( $i=1, \dots, 6$ ) によって  $\pi(L)$  から  $PSL(2, C)$  への non-abelian 表現が定まる。

証明  $i = 1, 3, 5 \pmod{6}$  に対し

$$U_i = \begin{pmatrix} u_{i+1} & u_{i+2} \\ u_{i+3} & u_{i+4} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_{i+2} - \lambda_i \lambda_{i-2}}{\lambda_{i+2}(\lambda_{i-2}^2 - 1)} & \lambda_1 \lambda_3 \lambda_5 - 1 \\ \frac{\lambda_i - \lambda_{i-2} \lambda_{i+2}}{\lambda_i(\lambda_{i-2}^2 - 1)(\lambda_{i+2}^2 - 1)} & \frac{\lambda_{i+2}(\lambda_{i-2} - \lambda_i \lambda_{i+2})}{\lambda_i(\lambda_{i+2}^2 - 1)} \end{pmatrix}$$

とおくと

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \frac{1}{\lambda_1} \end{pmatrix} U_5 \begin{pmatrix} \lambda_3 & 0 \\ 0 & \frac{1}{\lambda_3} \end{pmatrix} U_1 \begin{pmatrix} \lambda_5 & 0 \\ 0 & \frac{1}{\lambda_5} \end{pmatrix} U_3 = E$$

$$U_1 U_3 U_5 = 1$$

$$u_{i+1} u_{i+3} = \{C(\lambda_{i-2}) - C(\lambda_{i-2})C(\lambda_{i+2}) + S(\lambda_{i-2})S(\lambda_{i+2})\} / \{2S(\lambda_{i-2})S(\lambda_{i+2})\},$$

$$u_{i+2} u_{i+4} = \{C(\lambda_i) - C(\lambda_{i-2})C(\lambda_{i+2}) - S(\lambda_{i-2})S(\lambda_{i+2})\} / \{2S(\lambda_{i-2})S(\lambda_{i+2})\}$$

(1)

が成り立つ。故に、 $A_1 = E$ ,  $A_3 = U_5$ ,  $A_5 = U_3^{-1}$  とおくと

$i = 1, 3, 5 \pmod{6}$  に対し  $U_i = A_{i+2}^{-1} A_{i-2}$  が成り立ち

$$\left\{ A_1 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \frac{1}{\lambda_1} \end{pmatrix} A_1^{-1} \right\} \left\{ A_3 \begin{pmatrix} \lambda_3 & 0 \\ 0 & \frac{1}{\lambda_3} \end{pmatrix} A_3^{-1} \right\} \left\{ A_5 \begin{pmatrix} \lambda_5 & 0 \\ 0 & \frac{1}{\lambda_5} \end{pmatrix} A_5^{-1} \right\} = E$$

となる。即ち、 $i = 1, 3, 5$  に対し

$$Y_i = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1} \quad X_i = A_i \begin{pmatrix} \mu_i & 0 \\ 0 & \frac{1}{\mu_i} \end{pmatrix} A_i^{-1}$$

とおけば  $Y_1 Y_3 Y_5 = E$  となる。これは、関係式  $y_1 y_3 y_5 = 1$  に対応する。つぎに、 $i = 2, 4, 6$  に対し、 $X_{i-1} X_{i+1}^{-1}$  の trace を計算すると

$$\begin{aligned} \text{Tr}(X_{i-1} X_{i+1}^{-1}) &= \text{Tr} \left( A_{i-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} A_{i-1}^{-1} A_{i+1} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} A_{i+1}^{-1} \right) \\ &= \text{Tr} \left( \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} A_{i-1}^{-1} A_{i+1} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} A_{i+1}^{-1} A_{i-1} \right) \\ &= \text{Tr} \left( \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} U_{i+3} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} U_{i+3}^{-1} \right) \end{aligned}$$

ところで

$$\begin{aligned}
& \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} U_{i+3} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} U_{i+3}^{-1} \\
&= \begin{pmatrix} \mu_{i-1} \left( \frac{1}{\mu_{i+1}} u_{i+31} u_{i+34} - \mu_{i+1} u_{i+32} u_{i+33} \right) & \mu_{i-1} \left( \mu_{i+1} - \frac{1}{\mu_{i+1}} \right) u_{i+31} u_{i+32} \\ \frac{1}{\mu_{i-1}} \left( \frac{1}{\mu_{i+1}} - \mu_{i+1} \right) u_{i+33} u_{i+34} & \frac{1}{\mu_{i-1}} \left( \mu_{i+1} u_{i+31} u_{i+34} - \frac{1}{\mu_{i+1}} u_{i+32} u_{i+33} \right) \end{pmatrix} \\
&\quad (2)
\end{aligned}$$

