

Expansions of relative operator entropies and operator valued α -divergence

遠山宏明 (Hiroaki TOHYAMA)⁽¹⁾, 伊佐浩史 (Hiroshi ISA)⁽²⁾, 伊藤公智 (Masatoshi ITO)⁽³⁾,
亀井栄三郎 (Eizaburo KAMEI)⁽⁴⁾, 渡邊雅之 (Masayuki WATANABE)⁽⁵⁾
(1), (2), (3), (5): 前橋工科大学 (Maebashi Institute of Technology)

1. Introduction

Throughout this paper, an operator means a bounded linear operator on a Hilbert space H . An operator T on H is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and an operator T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

For strictly positive operators A and B , and for $x \in \mathbf{R}$, a path passing through A and B is defined as follows ([4], [5], [11] etc.):

$$A \natural_x B \equiv A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^x A^{\frac{1}{2}}.$$

If $x \in [0, 1]$, then the path becomes weighted geometric operator mean denoted by $A \natural_x B$. Weighted arithmetic operator mean is defined as $A \nabla_x B \equiv (1-x)A + xB$ for $x \in [0, 1]$.

Fujii and Kamei [3] defined relative operator entropy as follows:

$$S(A|B) \equiv A^{\frac{1}{2}} \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}.$$

Furuta [8] gave a generalized form of relative operator entropy. Furuta's one is called generalized relative operator entropy and is defined as follows:

$$S_t(A|B) \equiv A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}, \quad t \in \mathbf{R}.$$

We remark that $S_0(A|B) = S(A|B)$ holds.

Yanagi, Kuriyama and Furuichi [13] defined Tsallis relative operator entropy as follows:

$$T_t(A|B) = \frac{A \natural_t B - A}{t}, \quad t \in (0, 1].$$

By replacing $A \natural_t B$ with $A \natural_x B$, Tsallis relative operator entropy can be extended as the notion for $t \in \mathbf{R}$. Since $\lim_{t \rightarrow 0} \frac{a^t - 1}{t} = \log a$ holds for $a > 0$, we have $T_0(A|B) \equiv \lim_{t \rightarrow 0} T_t(A|B) = S(A|B)$.

For these relative operator entropies, we can give geometrical interpretations. By the derivative of the path with respect to x at t , we can get generalized relative operator entropy, that is, the following holds:

$$\left. \frac{d}{dx} A \natural_x B \right|_{x=t} = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}} = S_t(A|B).$$

Therefore, we can regard $S_t(A|B)$ as the slope of the tangent line at $x = t$ of the path. From this interpretation, we can regard $S(A|B)$ as the slope of the tangent line at $x = 0$ of the path. Tsallis relative operator entropy can be regarded as the slope of the line passing through points A and $A \natural_t B$ on the path.

Amari [1] defined α -divergence as a notion to measure the difference between two probability distributions. Based on this notion, Fujii [2] defined operator valued α -divergence as follows: For strictly positive operators A and B , and for $\alpha \in (0, 1)$,

$$D_\alpha(A|B) \equiv \frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1 - \alpha)}.$$

Petz [12] introduced the operator divergence $D_{FK}(A|B) \equiv B - A - S(A|B)$. Fujii et al. [6, 7] showed the following relation between $D_{FK}(A|B)$ and operator valued α -divergences at end points for interval $(0, 1)$.

$$\begin{aligned} D_0(A|B) &\equiv \lim_{t \rightarrow 0} D_t(A|B) = B - A - S(A|B), \\ D_1(A|B) &\equiv \lim_{t \rightarrow 1} D_t(A|B) = A - B + S_1(A|B). \end{aligned}$$

For the quantity $D_0(A|B)$, we give a geometrical interpretation shown in Figure 1.

