

Relative t -designs in Johnson association schemes for P-polynomial structure

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1 Introduction

Relative t -designs were defined in Q-polynomial and P-polynomial association schemes respectively by Delsarte [6] in 1977 and Bannai–Bannai–Suda–Tanaka [1] in 2015. Actually, relative t -designs in P-polynomial association schemes were first introduced by Delsarte–Seidel [7] in 1998 for binary Hamming association scheme $H(n, 2)$. They called such designs regular t -wise balanced designs which are equivalent to relative t -designs in $H(n, 2)$ for P-polynomial structure with respect to the fixed point $(0, 0, \dots, 0)$. In 2015, Bannai–Bannai–Suda–Tanaka [1] proposed the definition for general P-polynomial association schemes and proved that the concepts of relative t -designs in Hamming association schemes $H(n, q)$ ($q \geq 2$) for P-polynomial structure and Q-polynomial structure are equivalent. Using this good property, Bannai–Bannai–Zhu [2] proved a necessary and sufficient condition for relative t -designs on two shells of $H(n, 2)$. In addition, they proved that if (Y, w) is a relative t -design in $H(n, 2)$ on p shells then the subset of (Y, w) on each shell must be a usual (weighted) combinatorial $(t+1-p)$ -design. It is an interesting question to ask how the situation is in the case of Johnson association scheme $J(v, k)$. Each nontrivial shell X_r of $J(v, k)$ is known to be a commutative association scheme which is the product of two smaller Johnson association schemes. Bannai–Zhu [4] studied relative t -designs on one shell X_r in $J(v, k)$ for Q-polynomial structure and proved that they are $\mathcal{T}$ -designs in X_r for $\mathcal{T} = \{(t_1, t_2) \mid 0 \leq t_1 + t_2 \leq t\}$. In particular, if the weight is constant, then relative t -designs on one shell in $J(v, k)$ are mixed t -designs in $J(k, k_1) \otimes J(v-k, k_2)$ which were introduced and studied by Martin [8]. It is well known that Johnson association schemes are both P- and Q-polynomial association schemes. Therefore it would be interesting to ask the similar question for P-polynomial structure whether we can expect to regard relative t -designs on one shell of $J(v, k)$ as weighted $\mathcal{T}$ -designs in X_r for some set $\mathcal{T}$.

In this report, we investigate relative t -designs in $J(v, k)$ for P-polynomial structure and give the answer to this question. The main result is that if (Y, w) is a relative t -design supported by one shell X_r in $J(v, k)$ for P-polynomial structure, then (Y, w) is a weighted $\mathcal{T}$ -design in X_r with $\mathcal{T} = \{(t_1, t_2) \mid 0 \leq t_1, t_2 \leq t\}$. We also discuss the existence problem of tight relative t -designs. We make an algorithm to construct tight relative 2-designs in one shell and obtain many examples. In addition, we can construct some tight relative 3-designs on one shell X_{2u} in $J(8u, 4u)$ for integer $u \geq 1$.

2 Preliminaries

In this section, we will give the definition of t -designs in Q-polynomial association schemes introduced by Delsarte [5], designs in product of Q-polynomial association schemes by Martin [9] and relative t -designs in P-polynomial association schemes by Bannai–Bannai–Suda–Tanaka [1]. (Please refer to [3] for more information on P-polynomial or Q-polynomial association schemes.)

Throughout this report (Y, w) is assumed to be a weighted subset of X , namely, Y is a non-empty finite subset of X and $w : Y \rightarrow \mathbb{R}_{>0}$.

2.1 Definition of t -designs

Definition 2.1 ([5, Theorem 3.10]). Let $\mathfrak{X} = (X, \{R_r\}_{r=0}^k)$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \dots, E_k$. A weighted subset (Y, w) of X is called a weighted t -design in $\mathfrak{X}$ if $E_j \chi_{(Y,w)} = 0$ for all $1 \leq j \leq t$, where $\chi_{(Y,w)}$ is the weighted characteristic vector of (Y, w) defined by

$$\chi_{(Y,w)} = \begin{cases} w(y), & \text{if } y \in Y, \\ 0, & \text{if } y \notin Y. \end{cases}$$

Definition 2.2. Let $X = \binom{V}{k}$ be the set of all k -subsets of V with $|V| = v$. A weighted subset (Y, w) of X is called a weighted t -(v, k, λ_t) design if for any $z \in \binom{V}{t}$ the following value

$$\sum_{y \in Y, z \subset y} w(y) = \lambda_t$$

is a constant depending only on t but not on the choice of z .

Remark 1. Delsarte [5, Theorem 4.7] proved that a t -design in Johnson association scheme $J(v, k)$ for Q-polynomial structure, which is a weighted t -design with constant weight $w = 1$, is equivalent to a combinatorial t -(v, k, λ_t) design. We should remark that this result is also true for (non-constant) weighted t -designs in $J(v, k)$ and weighted t -(v, k, λ_t) designs.