であるから

$$\begin{aligned}
& \text{Tr} (X_{i-1} X_{i+1}^{-1}) = \\
& \mu_{i-1} \left( \frac{1}{\mu_{i+1}} u_{i+31} u_{i+34} - \mu_{i+1} u_{i+32} u_{i+33} \right) + \frac{1}{\mu_{i-1}} \left( \mu_{i+1} u_{i+31} u_{i+34} - \frac{1}{\mu_{i+1}} u_{i+32} u_{i+33} \right) \\
&= \left( \frac{\mu_{i-1}}{\mu_{i+1}} + \frac{\mu_{i+1}}{\mu_{i-1}} \right) u_{i+31} u_{i+34} - \left( \mu_{i-1} \mu_{i+1} + \frac{1}{\mu_{i-1} \mu_{i+1}} \right) u_{i+32} u_{i+33} \\
&= \frac{(\mu_{i-1}/\mu_{i+1} + \mu_{i+1}/\mu_{i-1}) \{C(\lambda_{i+3}) - C(\lambda_{i+1})C(\lambda_{i-1}) + S(\lambda_{i+1})S(\lambda_{i-1})\}}{2S(\lambda_{i+1})S(\lambda_{i-1})} \\
&\quad - \frac{(\mu_{i-1}\mu_{i+1} + 1/(\mu_{i-1}\mu_{i+1})) \{C(\lambda_{i+3}) - C(\lambda_{i+1})C(\lambda_{i-1}) - S(\lambda_{i+1})S(\lambda_{i-1})\}}{2S(\lambda_{i+1})S(\lambda_{i-1})}
\end{aligned}$$

(1) による。

ここで

$$\frac{\mu_{i-1}}{\mu_{i+1}} + \frac{\mu_{i+1}}{\mu_{i-1}} = 2 (C(\mu_{i-1}) C(\mu_{i+1}) - S(\mu_{i-1}) S(\mu_{i+1}))$$

$$\mu_{i-1}\mu_{i+1} + \frac{1}{\mu_{i-1}\mu_{i+1}} = 2 (C(\mu_{i-1}) C(\mu_{i+1}) + S(\mu_{i-1}) S(\mu_{i+1}))$$

であるから

$$\text{Tr}(X_{i-1}X_{i+1}^{-1}) = 2(C(\mu_{i-1})C(\mu_{i+1}) - S(\mu_{i-1})S(\mu_{i+1}))$$

$$(C(\lambda_{i+3}) - C(\lambda_{i+1})C(\lambda_{i-1}) + S(\lambda_{i+1})S(\lambda_{i-1})) / \{2S(\lambda_{i+1})S(\lambda_{i-1})\}$$

$$-2(C(\mu_{i-1})C(\mu_{i+1}) + S(\mu_{i-1})S(\mu_{i+1}))(C(\lambda_{i+3}) -$$

$$C(\lambda_{i+1})C(\lambda_{i-1}) - S(\lambda_{i+1})S(\lambda_{i-1})) / \{2S(\lambda_{i+1})S(\lambda_{i-1})\}$$

$$= 2\{C(\mu_{i-1})C(\mu_{i+1}) + (C(\lambda_{i-1})C(\lambda_{i+1}) - C(\lambda_{i+3}))S(\mu_{i-1})$$

$$S(\mu_{i+1}) / (S(\lambda_{i-1})S(\lambda_{i+1}))$$

これは、定理の仮定から、 $2C(\lambda_i) = \lambda_i + \frac{1}{\lambda_i}$  に等しい。そして

$\lambda_i \neq \pm 2$ 。  $X_{i-1}X_{i+1}^{-1}$  の trace がこれに等しいのであるから

Jordan 標準形の定理により

$$X_{i-1}X_{i+1}^{-1} = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1}$$

となる  $A_i$  が  $SL(2, C)$  のなかに存在する。故に  $i = 2, 4, 6$

に対しても

$$Y_i = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1}, \quad X_i = A_i \begin{pmatrix} \mu_i & 0 \\ 0 & \frac{1}{\mu_i} \end{pmatrix} A_i^{-1}$$