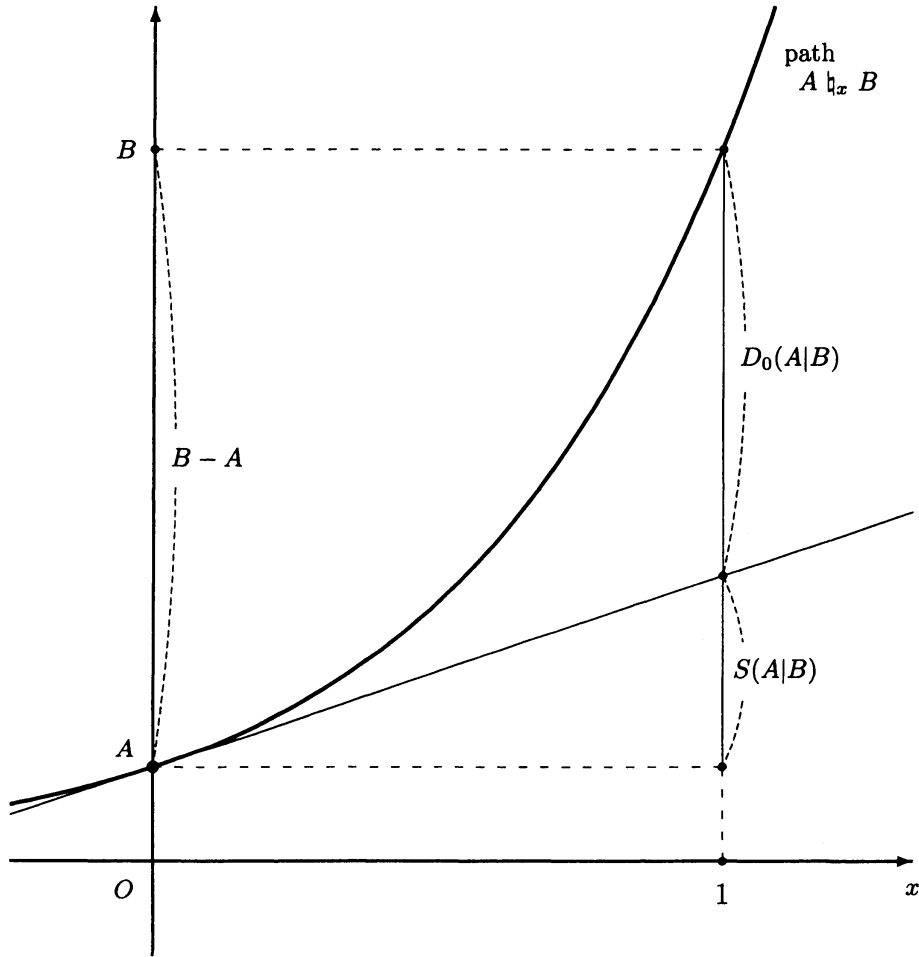


Figure 1: An interpretation for $D_0(A|B) = B - A - S(A|B)$.

In [11], Kamei showed that relative operator entropy has some kind of additivity as follows: For strictly positive operators A and B , and for $s \in \mathbf{R}$,

$$(\star) \quad S(A|A \sharp_s B) = sS(A|B).$$

In [9], we gave a viewpoint of operator valued distance for $S(A|B)$. Here, we give the following geometrical interpretation for this relation: The second component $A \natural_s B$ of the left hand side is an arbitrary point on the path. So, we can regard the relation $(\star)$ as relative operator entropy for a fixed point A and any point on the path.

For strictly positive operators A and B , and for $t \in [0, 1]$ and $r \in [-1, 1]$, operator power mean is defined as follows:

$$A \natural_{t,r} B \equiv A^{\frac{1}{2}} \left\{ (1-t)I + t \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^r \right\}^{\frac{1}{r}} A^{\frac{1}{2}} = A \natural_{\frac{1}{r}} \{ A \nabla_t (A \natural_r B) \}.$$

To preserve $(1-t)I + t \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^r \geq 0$, we have to impose t in $[0, 1]$. Operator power mean compounds the arithmetic, geometric and harmonic means, that is, the following holds.

arithmetic operator mean

$$A \nabla_t B = (1-t)A + tB$$

$\uparrow_{r=1}$

$$A \natural_{t,r} B \xrightarrow{r \rightarrow 0}$$

geometric operator mean

$$A \natural_t B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}}$$

$\downarrow_{r=-1}$

harmonic operator mean

$$A \Delta_t B = (A^{-1} \nabla_t B^{-1})^{-1}$$

We treat this operator power mean as an expanded path which links point A with point B . As the corresponding notions to relative operator entropies and operator valued α -divergence, we introduce expanded relative operator entropy, expanded Tsallis relative operator entropy and expanded operator valued α -divergence.

In this report, we aim at getting results on these expanded notions. In section 2, we show some results on relative operator entropies for two points on the path $A \natural_x B$. In section 3, we show results on expanded relative operator entropies, and in section 4, we show results on expanded operator valued α -divergence.

2. Relative operator entropies

Based on the relation $(\star)$, we show some basic results on relative operator entropies for two points on the path $A \natural_x B$.

To show the results in this section, we prepare the following properties of the path.

Lemma 2.1. *Let A and B be strictly positive operators. Then,*

- (1) $A \natural_t (A \natural_s B) = A \natural_{st} B,$
- (2) $(A \natural_t B) \natural_s A = A \natural_{(1-s)t} B$

hold for $s, t \in \mathbb{R}$.