2.2 Designs in product of Q-polynomial association schemes

In this subsection, we recall the concept of designs in product of Q-polynomial association schemes introduced by Martin [9]. For any positive integer k_i , let $\mathcal{C}_i$ be the totally ordered chain on $\{0, 1, \dots, k_i\}$ and set $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$. Consider the poset $(\mathcal{C}, \leq)$ defined by

$$\mathcal{C} = \{\underline{\ell} = (\ell_1, \ell_2) \mid 0 \leq \ell_i \leq k_i, i = 1, 2\}$$

with partial order $\underline{\ell}' \leq \underline{\ell}$ if $\ell'_1 \leq \ell_1$ and $\ell'_2 \leq \ell_2$. A subset $\mathcal{T}$ of $\mathcal{C}$ is called a *downset* in $(\mathcal{C}, \leq)$ if $\underline{\ell} \in \mathcal{T}$ and $\underline{\ell}' \leq \underline{\ell}$ imply $\underline{\ell}' \in \mathcal{T}$. For any set $\mathcal{E} \subset \mathcal{C}$, denote

$$\mathcal{E} + \mathcal{E} := \{(\ell_1 + \ell'_1, \ell_2 + \ell'_2) \mid (\ell_1, \ell_2), (\ell'_1, \ell'_2) \in \mathcal{E}\}.$$

Let $\mathfrak{X}^{(i)} = (X^{(i)}, \{R_r^{(i)}\}_{r=0}^{k_i})$ be a Q-polynomial association scheme and $E_0^{(i)}, E_1^{(i)}, \dots, E_{k_i}^{(i)}$ its Q-polynomial ordering of primitive idempotents for $i = 1, 2$. For two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ from $X^{(1)} \times X^{(2)}$, define a relation $R_{(h_1, h_2)}$ on $X^{(1)} \times X^{(2)}$ by

$$(x, y) \in R_{(h_1, h_2)} \quad \text{if} \quad (x_1, y_1) \in R_{h_1}^{(1)}, (x_2, y_2) \in R_{h_2}^{(2)}.$$

Define $\mathfrak{X} = \mathfrak{X}^{(1)} \otimes \mathfrak{X}^{(2)} = (X^{(1)} \times X^{(2)}, \mathcal{R})$ as the product of these two association schemes, where the relations set $\mathcal{R}$ is

$$\mathcal{R} = \{R_{(h_1, h_2)} \mid 0 \leq h_1 \leq k_1, 0 \leq h_2 \leq k_2\}.$$

In particular, if $\mathfrak{X} = J(v_1, k_1) \otimes J(v_2, k_2)$, then, for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \binom{V_1}{k_1} \times \binom{V_2}{k_2}$, $(x, y) \in R_{(h_1, h_2)}$ means $|x_1 \cap y_1| = k_1 - h_1$ and $|x_2 \cap y_2| = k_2 - h_2$. Moreover, for a set $Y \subset \binom{V_1}{k_1} \times \binom{V_2}{k_2}$, we define the *distance set* of Y by

$$A(Y) = \{(h_1, h_2) \mid (x, y) \in R_{(h_1, h_2)}, x, y \in Y, x \neq y\}.$$

Note that $(0, 0) \notin A(Y)$.

Definition 2.3 ([9, Theorem 2.3]). Let $\mathcal{T}$ be a downset of $\mathcal{C}$. The weighted subset (Y, w) of $X = X^{(1)} \times X^{(2)}$ is called a weighted $\mathcal{T}$ -design in $\mathfrak{X} = \mathfrak{X}^{(1)} \otimes \mathfrak{X}^{(2)}$ if

$$(E_{t_1}^{(1)} \otimes E_{t_2}^{(2)})\chi_{(Y, w)} = 0, \quad \text{for all } (t_1, t_2) \in \mathcal{T} \setminus \{(0, 0)\}.$$

In particular, if $\mathfrak{X} = J(v_1, k_1) \otimes J(v_2, k_2)$ with points set $X = \binom{V_1}{k_1} \times \binom{V_2}{k_2}$, then we have another equivalent definition of $\mathcal{T}$ -designs given by Martin [9, Lemma 2.2]. Define a partial order $\preceq$ on the set $\tilde{X} = \binom{V_1}{\leq k_1} \times \binom{V_2}{\leq k_2}$ by $z \preceq y$ if $z_1 \subseteq y_1$ and $z_2 \subseteq y_2$ for $z = (z_1, z_2)$ and $y = (y_1, y_2)$ in $\tilde{X}$.

Definition 2.4 ([8]). Let (Y, w) be a weighted subset of $\binom{V_1}{k_1} \times \binom{V_2}{k_2}$. The pair (Y, w) is called a weighted (t_1, t_2) -($v_1, k_1, v_2, k_2, \lambda_{(t_1, t_2)}$) design if for any $(z_1, z_2) \in \binom{V_1}{t_1} \times \binom{V_2}{t_2}$, the following value

$$\sum_{\substack{(y_1, y_2) \in Y \\ (z_1, z_2) \preceq (y_1, y_2)}} w(y_1, y_2) = \lambda_{(t_1, t_2)}$$

is a constant depending only on the pair (t_1, t_2) but not on the choice of (z_1, z_2) .