とおけば  $Y_i = X_{i-1} X_{i+1}^{-1}$  となる。これは関係式  $y_i = x_{i-1} x_{i+1}^{-1}$

に対応する。また、明らかに、 $i = 1, 2, 3, 4, 5, 6$  に対し、

$[X_i, Y_i] = E$  であるから、あとは  $i = 1, 3, 5$  に対して

$Y_i \approx X_{i-1} X_{i+1}^{-1}$  を示せばよいことになる。但し  $\approx$  はスカラー倍を除いて

等しいことを表わす。即ち

$$A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1} \approx \left\{ A_{i-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} A_{i-1}^{-1} \right\} \left\{ A_{i+1} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} A_{i+1}^{-1} \right\}$$

であるが、これは  $V_i = A_{i-1}^{-1} A_i$  とおけば

$$\begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} V_{i+1} \begin{pmatrix} \mu_{i+1} & 0 \\ 0 & \frac{1}{\mu_{i+1}} \end{pmatrix} V_{i+1}^{-1} \approx V_{i-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} V_{i-1}^{-1} \quad (3)$$

と同値である。

これを示す為の準備をする。  $i = 2, 4, 6$  に対して

$$X_{i-1}^{-1} X_{i+1}^{-1} = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1} \quad (4)$$

であるから、この両辺の左から  $A_{i-1}^{-1}$  右から  $A_{i+1}$  を掛けると

$$\begin{aligned} A_{i-1}^{-1} X_{i-1}^{-1} X_{i+1}^{-1} A_{i+1} &= A_{i-1}^{-1} A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1} A_{i+1} \\ &= V_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} V_i^{-1} \end{aligned}$$

また

$$\begin{aligned} A_{i-1}^{-1} X_{i-1}^{-1} X_{i+1}^{-1} A_{i+1} &= \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} A_{i-1}^{-1} A_{i+1} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} A_{i+1}^{-1} A_{i-1} \\ &= \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} U_{i+3} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} U_{i+3}^{-1} \end{aligned}$$

ゆえに

$$V_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} V_i^{-1} = \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} U_{i+3} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} U_{i+3}^{-1}$$

となる。  $V_i = \begin{pmatrix} v_{i1} & v_{i2} \\ v_{i3} & v_{i4} \end{pmatrix}$  とおけば左辺は

$$\begin{pmatrix} \lambda_i v_{i1} v_{i4} - \frac{1}{\lambda_i} v_{i2} v_{i3} & (\frac{1}{\lambda_i} - \lambda_i) v_{i1} v_{i2} \\ (\lambda_i - \frac{1}{\lambda_i}) v_{i3} v_{i4} & \frac{1}{\lambda_i} v_{i1} v_{i4} - \lambda_i v_{i2} v_{i3} \end{pmatrix}$$

となる。右辺は (2) により



$$\begin{pmatrix} \mu_{i-1} \left( \frac{1}{\mu_{i+1}} u_{i+31} u_{i+34} - \mu_{i+1} u_{i+32} u_{i+33} \right) & \mu_{i-1} \left( \mu_{i+1} - \frac{1}{\mu_{i+1}} \right) u_{i+31} u_{i+32} \\ \frac{1}{\mu_{i-1}} \left( \frac{1}{\mu_{i+1}} - \mu_{i+1} \right) u_{i+33} u_{i+34} & \frac{1}{\mu_{i-1}} \left( \mu_{i+1} u_{i+31} u_{i+34} - \frac{1}{\mu_{i+1}} u_{i+32} u_{i+33} \right) \end{pmatrix}$$

に等しい。故に

$$\begin{aligned} V_{i+1} V_{i+2} &= - \frac{S(\mu_{i+1}) \mu_{i-1} (\lambda_{i-1} - \lambda_{i+3} \lambda_{i+1}) (\lambda_i \lambda_3 \lambda_5 - 1)}{S(\lambda_i) \lambda_{i-1} (\lambda_{i+1}^2 - 1)} \\ V_{i+3} V_{i+4} &= - \frac{S(\mu_{i+1}) \lambda_{i-1} (\lambda_{i+3} - \lambda_{i-1} \lambda_{i+1}) (\lambda_{i+1} - \lambda_{i-1} \lambda_{i+3})}{\mu_{i-1} S(\lambda_i) \lambda_{i+3}^2 (\lambda_{i-1}^2 - 1)^2 (\lambda_{i+1}^2 - 1)} \\ V_{i+1} V_{i+4} + V_{i+2} V_{i+3} &= \frac{C(\mu_{i-1}) S(\mu_{i+1}) \{C(\lambda_{i+1}) C(\lambda_{i-1}) - C(\lambda_{i+3})\}}{S(\lambda_i) S(\lambda_{i+1}) S(\lambda_{i-1})} + \frac{S(\mu_{i-1}) C(\mu_{i+1})}{S(\lambda_i)} \end{aligned}$$