Lemma 2.2. (Lemma 3.6 in [10]) *Let A and B be strictly positive operators. Then,*

$$(A \natural_u B) \natural_w (A \natural_{u+v} B) = A \natural_{u+vw} B$$

holds for $u, v, w \in \mathbb{R}$.

For generalized relative operator entropy and Tsallis relative operator entropy, we have the following result corresponding to the relation $(\star)$.

Theorem 2.3. *Let A and B be strictly positive operators. Then,*

$$\begin{aligned} (1) \quad & S_t(A|A \natural_s B) = {}_s S_{st}(A|B), \\ (2) \quad & T_t(A|A \natural_s B) = {}_s T_{st}(A|B) \end{aligned}$$

hold for $s, t \in \mathbf{R}$.

Proof. (1) If $s = 0$, then it is obvious that the both sides equal zero, and if $t = 0$, then this equality becomes the relation $(\star)$. Otherwise, we get

$$\begin{aligned} & S_t(A|A \natural_s B) \\ &= A^{\frac{1}{2}} \left\{ A^{-\frac{1}{2}} A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^s A^{\frac{1}{2}} A^{-\frac{1}{2}} \right\}^t \log \left\{ A^{-\frac{1}{2}} A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^s A^{\frac{1}{2}} A^{-\frac{1}{2}} \right\} A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{st} \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^s A^{\frac{1}{2}} = {}_s S_{st}(A|B). \end{aligned}$$

(2) By (1) in Lemma 2.1, we have

$$T_t(A|A \natural_s B) = \frac{A \natural_t (A \natural_s B) - A}{t} = {}_s \frac{A \natural_{st} B - A}{st} = {}_s T_{st}(A|B).$$

□

For the relation (1) in Theorem 2.3, we can give a geometrical interpretation shown in Figure 2. We remark that two tangent lines drawn in this figure intersect on the axis of the vertical direction.

In [10], we showed the following result on translation of generalized relative operator entropy.

Proposition 2.4. (Proposition 3.1 in [10]) *Let A and B be strictly positive operators. Then,*

$$S_{u+v}(A|B) = (A \natural_{u+v} B)(A \natural_u B)^{-1} S_u(A|B)$$

holds for $u, v \in \mathbf{R}$.

When we regard $S_u(A|B)$ and $S_{u+v}(A|B)$ as tangent vectors at u and $u + v$ on the path $A \natural_w B$, respectively, Proposition 2.4 means that $S_{u+v}(A|B)$ is parallelly transferring $S_u(A|B)$ by v along the path.

Here, we define the following noncommutative ratio on the path $A \natural_w B$.

Definition 2.5. *For strictly positive operators A and B , and for $u, v \in \mathbf{R}$, noncommutative ratio on the path $A \natural_w B$ is defined as follows:*

$$\mathcal{R}(u, v; A, B) \equiv (A \natural_{u+v} B)(A \natural_u B)^{-1}.$$

For the noncommutative ratio, the following property holds.

Proposition 2.6. (Proposition 3.3 in [10]) *Let A and B be strictly positive operators. Then,*

$$(A \natural_{u+v} B)(A \natural_u B)^{-1} = (A \natural_v B)A^{-1},$$

that is,

$$\mathcal{R}(u, v; A, B) = \mathcal{R}(0, v; A, B) = (A \natural_v B)A^{-1}$$

holds for $u, v \in \mathbf{R}$.