By Definition 2.4, we obtain the following lemma (see also [8]).

Lemma 2.5. *Let (Y, w) be a weighted (t_1, t_2) -($v_1, k_1, v_2, k_2, \lambda_{(t_1, t_2)}$) design. Then it is also a weighted (s_1, s_2) -($v_1, k_1, v_2, k_2, \lambda_{(s_1, s_2)}$) design satisfying*

$$\binom{v_1 - s_1}{t_1 - s_1} \binom{v_2 - s_2}{t_2 - s_2} \lambda_{(t_1, t_2)} = \binom{k_1 - s_1}{t_1 - s_1} \binom{k_2 - s_2}{t_2 - s_2} \lambda_{(s_1, s_2)}, \quad (2.1)$$

whenever $0 \leq s_1 \leq t_1$ and $0 \leq s_2 \leq t_2$. In particular, $\lambda_{(0, 0)} = |Y|$.

Lemma 2.5 implies that a weighted (t, t) -($v_1, k_1, v_2, k_2, \lambda_{(t, t)}$) design is exactly a weighted $\mathcal{T}$ -design in $J(v_1, k_1) \otimes J(v_2, k_2)$ with $\mathcal{T} = \{(t_1, t_2) \mid 0 \leq t_1, t_2 \leq t\}$.

2.3 Relative t -designs in P-polynomial association schemes

Now let us give the definition of relative t -designs in P-polynomial association schemes introduced by Bannai–Bannai–Suda–Tanaka [1]. Let $\mathfrak{X} = (X, \{R_r\}_{r=0}^k)$ be a P-polynomial association scheme and u_0 a fixed point in X . For $0 \leq r \leq k$, we call the subset $X_r := \{x \in X \mid (x, u_0) \in R_r\}$ the r -th shell of $\mathfrak{X}$. Let $\mathbb{R}^{|X|}$ be the space consisting of column vectors indexed by the points in X . Given a point $z \in X_j$, define a vector $f_z \in \mathbb{R}^{|X|}$ whose x -th entry is

$$f_z(x) = \begin{cases} 1 & \text{if } x \in X_i, (x, z) \in R_{i-j}, i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\text{Hom}_j(X) := \text{span}\{f_z \mid z \in X_j\}$ for $0 \leq j \leq k$, then

$$\mathbb{R}^{|X|} = \text{Hom}_0(X) + \text{Hom}_1(X) + \cdots + \text{Hom}_k(X).$$

We say Y is a subset of X on p shells $X_{r_1} \cup \cdots \cup X_{r_p}$ if $\{r \mid Y \cap X_r \neq \emptyset\} = \{r_1, r_2, \dots, r_p\}$. Denote $Y_{r_\nu} = Y \cap X_{r_\nu}$ for $1 \leq \nu \leq p$.

Definition 2.6 ([1, Definition 1.1]). Let $\mathfrak{X} = (X, \{R_r\}_{r=0}^k)$ be a P-polynomial association scheme and u_0 a fixed point in X . A weighted subset (Y, w) is called a relative t -design on p shells $X_{r_1} \cup \cdots \cup X_{r_p}$ in $\mathfrak{X}$ with respect to u_0 if the following

$$\sum_{\nu=1}^p \frac{W(Y_{r_\nu})}{|X_{r_\nu}|} \sum_{x \in X_{r_\nu}} f(x) = \sum_{y \in Y} w(y) f(y) \quad (2.2)$$

holds for any $f \in \text{Hom}_0(X) + \text{Hom}_1(X) + \cdots + \text{Hom}_t(X)$, where $W(Y_{r_\nu}) = \sum_{y \in Y_{r_\nu}} w(y)$.

3 Relative t -designs in one shell of $J(v, k)$

In this section, we discuss relative t -designs on one shell of Johnson association scheme $J(v, k)$ for P-polynomial structure. We first describe the structure of each nontrivial shell of $J(v, k)$. Let $X = \binom{V}{k}$ be the set of all k -subsets of $V = \{1, \dots, v\}$ with $v \geq 2k$ and $J(v, k) = (X, \{R_r\}_{r=0}^k)$ Johnson association scheme. For any fixed point $u_0 \in X$, denote $X_r := \{x \in X \mid |x \cap u_0| = k - r\}$. Without loss of generality, we may assume $u_0 = \{1, 2, \dots, k\}$. Let $V_1 = u_0$, $V_2 = V \setminus u_0$ and denote $v_i = |V_i|$ for $i = 1, 2$.