となる。これは  $i=2, 4, 6$  に対してだから  $i=1, 3, 5$  に対しては

$$V_{i+1} V_{i+2} = - \frac{S(\mu_{i+2}) \mu_i (\lambda_i - \lambda_{i-2} \lambda_{i+2}) (\lambda_i \lambda_3 \lambda_5 - 1)}{S(\lambda_{i+1}) \lambda_i (\lambda_{i+2}^2 - 1)}$$

$$\begin{aligned}
V_{i+1} V_{i+4} + V_{i+2} V_{i+3} = & \frac{S(\mu_{i+2}) \lambda_i (\lambda_{i-2} - \lambda_i \lambda_{i+2}) (\lambda_{i+2} - \lambda_i \lambda_{i-2})}{\mu_i S(\lambda_{i+1}) \lambda_{i-2}^2 (\lambda_i^2 - 1)^2 (\lambda_{i+2}^2 - 1)} \\
& \frac{C(\mu_i) S(\mu_{i+2}) \{C(\lambda_{i+2}) C(\lambda_i) - C(\lambda_{i-2})\}}{S(\lambda_{i+1}) S(\lambda_{i+2}) S(\lambda_i)} \\
& + \frac{S(\mu_i) C(\mu_{i+2})}{S(\lambda_{i+1})} \quad (5)
\end{aligned}$$

となる。次に  $i=2, 4, 6$  に対して (4) の両辺の左から  $A_{i+1}^{-1}$ 、右から  $A_{i+1}$  を掛けると

$$\begin{aligned}
A_{i+1}^{-1} X_{i-1} X_{i+1}^{-1} A_{i+1} &= A_{i+1}^{-1} A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} A_i^{-1} A_{i+1} \\
&= V_{i+1}^{-1} \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} V_{i+1}
\end{aligned}$$

また

$$\begin{aligned}
A_{i+1}^{-1} X_{i-1} X_{i+1}^{-1} A_{i+1} &= A_{i+1}^{-1} A_{i-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} A_{i-1}^{-1} A_{i+1} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} \\
&= U_{i+3}^{-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} U_{i+3} \begin{pmatrix} \frac{1}{\mu_{i+1}} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\mu_{i+1}} (u_{i+31} u_{i+34} \mu_{i-1} - u_{i+32} u_{i+33} \frac{1}{\mu_{i-1}}) & \mu_{i+1} (\mu_{i-1} - \frac{1}{\mu_{i-1}}) u_{i+32} u_{i+34} \\ \frac{1}{\mu_{i+1}} (\frac{1}{\mu_{i-1}} - \mu_{i-1}) u_{i+31} u_{i+33} & \mu_{i+1} (u_{i+31} u_{i+34} \frac{1}{\mu_{i-1}} - u_{i+32} u_{i+33} \mu_{i-1}) \end{pmatrix}
\end{aligned}$$

であるから (1) を用いて

$$v_{i+2} v_{i+4} = \frac{S(\mu_{i-1}) \mu_{i+1} \lambda_{i-1} (\lambda_{i+1} - \lambda_{i-1} \lambda_{i+3}) (\lambda_1 \lambda_3 \lambda_5 - 1)}{S(\lambda_i) \lambda_{i+3} (\lambda_{i-1}^2 - 1)}$$

$$v_{i+1} v_{i+3} = \frac{S(\mu_{i-1}) (\lambda_{i-1} - \lambda_{i+3} \lambda_{i+1}) (\lambda_{i+3} - \lambda_{i-1} \lambda_{i+1})}{S(\lambda_i) \mu_{i+1} \lambda_{i-1} \lambda_{i+3} (\lambda_{i-1}^2 - 1) (\lambda_{i+1}^2 - 1)^2}$$

$$\begin{aligned} v_{i+1} v_{i+4} + v_{i+2} v_{i+3} &= \frac{C(\mu_{i+1}) S(\mu_{i-1}) \{C(\lambda_{i+3}) - C(\lambda_{i+1}) C(\lambda_{i-1})\}}{S(\lambda_i) S(\lambda_{i+1}) S(\lambda_{i-1})} \\ &\quad - \frac{S(\mu_{i+1}) C(\mu_{i-1})}{S(\lambda_i)} \end{aligned}$$