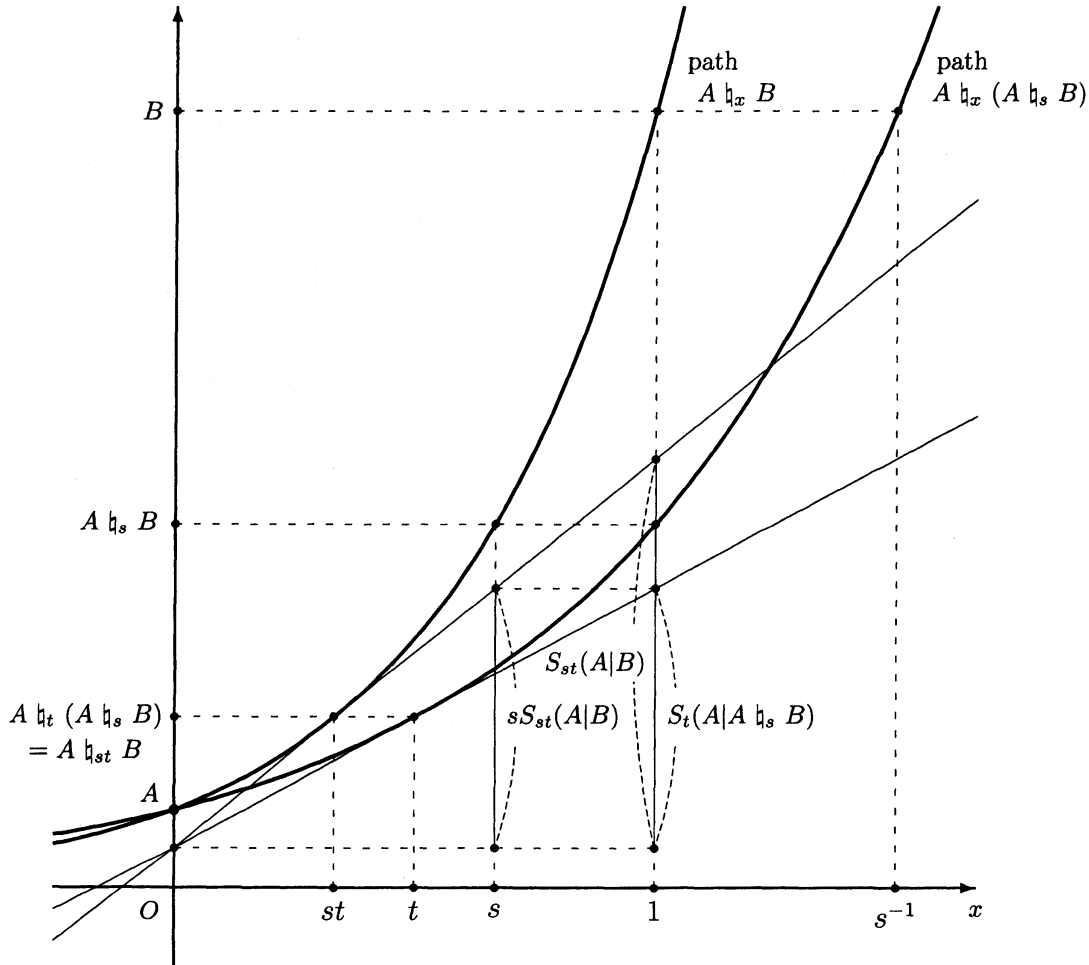


Figure 2: An interpretation for $S_t(A|A \Vdash_s B) = sS_{st}(A|B)$.

By Proposition 2.6, $\mathcal{R}(u, v; A, B)$ does not depend on u . So, we denote $\mathcal{R}(u, v; A, B)$ by $\mathcal{R}(v)$ in the rest of this report simply. We call multiplying by $\mathcal{R}(v)$ from the left side noncommutative ratio translation.

From Proposition 2.4 and Definition 2.5, we get the following immediately.

Corollary 2.7. (Corollary 3.4 in [10]) *Let A and B be strictly positive operators. Then,*

$$S_{u+v}(A|B) = \mathcal{R}(v)S_u(A|B)$$

holds for $u, v \in \mathbf{R}$.

Remark 1. By putting $u = 0$ in Corollary 2.7, we have $S_v(A|B) = \mathcal{R}(v)S(A|B)$.

The following is an extension of the relation $(\star)$. This is a result of generalized relative operator entropy for any two points on the path.

Proposition 2.8. (Proposition 3.7 in [10]) *Let A and B be strictly positive operators. Then,*

$$S_t(A \Vdash_v B | A \Vdash_{v+s} B) = {}_s S_{v+st}(A | B)$$

hold for $s, t, v \in \mathbf{R}$.

By using noncommutative ratio, we can represent the results of relative operator entropies for any two points on the path as follows:

Theorem 2.9. (Theorem 3.11 in [10]) *Let A and B be strictly positive operators. Then,*

$$\begin{aligned} (1) \quad & S_t(A \natural_v B | A \natural_{v+s} B) = s\mathcal{R}(v)S_{st}(A|B), \\ (2) \quad & T_t(A \natural_v B | A \natural_{v+s} B) = s\mathcal{R}(v)T_{st}(A|B), \\ (3) \quad & S(A \natural_v B | A \natural_{v+s} B) = s\mathcal{R}(v)S(A|B) \end{aligned}$$

hold for $s, t, v \in \mathbf{R}$.