Proposition 3.1. *Each nontrivial shell X_r of $J(v, k)$ is identified with the product of two smaller Johnson association schemes $J(k, k_1) \otimes J(v - k, k_2)$. More explicitly,*

- (1) $(X_r, \{R_h\}_{h=0}^k) \cong J(k, r) \otimes J(v - k, r)$ if $1 \leq r \leq \frac{k}{2}$.
- (2) $(X_r, \{R_h\}_{h=0}^k) \cong J(k, k - r) \otimes J(v - k, r)$ if $\frac{k}{2} < r \leq \frac{v-k}{2}$.
- (3) $(X_r, \{R_h\}_{h=0}^k) \cong J(k, k - r) \otimes J(v - k, v - k - r)$ if $\frac{v-k}{2} < r \leq k$.

Proof. For cases (1), (2) and (3), define the bijection $\phi : X_r \longrightarrow \binom{V_1}{k_1} \times \binom{V_2}{k_2}$ respectively by

$$\phi(x) = (V_1 \setminus x, V_2 \cap x), \quad (V_1 \cap x, V_2 \cap x), \quad (V_1 \setminus x, V_2 \setminus x).$$

We can check that, for $x, y \in X_r$, $(x, y) \in R_{h_1+h_2}$ if and only if $(\phi(x), \phi(y)) \in R_{(h_1, h_2)}$. Therefore $(X_r, \{R_h\}_{h=0}^k) \cong J(k, k_1) \otimes J(v - k, k_2)$. $\square$

Remark 2. Without confusion, we identify X_r with the product association scheme $J(k, k_1) \otimes J(v - k, k_2)$ whenever we mention designs in X_r . In addition, if (Y, w) is a weighted subset of one shell X_r of $J(v, k)$, then the above map ϕ preserves the weight as well, namely, $w(y) = w(\phi(y))$ for any $y \in Y$.

Now we are ready to give our main theorem. By definition 2.6, if $p = 1$, i.e., $Y \subset X_r$, then we have the specific condition for relative t -designs supported by one shell X_r as follows.

$$\frac{W(Y_r)}{|X_r|} \sum_{x \in X_r} f_z(x) = \sum_{y \in Y} w(y) f_z(y), \quad \forall z \in X_0 \cup X_1 \cup \cdots \cup X_t. \quad (3.1)$$

Theorem 3.2. *If (Y, w) is a relative t -design in $J(v, k)$ supported by one shell X_r , then (Y, w) is a weighted $\mathcal{T}$ -design in X_r with $\mathcal{T} = \{(t_1, t_2) \mid 0 \leq t_1, t_2 \leq t\}$.*

Using Lemma 2.5, it is enough to prove that (Y, w) is a (t, t) -($v_1, k_1, v_2, k_2, \lambda_{(t,t)}$) design for some constant $\lambda_{(t,t)}$ if $(X_r, \{R_h\}_{h=0}^k) \cong J(v_1, k_1) \otimes J(v_2, k_2)$. More precisely, we need to prove that $\sum_{\substack{y \in \phi(Y) \\ (z_1, z_2) \preceq y}} w(y)$ is constant for any $(z_1, z_2) \in \binom{V_1}{t} \times \binom{V_2}{t}$.

Remark 3. In Theorem 3.2, (Y, w) is a weighted $\mathcal{T}$ -design in X_r means $(\phi(Y), w)$ is a weighted $\mathcal{T}$ -design in $J(k, k_1) \otimes J(v - k, k_2)$, where ϕ is the bijection defined in the proof of Proposition 3.1.

4 Lower bound for relative t -designs on one shell

In this section, we give the lower bound for relative t -designs on one shell X_r of $J(v, k)$. By Theorem 3.2, it is equivalent to obtain the lower bound for weighted $\mathcal{T}$ -designs in product association scheme $J(k, k_1) \otimes J(v - k, k_2)$ with $\mathcal{T} = \{(t_1, t_2) \mid 0 \leq t_1, t_2 \leq t\}$. The following lower bound of designs in product association schemes was proved by Martin [8].

Lemma 4.1 ([8, Theorem 3.2]). *Let $\mathcal{T}$ be a downset in $(\mathcal{C}, \preceq)$ and $\mathcal{E}$ a set satisfying $(\mathcal{E} + \mathcal{E}) \cap \mathcal{C} \subseteq \mathcal{T}$. If Y is a $\mathcal{T}$ -design in $J(v_1, k_1) \otimes J(v_2, k_2)$, then*

$$|Y| \geq \sum_{(j_1, j_2) \in \mathcal{E}} \left[\binom{v_1}{j_1} - \binom{v_1}{j_1 - 1} \right] \cdot \left[\binom{v_2}{j_2} - \binom{v_2}{j_2 - 1} \right]. \quad (4.1)$$

Moreover, if equality holds, then for any $(h_1, h_2) \in A(Y)$ we have

$$\sum_{(j_1, j_2) \in \mathcal{E}} Q_{j_1}^{(1)}(h_1) Q_{j_2}^{(2)}(h_2) = 0,$$

where $Q_{j_i}^{(i)}(x)$ is the following Hahn polynomial corresponding to $J(v_i, k_i)$

$$Q_j^{(i)}(x) = \left(\binom{v_i}{j} - \binom{v_i}{j-1} \right) {}_3F_2 \left(\begin{matrix} -x, -j, -v_i - 1 + j \\ -k_i, -v_i + k_i \end{matrix}; 1 \right).$$

From the proof of Theorem 3.2 in [8], one can check that inequality (4.1) also holds for weighted $\mathcal{T}$ -designs in $J(v_1, k_1) \otimes J(v_2, k_2)$.