これは  $i=2, 4, 6$  に対してだから  $i=1, 3, 5$  に対しては

$$\begin{aligned} v_{i2} v_{i4} &= \frac{S(\mu_{i-2}) \mu_i \lambda_{i-2} (\lambda_i - \lambda_{i-2} \lambda_{i+2}) (\lambda_1 \lambda_3 \lambda_5 - 1)}{S(\lambda_{i-1}) \lambda_{i+2} (\lambda_{i-2}^2 - 1)} \\ v_{i1} v_{i3} &= \frac{S(\mu_{i-2}) (\lambda_{i-2} - \lambda_{i+2} \lambda_i) (\lambda_{i+2} - \lambda_{i-2} \lambda_i)}{S(\lambda_{i-1}) \mu_i \lambda_{i-2} \lambda_{i+2} (\lambda_{i-2}^2 - 1) (\lambda_i^2 - 1)^2} \quad (6) \end{aligned}$$

$$v_{i1} v_{i4} + v_{i2} v_{i3} = \frac{C(\mu_i) S(\mu_{i-2}) \{C(\lambda_{i+2}) - C(\lambda_i) C(\lambda_{i-2})\}}{S(\lambda_{i-1}) S(\lambda_i) S(\lambda_{i-2})} \\ - \frac{S(\mu_i) C(\mu_{i-2})}{S(\lambda_{i-1})}$$

となる。さて (3) を示そう。

$$\begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} v_{i+1} \begin{pmatrix} \mu_{i+1} & 0 \\ 0 & \frac{1}{\mu_{i+1}} \end{pmatrix} v_{i+1}^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

$$v_i^{-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & \frac{1}{\mu_{i-1}} \end{pmatrix} v_i = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

と置き

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \approx \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

を示す。

$$a_1 = \lambda_i \left( v_{i+1} v_{i+4} \mu_{i+1} - v_{i+2} v_{i+3} \frac{1}{\mu_{i+1}} \right)$$

$$b_i = \lambda_i \left( \frac{1}{\mu_{i+1}} - \mu_{i+1} \right) v_{i+11} v_{i+12}$$

$$c_i = \frac{1}{\lambda_i} \left( \mu_{i+1} - \frac{1}{\mu_{i+1}} \right) v_{i+13} v_{i+14}$$

$$d_i = \frac{1}{\lambda_i} \left( v_{i+11} v_{i+14} \frac{1}{\mu_{i+1}} - v_{i+12} v_{i+13} \mu_{i+1} \right)$$

$$a_2 = v_{i1} v_{i4} \mu_{i-1} - v_{i2} v_{i3} \frac{1}{\mu_{i-1}}$$

$$b_2 = \left( \mu_{i-1} - \frac{1}{\mu_{i-1}} \right) v_{i2} v_{i4}$$

$$c_2 = \left( \frac{1}{\mu_{i-1}} - \mu_{i-1} \right) v_{i1} v_{i3}$$

$$d_2 = v_{i1} v_{i4} \frac{1}{\mu_{i-1}} - v_{i2} v_{i3} \mu_{i-1}$$

である。(5)と(6)を用いて

$$b_i = \lambda_i (-2S(\mu_{i+1})) \frac{-S(\mu_{i+2}) \mu_i (\lambda_i - \lambda_{i-2} \lambda_{i+2}) (\lambda_1 \lambda_3 \lambda_5 - 1)}{S(\lambda_{i+1}) \lambda_i (\lambda_{i+2}^2 - 1)}$$

$$= \frac{S(\mu_{i+1}) S(\mu_{i+2})}{S(\lambda_{i+1}) S(\lambda_{i+2})} \frac{\mu_i (\lambda_i - \lambda_{i-2} \lambda_{i+2}) (\lambda_1 \lambda_3 \lambda_5 - 1)}{\lambda_{i+2}}$$

$$b_2 = 2S(\mu_{i-1}) \frac{S(\mu_{i-2}) \mu_i \lambda_{i-2} (\lambda_i - \lambda_{i-2} \lambda_{i+2}) (\lambda_1 \lambda_3 \lambda_5 - 1)}{S(\lambda_{i-1}) \lambda_i (\lambda_{i-2}^2 - 1)}$$

$$= \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})} \cdot \frac{\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_1\lambda_3\lambda_5 - 1)}{\lambda_{i+2}}$$

であるから

$$b = \mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_1\lambda_3\lambda_5 - 1)/\lambda_{i+2}$$

と置けば

$$b_1 = \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})} \quad b_3, \quad b_2 = \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})} \quad b_3 \quad (7)$$