Proof. (1) By Proposition 2.8 and Corollary 2.7, we have

$$S_t(A \natural_v B | A \natural_{v+s} B) = sS_{v+st}(A|B) = s\mathcal{R}(v)S_{st}(A|B).$$

(2) By Lemma 2.2 and Proposition 2.6, we get

$$\begin{aligned} T_t(A \natural_v B | A \natural_{v+s} B) &= \frac{(A \natural_v B) \natural_t (A \natural_{v+s} B) - A \natural_v B}{t} \\ &= \frac{A \natural_{v+st} B - A \natural_v B}{t} = s \frac{(A \natural_v B)A^{-1}(A \natural_{st} B) - (A \natural_v B)A^{-1}A}{st} \\ &= s(A \natural_v B)A^{-1}T_{st}(A|B) = s\mathcal{R}(v)T_{st}(A|B). \end{aligned}$$

(3) This equality can be obtained by putting $t = 0$ for (1). □

Remark 2. We can get Theorem 2.3 by putting $v = 0$ for the relations (1) and (2) in Theorem 2.9.

3. Expanded relative operator entropies

In this section, we show the results of expanded relative operator entropies for two points on the expanded path $A \natural_{t,r} B$. Similarly to $S_t(A|B)$, in [9], expanded relative operator entropy $S_{t,r}(A|B)$ is defined by the derivative of expanded path with respect to x at t as follows: For strictly positive operators A and B , and for $t \in [0, 1]$ and $r \in [-1, 1]$,

$$\begin{aligned} S_{t,r}(A|B) &\equiv \left. \frac{d}{dx} A \natural_{x,r} B \right|_{x=t} \\ &= A^{\frac{1}{2}} \left[\left\{ (1-t)I + t \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^r \right\}^{\frac{1}{r}-1} \frac{\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^r - I}{r} \right] A^{\frac{1}{2}} \\ &= \left[A \natural_{\frac{1}{r}-1} \{ A \nabla_t (A \natural_r B) \} \right] A^{-1} T_r(A|B). \end{aligned}$$

We remark that expanded relative operator entropy has the following relations [9]:

$$\begin{array}{c}
B - A \\
\uparrow_{r=1} \\
T_r(A|B) \xleftarrow[t=0]{} S_{t,r}(A|B) \xrightarrow[t=1]{} -T_r(B|A) \\
\downarrow_{r \rightarrow 0} \\
S_t(A|B)
\end{array}$$

By replacing weighted geometric operator mean with operator power mean, we obtain the definition of expanded Tsallis relative operator entropy [9]: For strictly positive operators A and B , and for $t \in (0, 1]$ and $r \in [-1, 1]$,

$$T_{t,r}(A|B) \equiv \frac{A \sharp_{t,r} B - A}{t}.$$

We remark that expanded Tsallis relative operator entropy also has the following relations:

$$\begin{array}{c}
B - A \\
\uparrow_{r=1} \\
T_r(A|B) \xleftarrow[t \rightarrow 0]{} T_{t,r}(A|B) \xrightarrow[t=1]{} B - A \\
\downarrow_{r \rightarrow 0} \\
T_t(A|B)
\end{array}$$

For expanded relative operator entropy, we can show the following result corresponding to (1) in Theorem 2.3.

Theorem 3.1. *Let A and B be strictly positive operators. Then,*

$$S_{t,r}(A|A \sharp_{s,r} B) = s(A \sharp_{st,r} B) \{A \nabla_{st} (A \natural_r B)\}^{-1} T_r(A|B) = sS_{st,r}(A|B)$$

holds for $t, s \in [0, 1]$ and $r \in [-1, 1]$.

In cases of $t \in \{0, 1\}$, we get the following relations.

Corollary 3.2. *Let A and B be strictly positive operators. Then,*

$$\begin{aligned}
(1) \quad S_{0,r}(A|A \sharp_{s,r} B) &= T_r(A|A \sharp_{s,r} B) = sT_r(A|B), \\
(2) \quad S_{1,r}(A|A \sharp_{s,r} B) &= -T_r(A \sharp_{s,r} B|A) \\
&= s(A \sharp_{s,r} B) \{A \nabla_s (A \natural_r B)\}^{-1} T_r(A|B)
\end{aligned}$$

hold for $s \in [0, 1]$ and $r \in [-1, 1]$.

To prove Theorem 3.1, we prepare the following lemmas. We omit their proofs.

Lemma 3.3. *Let A and B be strictly positive operators. Then,*

$$\begin{aligned} (1) \quad & A \sharp_r (A \sharp_{t,r} B) = A \nabla_t (A \sharp_r B), \\ (2) \quad & A \sharp_{t,r} (A \sharp_{s,r} B) = A \sharp_{st,r} B, \\ (3) \quad & A \sharp_{t,r} B = B \sharp_{1-t,r} A \end{aligned}$$

hold for $t, s \in [0, 1]$ and $r \in [-1, 1]$.