Corollary 4.2. *If (Y, w) is a relative $2e$ -design in $J(v, k)$ on one shell X_r with P -polynomial structure, then*

$$|Y| \geq \binom{k}{e} \binom{v-k}{e}. \quad (4.2)$$

It follows from Theorem 3.2 that (Y, w) is a weighted $\mathcal{T}$ -design for $\mathcal{T} = \{(t_1, t_2) \mid 0 \leq t_1, t_2 \leq 2e\}$. Take $\mathcal{E} = \{(j_1, j_2) \mid 0 \leq j_1, j_2 \leq e\}$. Using Lemma 4.1, we obtain

$$|Y| \geq \sum_{j_1=0}^e \sum_{j_2=0}^e \left[\binom{k}{j_1} - \binom{k}{j_1-1} \right] \cdot \left[\binom{v-k}{j_2} - \binom{v-k}{j_2-1} \right] = \binom{k}{e} \binom{v-k}{e}.$$

Proposition 4.3. *If (Y, w) is a relative $(2e+1)$ -design in $J(v, k)$ on one shell X_r with P -polynomial structure, then*

$$|Y| \geq 4 \binom{k-1}{e} \binom{v-k-1}{e}. \quad (4.3)$$

A relative t -design on one shell of $J(v, k)$ is called *tight* if equality holds in (4.2) or (4.3).

At the end of this subsection, we introduce the concept of projections for designs which will be used later. Given a (t_1, t_2) -($v_1, k_1, v_2, k_2, \lambda_{(t_1, t_2)}$) design Y , define the left and right projection of Y as follows.

$$\begin{aligned} Y^{(L)} &= \{y^{(L)} \mid (y^{(L)}, y^{(R)}) \in Y\}. \\ Y^{(R)} &= \{y^{(R)} \mid (y^{(L)}, y^{(R)}) \in Y\}. \end{aligned}$$

Then $Y^{(L)}$ are t_1 -($v_1, k_1, \lambda_{(t_1, 0)}$) designs and $Y^{(R)}$ are t_2 -($v_2, k_2, \lambda_{(0, t_2)}$) designs.

5 Tight relative 2-designs

In this section, we give an algorithm to construct tight relative 2-designs (with constant weight) on one shell of $J(v, k)$. We also provide two explicit examples.

Assume the weight function is constant, i.e., $w = 1$. Let $(Y, 1)$ be a tight relative 2-design on one shell X_r of $J(v, k)$. Using Lemma 4.1, we obtain the distance set $A(Y)$ from the zeros of following equation.

$$F(h_1, h_2) = \sum_{\underline{j} \in \mathcal{E}} Q_{\underline{j}}(\underline{h}) = 1 + Q_1^{(1)}(h_1) + Q_1^{(2)}(h_2) + Q_1^{(1)}(h_1)Q_1^{(2)}(h_2),$$

where $\mathcal{E} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. More explicitly, we have

$$F(h_1, h_2) = \frac{v_1 v_2 (v_1 h_1 - h_1 - v_1 k_1 + k_1^2) (h_2 v_2 - h_2 - k_2 v_2 + k_2^2)}{k_1 k_2 (v_1 - k_1) (v_2 - k_2)}. \quad (5.1)$$

Then we know that

$$A(Y) \subseteq \left\{ \left(\frac{k_1(v_1 - k_1)}{v_1 - 1}, h_2 \right), \left(h_1, \frac{k_2(v_2 - k_2)}{v_2 - 1} \right) \mid 0 \leq h_1 \leq k_1, 0 \leq h_2 \leq k_2, (h_1, h_2) \neq (0, 0) \right\}$$

The construction of explicit tight relative 2-designs on one shell X_r in $J(v, k)$ (i.e. tight $(2, 2)$ -($v_1, k_1, v_2, k_2, \lambda_{(2,2)}$) designs) Y is equivalent to the following problem.