となる。同様な計算を  $c_1, c_2$  について行い、

$$c_3 = - \frac{(\lambda_{i-2} - \lambda_i\lambda_{i+2})(\lambda_{i+2} - \lambda_i\lambda_{i-2})}{\mu_i^2 \lambda_{i-2} \lambda_{i+2} (\lambda_i^2 - 1)^2}$$

と置けば

$$c_1 = \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})} \quad c_3, \quad c_2 = \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})} \quad c_3 \quad (8)$$

となる。次に  $a_1$  と  $a_2$  を計算する。

$$\begin{aligned}
a_i &= \lambda_i (v_{i+1} v_{i+4} \mu_{i+1} - v_{i+2} v_{i+3} / \mu_{i+1}) \\
&= \lambda_i \{ (v_{i+1} v_{i+4} + v_{i+2} v_{i+3}) S(\mu_{i+1}) \\
&\quad + (v_{i+1} v_{i+4} - v_{i+2} v_{i+3}) C(\mu_{i+1}) \} \\
&= \lambda_i \{ (v_{i+1} v_{i+4} + v_{i+2} v_{i+3}) S(\mu_{i+1}) + C(\mu_{i+1}) \} \\
&= \lambda_i S(\mu_{i+1}) \{ (v_{i+1} v_{i+4} + v_{i+2} v_{i+3}) + C(\mu_{i+1}) / S(\mu_{i+1}) \} \\
&= \lambda_i S(\mu_{i+1}) \{ \frac{C(\mu_i) S(\mu_{i+2}) \{ C(\lambda_{i+2}) C(\lambda_i) - C(\lambda_{i-2}) \}}{S(\lambda_{i+1}) S(\lambda_{i+2}) S(\lambda_i)} \\
&\quad + \frac{S(\mu_i) C(\mu_{i+2})}{S(\lambda_{i+1})} + \frac{C(\mu_{i+1})}{S(\mu_{i+1})} \} \\
&= \lambda_i \frac{S(\mu_{i+1})}{S(\lambda_{i+1})} \{ \frac{C(\mu_i) S(\mu_{i+2}) \{ C(\lambda_{i+2}) C(\lambda_i) - C(\lambda_{i-2}) \}}{S(\lambda_{i+2}) S(\lambda_i)} \\
&\quad + S(\mu_i) C(\mu_{i+2}) + \frac{S(\lambda_{i+1}) C(\mu_{i+1})}{S(\mu_{i+1})} \}
\end{aligned}$$

(5) を用いた。ここで、定理の仮定より

$$S(\lambda_{i+1})C(\mu_{i+1})/S(\mu_{i+1}) = -S(\mu_i)C(\lambda_i)C(\mu_{i+2})/S(\lambda_i)$$

$$-S(\mu_{i+2})C(\mu_i)C(\lambda_{i+2})/S(\lambda_{i+2})$$

$$-S(\lambda_{i-2})S(\mu_i)S(\mu_{i+2})C(\mu_{i-2})/(S(\mu_{i-2})S(\lambda_i)S(\lambda_{i+2}))$$

であるから

$$\begin{aligned} a_i = \lambda_i & \frac{S(\mu_{i+1})}{S(\lambda_{i+1})} \left\{ \frac{C(\mu_i)S(\mu_{i+2})\{C(\lambda_{i+2})C(\lambda_i) - C(\lambda_{i-2})\}}{S(\lambda_{i+2})S(\lambda_i)} \right. \\ & + S(\mu_i)C(\mu_{i+2}) - \frac{S(\mu_i)}{S(\lambda_i)} C(\lambda_i)C(\mu_{i+2}) - \frac{S(\mu_{i+2})}{S(\lambda_{i+2})} C(\mu_i)C(\lambda_{i+2}) \\ & \left. - \frac{S(\lambda_{i-2})S(\mu_i)S(\mu_{i+2})}{S(\mu_{i-2})S(\lambda_i)S(\lambda_{i+2})} C(\mu_{i-2}) \right\} \\ = & \frac{S(\mu_{i+1})S(\mu_{i+2})\lambda_i}{S(\lambda_{i+1})S(\lambda_{i+2})S(\lambda_i)} \{C(\mu_i)C(\lambda_{i+2})C(\lambda_i) - C(\mu_i)C(\lambda_{i-2})\} \\ & + S(\mu_i) \frac{C(\mu_{i+2})S(\lambda_{i+2})S(\lambda_i)}{S(\mu_{i+2})} - \frac{S(\mu_i)C(\lambda_i)C(\mu_{i+2})S(\lambda_{i+2})}{S(\mu_{i+2})} \end{aligned}$$