Lemma 3.4. *Let A and B be strictly positive operators. Then,*

$$\begin{aligned} S_{t,r}(A|B) &= (A \sharp_{t,r} B) \{A \nabla_t (A \sharp_r B)\}^{-1} T_r(A|B) \\ &= \left[A \sharp_{\frac{1}{r}} \{A \nabla_t (A \sharp_r B)\} \right] \{A \nabla_t (A \sharp_r B)\}^{-1} T_r(A|B) \end{aligned}$$

holds for $t \in [0, 1]$ and $r \in [-1, 1]$.

Proof of Theorem 3.1. By Lemma 3.3 and Lemma 3.4, the following holds:

$$\begin{aligned} S_{t,r}(A|A \sharp_{s,r} B) &= \{A \sharp_{t,r} (A \sharp_{s,r} B)\} [A \nabla_t \{A \sharp_r (A \sharp_{s,r} B)\}]^{-1} T_r(A|A \sharp_{s,r} B) \\ &= (A \sharp_{st,r} B) [A \nabla_t \{A \nabla_s (A \sharp_r B)\}]^{-1} T_r(A|A \sharp_{s,r} B) \\ &= (A \sharp_{st,r} B) \{A \nabla_{st} (A \sharp_r B)\}^{-1} \frac{A \sharp_r (A \sharp_{s,r} B) - A}{r} \\ &= (A \sharp_{st,r} B) \{A \nabla_{st} (A \sharp_r B)\}^{-1} \frac{A \nabla_s (A \sharp_r B) - A}{r} \\ &= (A \sharp_{st,r} B) \{A \nabla_{st} (A \sharp_r B)\}^{-1} \frac{(1-s)A + s(A \sharp_r B) - A}{r} \\ &= s(A \sharp_{st,r} B) \{A \nabla_{st} (A \sharp_r B)\}^{-1} T_r(A|B) = sS_{st,r}(A|B). \end{aligned}$$

□

The following (1) in Theorem 3.5 is a corresponding result to (2) in Theorem 2.3.

Theorem 3.5. *Let A and B be strictly positive operators. Then,*

$$\begin{aligned} (1) \quad & T_{t,r}(A|A \sharp_{s,r} B) = sT_{st,r}(A|B) \quad (t \in (0, 1]), \\ (2) \quad & T_{0,r}(A|A \sharp_{s,r} B) = sT_r(A|B), \\ (3) \quad & T_{1,r}(A|A \sharp_{s,r} B) = sT_{s,r}(A|B), \\ (4) \quad & T_{1-t,r}(A \sharp_{s,r} B|A) = s \frac{tT_{st,r}(A|B) - T_{s,r}(A|B)}{1-t} \quad (t \in [0, 1)) \end{aligned}$$

hold for $s \in [0, 1]$ and $r \in [-1, 1]$.

Proof. (1) By (2) in Lemma 3.3, these can be shown as follows:

$$T_{t,r}(A|A \sharp_{s,r} B) = \frac{A \sharp_{t,r} (A \sharp_{s,r} B) - A}{t} = s \frac{A \sharp_{st,r} B - A}{st} = sT_{st,r}(A|B).$$

(2), (3) For (1), by putting $t = 0$ and $t = 1$, we can get these relations, respectively.

(4) Since $A \sharp_{t,r} B = B \sharp_{1-t,r} A$ holds for $t \in [0, 1]$ and $r \in [-1, 1]$, we have

$$\begin{aligned} T_{1-t,r}(A \sharp_{s,r} B|A) &= \frac{(A \sharp_{s,r} B) \sharp_{1-t,r} A - A \sharp_{s,r} B}{1-t} \\ &= \frac{A \sharp_{t,r} (A \sharp_{s,r} B) - A \sharp_{s,r} B}{1-t} = \frac{A \sharp_{st,r} B - A \sharp_{s,r} B}{1-t} \\ &= \frac{(A \sharp_{st,r} B - A) - (A \sharp_{s,r} B - A)}{1-t} = s \frac{tT_{st,r}(A|B) - T_{s,r}(A|B)}{1-t}. \end{aligned}$$

□

4. Expanded operator valued α -divergence

By Theorem 2.5 in [10] and the results in [6, 7], the following relation between operator valued α -divergence and Tsallis relative operator entropy was shown.