Problem. For a point set $X = X^{(L)} \times X^{(R)} = \binom{V_1}{k_1} \times \binom{V_2}{k_2}$, find a pair of 2-($v_1, k_1, \lambda_{(2,0)}$) design $Y^{(L)} = \{y_1^{(L)}, \dots, y_{\lambda_{(0,0)}}^{(L)}\} \subset X^{(L)}$ and 2-($v_2, k_2, \lambda_{(0,2)}$) design $Y^{(R)} = \{y_1^{(R)}, \dots, y_{\lambda_{(0,0)}}^{(R)}\} \subset X^{(R)}$ with the same cardinality, so that $Y = \{(y_i^{(L)}, y_i^{(R)}) \mid 1 \leq i \leq \lambda_{(0,0)}\} \subset Y^{(L)} \times Y^{(R)}$ satisfies the condition that, for each $\{p_1, q_1\} \subset V_1$ and $\{p_2, q_2\} \subset V_2$, there exists exactly $\lambda_{(2,2)}$ element(s) $y \in Y$ such that $(\{p_1, q_1\}, \{p_2, q_2\}) \preceq y$. The partial order $\preceq$ means $\{p_1, q_1\} \subset y^{(L)}$ and $\{p_2, q_2\} \subset y^{(R)}$. Moreover, the parameters $\lambda_{(0,0)}$, $\lambda_{(2,0)}$, $\lambda_{(0,2)}$ and $\lambda_{(2,2)}$ satisfy the relation given by Eq. (2.1).

5.1 Basic idea for the construction

Now we explain the basic idea of an algorithm to construct $(2, 2)$ -($v_1, k_1, v_2, k_2, \lambda_{(2,2)}$) designs for a given 2-($v_1, k_1, \lambda_{(2,0)}$) design as the left projection $Y^{(L)}$. The algorithm is applicable for cases when the size of 2-($v_2, k_2, \lambda_{(2,2)}$) design is equal to $\lambda_{(2,0)}$. Define $\tilde{Y}^{(L)} = (y_1^{(L)}, y_2^{(L)}, \dots, y_{\lambda_{(0,0)}}^{(L)})$ and $\tilde{Y}^{(R)} = (y_1^{(R)}, y_2^{(R)}, \dots, y_{\lambda_{(0,0)}}^{(R)})$. We introduce the notation of $\rho(p, q; z)$ for a given $z = (z_1, z_2, \dots, z_{\lambda_{(0,0)}})$ and $p, q \in V_1$ ($p \neq q$), defined as

$$\rho(p, q; z) = (z_{i_1}, z_{i_2}, \dots, z_{i_{\lambda_{(2,0)}}}) \text{ for } \{i_1, i_2, \dots, i_{\lambda_{(2,0)}}\} = \{i \mid \{p, q\} \subset y_i^{(L)} \in \tilde{Y}^{(L)}\}.$$

We observe that $\rho(p, q; \tilde{Y}^{(R)})$ is a 2-($v_2, k_2, \lambda_{(2,2)}$) design. This will be the key fact in the algorithm. We use this fact as a condition to fix the right (ordered) projection $\tilde{Y}^{(R)}$.

The essential idea of the algorithm is the following. We choose one 2-($v_1, k_1, \lambda_{(2,0)}$) design as the left (ordered) projection $Y^{(L)}$. The right (ordered) projection $\tilde{Y}^{(R)}$ is unknown. Denote $c = (\emptyset, \dots, \emptyset)$ of length $\lambda_{(0,0)}$. Since $\rho(p, q; \tilde{Y}^{(R)})$ for $\{p, q\} \subset V_1$ is a 2-($v_2, k_2, \lambda_{(2,2)}$) design, we assign one 2-($v_2, k_2, \lambda_{(2,2)}$) design to $\rho(p, q; c)$. We do this for all choices of $\{p, q\} \subset V_1$. If we find a c that is consistent with all conditions from $\rho(p, q; c)$, then we take it as $\tilde{Y}^{(R)}$.

5.2 Examples for tight relative 2-designs

Example 1. One trivial example for tight relative 2-design (i.e., $(2, 2)$ -($v_1, k_1, v_2, k_2, \lambda_{(2,2)}$) design) Y is the product of a symmetric 2-(v_1, k_1, λ_1) design $Y^{(L)}$ and a symmetric 2-(v_2, k_2, λ_2) design $Y^{(R)}$ with the same cardinality and $\lambda_1 \cdot \lambda_2 = \lambda_{(2,2)}$. Namely,

$$Y = \{(y_i, y_j) \mid y_i \in Y^{(L)}, y_j \in Y^{(R)}, 1 \leq i \leq v_1, 1 \leq j \leq v_2\}.$$

Example 2. Tight relative 2-designs on X_3 in $J(14, 7)$, i.e., tight $(2, 2)$ -($7, 3, 7, 3, 1$) designs.