$$\begin{aligned}
& -C(\mu_i)C(\lambda_{i+2})S(\lambda_i) - S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})/S(\mu_{i-2})\} \\
& = \frac{S(\mu_{i+1})S(\mu_{i+2})\lambda_i}{S(\lambda_{i+1})S(\lambda_{i+2})S(\lambda_i)} \{C(\mu_i)C(\lambda_{i+2})\{C(\lambda_i) - S(\lambda_i)\} \\
& + \frac{S(\mu_i)C(\mu_{i+2})S(\lambda_{i+2})(S(\lambda_i) - C(\lambda_i))}{S(\mu_{i+2})} - C(\mu_i)C(\lambda_{i-2}) \\
& - S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})/S(\mu_{i-2})\}
\end{aligned}$$

ここで  $C(\lambda_i) - S(\lambda_i) = 1/\lambda_i$  であるから

$$\begin{aligned}
a_i & = \frac{S(\mu_{i+1})S(\mu_{i+2})\lambda_i}{S(\lambda_{i+1})S(\lambda_{i+2})S(\lambda_i)} \{C(\mu_i)C(\lambda_{i+2})/\lambda_i \\
& - \frac{S(\mu_i)C(\mu_{i+2})S(\lambda_{i+2})}{S(\mu_{i+2})\lambda_i} - C(\mu_i)C(\lambda_{i-2}) \\
& - \frac{S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})}{S(\mu_{i-2})} \}
\end{aligned}$$

$$\begin{aligned}
&= \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})S(\lambda_i)} \{C(\mu_i)C(\lambda_{i+2}) - \\
&\quad \frac{S(\mu_i)C(\mu_{i+2})S(\lambda_{i+2})}{S(\mu_{i+2})} - \lambda_i C(\mu_i)C(\lambda_{i-2}) \\
&\quad - \frac{\lambda_i S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})}{S(\mu_{i-2})} \}
\end{aligned}$$

となり

$$\begin{aligned}
a_3 &= \frac{1}{S(\lambda_i)} \{C(\mu_i)C(\lambda_{i+2}) - S(\mu_i)C(\mu_{i+2})S(\lambda_{i+2})/S(\mu_{i+2}) \\
&\quad - \lambda_i C(\mu_i)C(\lambda_{i-2}) - \lambda_i S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})/S(\mu_{i-2})\}
\end{aligned}$$

と置けば

$$a_1 = \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})} a_3 \quad (9)$$

となる。また

$$\begin{aligned}
a_2 &= v_{i1} v_{i4} \mu_{i-1} - v_{i2} v_{i3} / \mu_{i-1} \\
&= (v_{i1} v_{i4} + v_{i2} v_{i3}) S(\mu_{i-1}) + (v_{i1} v_{i4} - v_{i2} v_{i3}) C(\mu_{i-1}) \\
&= (v_{i1} v_{i4} + v_{i2} v_{i3}) S(\mu_{i-1}) + C(\mu_{i-1}) \\
&= S(\mu_{i-1}) \{ (v_{i1} v_{i4} + v_{i2} v_{i3}) + C(\mu_{i-1}) / S(\mu_{i-1}) \} \\
&= S(\mu_{i-1}) \{ \frac{C(\mu_i) S(\mu_{i-2}) \{ C(\lambda_{i+2}) - C(\lambda_i) C(\lambda_{i-2}) \}}{S(\lambda_{i-1}) S(\lambda_i) S(\lambda_{i-2})} \\
&\quad - \frac{S(\mu_i) C(\mu_{i-1})}{S(\lambda_{i-1})} + \frac{C(\mu_{i-1})}{S(\mu_{i-1})} \} \\
&= \frac{S(\mu_{i-1})}{S(\lambda_{i-1})} \{ \frac{C(\mu_i) S(\mu_{i-2}) \{ C(\lambda_{i+2}) - C(\lambda_i) C(\lambda_{i-2}) \}}{S(\lambda_i) S(\lambda_{i-2})} \\
&\quad - S(\mu_i) C(\mu_{i-2}) + C(\mu_{i-1}) S(\lambda_{i-1}) / S(\mu_{i-1}) \}
\end{aligned}$$