Theorem 4.1. ([6, 7, 10]) *Let A and B be strictly positive operators. Then,*

$$D_t(A|B) = -T_{1-t}(B|A) - T_t(A|B)$$

holds for $t \in [0, 1]$.

Theorem 4.1 gives a geometrical interpretation for operator valued α -divergence. Tsallis relative operator entropy $T_t(A|B)$ can be regarded as the slope of the line passing through points A and $A \sharp_t B$. Since $-T_{1-t}(B|A) = -\frac{B \sharp_{1-t} A - B}{1-t} = \frac{B - A \sharp_t B}{1-t}$, we can regard this operator value as the slope of the line passing through points $A \sharp_t B$ and B . Therefore, $D_t(A|B)$ gives the difference between the slopes of these two lines. We can illustrate the quantity corresponding to $D_t(A|B)$ by bold straight line in Figure 3.

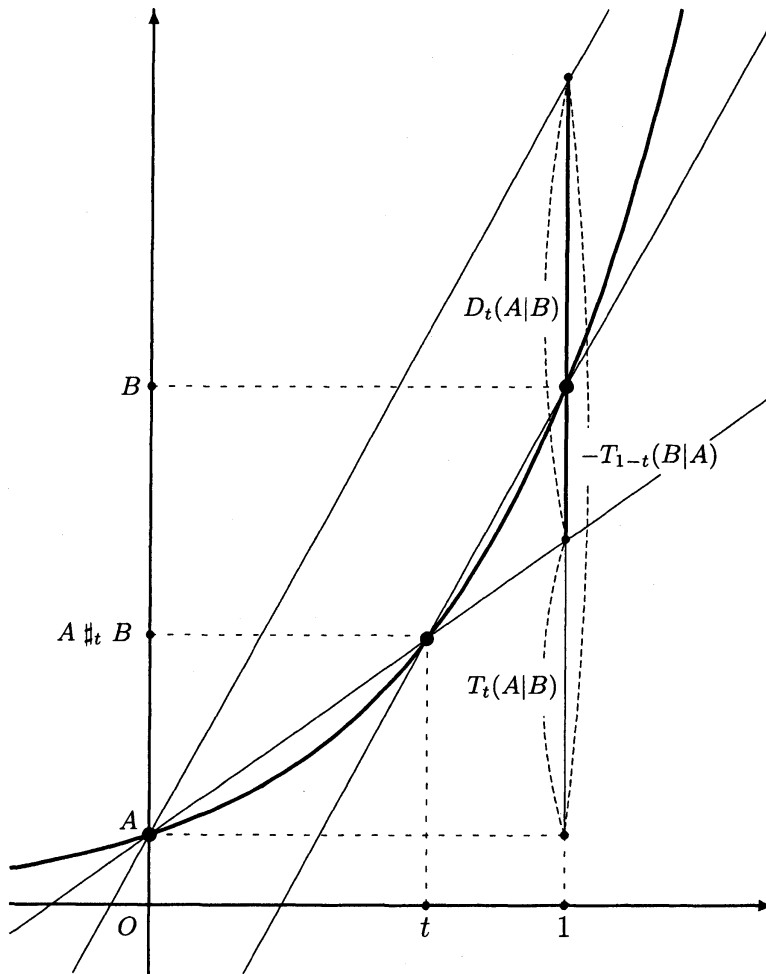


Figure 3: An interpretation for $D_t(A|B) = -T_{1-t}(B|A) - T_t(A|B)$.

Based on Theorem 4.1, we define expanded operator valued α -divergence.

Definition 4.2. *For strictly positive operators A and B , and for $t \in [0, 1]$, $r \in [-1, 1]$, expanded operator valued α -divergence is defined as follows:*

$$D_{t,r}(A|B) \equiv -T_{1-t,r}(B|A) - T_{t,r}(A|B).$$

Remark 3. It is obvious that $D_{1-t,r}(B|A) = D_{t,r}(A|B)$ holds for $t \in (0, 1)$ and $r \in [-1, 1]$.

We get the following relations for expanded operator valued α -divergence immediately.

Proposition 4.3. Let A and B be strictly positive operators. Then,

$$\begin{aligned} (1) \quad & D_{t,0}(A|B) = D_t(A|B), \\ (2) \quad & D_{t,1}(A|B) = 0, \\ (3) \quad & D_{0,r}(A|B) = B - A - T_r(A|B), \\ (4) \quad & D_{1,r}(A|B) = A - B - T_r(B|A) \end{aligned}$$

hold for $t \in [0, 1]$ and $r \in [-1, 1]$.