Using the algorithm, we obtain many tight $(2, 2)-(7, 3, 7, 3, 1)$ designs. Since both the left and right projections of a $(2, 2)-(7, 3, 7, 3, 1)$ design are $2-(7, 3, 7)$ designs, take the second $2-(7, 3, 7)$ design on Spence's homepage [10] as the left projection. We give two examples which is constructed from that $2-(7, 3, 7)$ design. Let

$$Y_1 = \{(y_{i_1}, y_{j_1}), (y_{i_2}, y_{j_2}), (y_{i_0}, y_{j_1}) \mid i_\ell \in I_\ell, j_\ell \in J_\ell, \ell = 1, 2\},$$

$$Y_2 = \{(y_{i_1}, y_{j_1}), (y_{i_2}, y_{j_2}), (y_{i_0}, y_{j_3}) \mid i_\ell \in I_\ell, j_m \in J_m, \ell = 1, 2, m = 1, 1, 3.\}$$

where $I_1 = \{1, 2, 3, 4, 7, 9\}$, $I_2 = \{5, 6, 8, 11\}$, $J_1 = \{1, 2, 3, 4, 7, 9\}$, $J_2 = \{10\}$, $J_3 = \{1, 12, 13, 14, 15, 9\}$. Here y_i denotes the i -th block of the list given below.

$$\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 4, 7\}, \{2, 5, 6\}, \{2, 5, 7\}, \{3, 4, 6\}, \{3, 4, 7\}, \{3, 5, 6\},$$

$$\{3, 5, 7\}, \{1, 4, 6\}, \{1, 5, 7\}, \{2, 4, 5\}, \{2, 6, 7\}.$$

Then both Y_1 and Y_2 are tight $(2, 2)-(7, 3, 7, 3, 1)$ designs with the following distance sets.

$$A(Y_1) = \{(0, 2), (2, 0), (2, 2), (1, 2), (3, 2)\},$$

$$A(Y_2) = \{(0, 2), (2, 0), (2, 2), (2, 1), (2, 3), (1, 2), (3, 2)\}.$$

Example 3. A tight relative 2-design on X_4 in $J(26, 13)$, i.e., tight $(2, 2)-(13, 4, 13, 4, 1)$ design.

Both the left and right projections of a $(2, 2)-(13, 4, 13, 4, 1)$ design are $2-(13, 4, 13)$ designs. It is known that there is a unique symmetric $2-(13, 4, 1)$ -design $\mathcal{D}_{13}$ up to automorphism. Take twelve copies of $\mathcal{D}_{13}$ and one more design by permuting the points of $\mathcal{D}_{13}$ labelled by 12 and 13, then we have a $2-(13, 4, 13)$ design as the left projection $Y^{(L)}$. Using the algorithm, we construct a tight $(2, 2)-(13, 4, 13, 4, 1)$ design Y in the following.

$$Y = \{(y_{i_1}, y_{j_1}), (y_{i_2}, y_{j_2}), (y_{i_3}, y_{18}) \mid i_\ell \in I_\ell, j_m \in J_m, \ell = 1, 2, 3, m = 1, 2\},$$

where $I_1 = \{1, 4, 7, 8, 10, 22, 30\}$, $I_2 = \{13, 16, 20, 23, 26, 29\}$, $I_3 = \{14, 15, 19, 24, 27, 28\}$, $J_1 = \{1, 4, 7, 8, 10, 13, 16, 20, 22, 23, 26, 29, 30\}$, $J_2 = \{2, 3, 5, 6, 9, 11, 12, 17, 21, 25, 31, 32\}$. Here y_i denotes the i -th block of the list given below

$$\{1, 2, 3, 4\}, \{1, 2, 3, 13\}, \{1, 4, 7, 12\}, \{1, 5, 6, 7\}, \{1, 5, 9, 11\}, \{1, 6, 8, 10\}, \{1, 8, 9, 10\},$$

$$\{1, 11, 12, 13\}, \{2, 4, 9, 10\}, \{2, 5, 8, 11\}, \{2, 5, 8, 12\}, \{2, 6, 7, 11\}, \{2, 6, 9, 12\}, \{2, 6, 9, 13\},$$

$$\{2, 7, 10, 12\}, \{2, 7, 10, 13\}, \{3, 4, 8, 11\}, \{3, 5, 7, 10\}, \{3, 5, 9, 12\}, \{3, 5, 9, 13\}, \{3, 6, 9, 12\},$$

$$\{3, 6, 10, 11\}, \{3, 7, 8, 12\}, \{3, 7, 8, 13\}, \{4, 5, 6, 13\}, \{4, 5, 10, 12\}, \{4, 5, 10, 13\}, \{4, 6, 8, 12\},$$

$$\{4, 6, 8, 13\}, \{4, 7, 9, 11\}, \{7, 8, 9, 13\}, \{10, 11, 12, 13\}.$$

The distance set of Y equals

$$A(Y) = \{(0, 3), (1, 3), (2, 3), (3, 0), (3, 1), (3, 2), (3, 3), (3, 4), (4, 3)\}.$$

5.3 Tight relative 3-designs on one shell of $J(v, k)$

In this subsection, we will discuss the existence problem of tight relative 3-designs (Y, w) in $J(v, k)$ on one shell with constant weight, i.e., $w = 1$.