ここで、定理の仮定より

$$\frac{C(\mu_{i-1}) S(\lambda_{i-1})}{S(\mu_{i-1})} = - \frac{S(\mu_{i-2})}{S(\lambda_{i-2})} C(\lambda_{i-2}) C(\mu_i)$$

$$- \frac{S(\mu_i)}{S(\lambda_i)} C(\mu_{i-2}) C(\lambda_i) - \frac{S(\lambda_{i+2}) S(\mu_{i-2}) S(\mu_i)}{S(\mu_{i+2}) S(\lambda_{i-2}) S(\lambda_i)} C(\mu_{i+2})$$

であるから

$$a_2 = \frac{S(\mu_{i-1})}{S(\lambda_{i-1})} \left\{ \frac{C(\mu_i) S(\mu_{i-2}) \{C(\lambda_{i+2}) - C(\lambda_i) C(\lambda_{i-2})\}}{S(\lambda_i) S(\lambda_{i-2})} \right.$$

$$- S(\mu_i) C(\mu_{i-2}) - \frac{S(\mu_{i-2})}{S(\lambda_{i-2})} C(\lambda_{i-2}) C(\mu_i)$$

$$- \frac{S(\mu_i)}{S(\lambda_i)} C(\mu_{i-2}) C(\lambda_i) - \frac{S(\lambda_{i+2}) S(\mu_{i-2}) S(\mu_i)}{S(\mu_{i+2}) S(\lambda_{i-2}) S(\lambda_i)} C(\mu_{i+2}) \}$$

$$= \frac{S(\mu_{i-1}) S(\mu_{i-2})}{S(\lambda_{i-1}) S(\lambda_{i-2}) S(\lambda_i)} \{C(\mu_i) C(\lambda_{i+2}) - C(\mu_i) C(\lambda_i) C(\lambda_{i-2})\}$$

$$- \frac{S(\mu_i) C(\mu_{i-2}) S(\lambda_{i-2}) S(\lambda_i)}{S(\mu_{i-2})} - S(\lambda_i) C(\lambda_{i-2}) C(\mu_i)$$

$$\begin{aligned}
& - \frac{S(\mu_{i-1})S(\lambda_{i-2})}{S(\mu_{i-2})} C(\mu_{i-2})C(\lambda_i) - \frac{S(\lambda_{i+2})S(\mu_i)C(\mu_{i+2})}{S(\mu_{i+2})} \\
& = \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})S(\lambda_i)} \{ C(\mu_i)C(\lambda_{i+2}) - C(\mu_i)C(\lambda_{i-2})S(\lambda_i) \\
& \quad + C(\lambda_i) \} - \frac{C(\mu_{i-2})S(\lambda_{i-2})S(\mu_i)\{S(\lambda_i) + C(\lambda_i)\}}{S(\mu_{i-2})} \\
& \quad - S(\lambda_i)C(\lambda_{i-2})C(\mu_i) - \frac{S(\lambda_{i+2})S(\mu_i)C(\mu_{i+2})}{S(\mu_{i+2})} \}
\end{aligned}$$

ここで  $S(\lambda_i) + C(\lambda_i) = \lambda_i$  であるから

$$\begin{aligned}
a_2 &= \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})S(\lambda_i)} \{ C(\mu_i)C(\lambda_{i+2}) - S(\mu_i)C(\mu_{i+2}) \\
& \quad S(\lambda_{i+2})/S(\mu_{i+2}) - \lambda_i C(\mu_i)C(\lambda_{i-2}) - \lambda_i S(\lambda_{i-2})S(\mu_i)C(\mu_{i-2})/S(\mu_{i-2}) \} \\
&= \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})} a_3 \quad (10)
\end{aligned}$$

となる。同様な計算により

$$d_3 = - \frac{C(\mu_i)C(\lambda_{i+2})S(\mu_{i+2}) + S(\mu_i)S(\lambda_{i+2})C(\mu_{i+2})}{S(\lambda_i)S(\mu_{i+2})} \\ + \frac{C(\mu_i)C(\lambda_{i-2})S(\mu_{i-2}) - S(\mu_i)S(\lambda_{i-2})C(\mu_{i-2})}{\lambda_i S(\lambda_i)S(\mu_{i-2})}$$

とおけば

$$d_1 = \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})} d_3, \quad d_2 = \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})} d_3 \quad (11)$$

となる。ゆえに、(7), (8), (9), (10), (11)により

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \frac{S(\mu_{i+1})S(\mu_{i+2})}{S(\lambda_{i+1})S(\lambda_{i+2})} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}, \\ \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \frac{S(\mu_{i-1})S(\mu_{i-2})}{S(\lambda_{i-1})S(\lambda_{i-2})} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$$

となり

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \approx \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

が示された。

(証明終わり)