Let $V_1 = u_0$, $V_2 = V \setminus u_0$. We introduce the notation

$$Y_{(W_1, W_2)}^{(U_1, U_2)} = \{(y_1 - W_1, y_2 - W_2) \mid (y_1, y_2) \in \phi(Y), W_i \subset y_i \subset V_i - U_i\},$$

where W_i and U_i are subsets of V_i such that $W_i \cap U_i = \emptyset$ for $i = 1, 2$. Using the proof of Proposition 4.3, if there exists a tight relative 3-design Y on one shell X_r of $J(v, k)$, then we have four tight $\{(2, 2)\}$ -designs $Y_{(\alpha, \beta)}^{(\emptyset, \emptyset)}$, $Y_{(\alpha, \emptyset)}^{(\emptyset, \beta)}$, $Y_{(\emptyset, \beta)}^{(\alpha, \emptyset)}$ and $Y_{(\emptyset, \emptyset)}^{(\alpha, \beta)}$ respectively in product association schemes $J(v'_1, k_1 - 1) \otimes J(v'_2, k_2 - 1)$, $J(v'_1, k_1 - 1) \otimes J(v'_2, k_2)$, $J(v'_1, k_1) \otimes J(v'_2, k_2 - 1)$ and $J(v'_1, k_1) \otimes J(v'_2, k_2)$, where $v'_1 = k - 1$, $v'_2 = v - k - 1$ and

$$(k_1, k_2) = \begin{cases} (r, r), & \text{if } 1 \leq r \leq \frac{k}{2}, \\ (k - r, r), & \text{if } \frac{k}{2} < r \leq \frac{v-k}{2}, \\ (k - r, v - k - r), & \text{if } \frac{v-k}{2} < r \leq k - 1. \end{cases}$$

According to the computer search, there is only one family of possible parameters, i.e., $(v, k, r) = (8u, 4u, 2u)$ when $v \leq 3000$.

We can construct a tight relative 3-design on one shell X_{2u} in $J(8u, 4u)$. Choose any tight relative 2-design on one shell X_{2u-1} in $J(8u - 2, 4u - 1)$ as $Y_{(\alpha, \beta)}^{(\emptyset, \emptyset)}$. Replacing the left (resp. right) projection of $Y_{(\alpha, \beta)}^{(\emptyset, \emptyset)}$ by its complementary design, then this new design is $Y_{(\alpha, \emptyset)}^{(\emptyset, \beta)}$ (resp. $Y_{(\emptyset, \beta)}^{(\alpha, \emptyset)}$). Replace both the left and right projection of $Y_{(\alpha, \beta)}^{(\emptyset, \emptyset)}$ by their complementary designs and denote this new design as $Y_{(\emptyset, \emptyset)}^{(\alpha, \beta)}$. Define the sets Y_1, Y_2, Y_3, Y_4 as follows.

$$\begin{aligned} Y_1 &= \{(Y' \cup \{\alpha\}, Y'' \cup \{\beta\}) \mid (Y', Y'') \in Y_{(\alpha, \beta)}^{(\emptyset, \emptyset)}\}, \\ Y_2 &= \{(Y' \cup \{\alpha\}, Y'') \mid (Y', Y'') \in Y_{(\alpha, \emptyset)}^{(\emptyset, \beta)}\}, \\ Y_3 &= \{(Y', Y'' \cup \{\beta\}) \mid (Y', Y'') \in Y_{(\emptyset, \beta)}^{(\alpha, \emptyset)}\}, \\ Y_4 &= Y_{(\emptyset, \emptyset)}^{(\alpha, \beta)}. \end{aligned}$$

Then $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$ is a tight relative 3-design on the shell X_{2u} in $J(8u, 4u)$.

5.4 Further problem

According to the computer search, if $r \leq \frac{k}{2}$, then we get the following table for the possible parameters of tight relative 2-designs on one shell X_r in $J(v, k)$, (i.e., tight $(2, 2)$ -($k, r, v - k, r, \lambda_{(2, 2)}$) designs).

v	14	22	26	29	30	32	32	37	37	38	...
k	7	11	13	13	15	11	16	15	16	19	...
r	3	5	4	6	7	5	6	7	6	9	...
$\lambda_{(2, 2)}$	1	4	1	5	9	2	4	6	3	16	...

There are two interesting two families in the table above, namely, $(k, r, \lambda_{(2,2)}) = (u^2 + u + 1, u + 1, 1)$ and $(k, r, \lambda_{(2,2)}) = (4u - 1, 2u - 1, (u - 1)^2)$ for integer $u \geq 2$. We obtained some non-trivial examples for the first family when $u = 2, 3, 4, 5$. **Problem one** is the general construction for tight relative 2-designs with the possible parameters from the above two families.

According to the computer search for $v \leq 3000$, there is only one family of parameters $(v, k, r) = (8u, 4u, 2u)$ for tight relative 3-designs and we can construct such designs if the tight relative 2-designs involved exist. **Problem two** is whether all the tight relative 3-designs on one shell of $J(v, k)$ have the parameter $(v, k, r) = (8u, 4u, 2u)$.

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