We can illustrate the relations in Proposition 4.3 as follows:

$$\begin{array}{c} 0 \\ \uparrow_{r=1} \\ B - A - T_r(A|B) \quad \xleftarrow[t=0]{} D_{t,r}(A|B) \quad \xrightarrow[t=1]{} A - B - T_r(B|A) \\ \downarrow_{r \rightarrow 0} \\ D_t(A|B) \end{array}$$

We can rewrite $D_{t,r}(A|B)$ as the difference between weighted arithmetic mean and operator power mean as follows.

Theorem 4.4. Let A and B be strictly positive operators. Then,

$$D_{t,r}(A|B) = \frac{A \nabla_t B - A \sharp_{t,r} B}{t(1-t)}$$

holds for $t \in (0, 1)$ and $r \in [-1, 1]$.

Proof. We can get this relation as follows:

$$\begin{aligned} D_{t,r}(A|B) &= -T_{1-t,r}(B|A) - T_{t,r}(A|B) = -\frac{B \sharp_{1-t,r} A - B}{1-t} - \frac{A \sharp_{t,r} B - A}{t} \\ &= \frac{-tA \sharp_{t,r} B + tB - (1-t)A \sharp_{t,r} B + (1-t)A}{t(1-t)} = \frac{A \nabla_t B - A \sharp_{t,r} B}{t(1-t)}. \end{aligned}$$

□

For expanded operator valued α -divergence, we are trying to obtain similar relations to Theorem 3.1 and Theorem 3.5. The followings are relations we have obtained until now.

Theorem 4.5. *Let A and B be strictly positive operators. Then,*

$$\begin{aligned} (1) \quad D_{t,r}(A|A \sharp_{s,r} B) &= s \frac{T_{s,r}(A|B) - T_{st,r}(A|B)}{1-t} \quad (t \in [0, 1)), \\ (2) \quad D_{0,r}(A|A \sharp_{s,r} B) &= s \{T_{s,r}(A|B) - T_r(A|B)\}, \\ (3) \quad D_{1,r}(A|A \sharp_{s,r} B) &= s \{S_{s,r}(A|B) - T_{s,r}(A|B)\}. \end{aligned}$$

hold for $s \in (0, 1)$ and $r \in [-1, 1]$.

Proof. (1) By Theorem 3.5, we have

$$\begin{aligned} D_{t,r}(A|A \sharp_{s,r} B) &= -T_{1-t,r}(A \sharp_{s,r} B|A) - T_{t,r}(A|A \sharp_{s,r} B) \\ &= s \left\{ \frac{1}{1-t} T_{s,r}(A|B) - \frac{t}{1-t} T_{st,r}(A|B) \right\} - s T_{st,r}(A|B) = s \frac{T_{s,r}(A|B) - T_{st,r}(A|B)}{1-t}. \end{aligned}$$

(2) We can get this result by putting $t = 0$ in (1).

(3) By Theorem 3.1 and Theorem 3.5, we have

$$\begin{aligned} D_{1,r}(A|A \sharp_{s,r} B) &= -T_r(A \sharp_{s,r} B|A) - T_{1,r}(A|A \sharp_{s,r} B) \\ &= S_{1,r}(A|A \sharp_{s,r} B) - T_{1,r}(A|A \sharp_{s,r} B) = s S_{s,r}(A|B) - s T_{s,r}(A|B) \\ &= s \{S_{st,r}(A|B) - T_{s,r}(A|B)\}. \end{aligned}$$

□

By the similar way to Theorem 4.5, we can obtain the results of operator valued α -divergence for fixed point A and any point on the path as follows:

Proposition 4.6. *Let A and B be strictly positive operators. Then,*

$$\begin{aligned} (1) \quad D_t(A|A \natural_s B) &= s \frac{T_s(A|B) - T_{st}(A|B)}{1-t} \quad (t \in [0, 1)), \\ (2) \quad D_0(A|A \natural_s B) &= s \{T_s(A|B) - S(A|B)\}, \\ (3) \quad D_1(A|A \natural_s B) &= s \{S_s(A|B) - T_s(A|B)\} \end{aligned}$$

hold for $s \in \mathbf{R}$.

By applying noncommutative ratio translation to the relations in Proposition 4.6, we can get the results of operator valued α -divergence for any two points on the path as follows:

Theorem 4.7. *Let A and B be strictly positive operators. Then,*

$$\begin{aligned} (1) \quad D_t(A \natural_v B|A \natural_{v+s} B) &= s\mathcal{R}(v) \frac{T_s(A|B) - T_{st}(A|B)}{1-t} \quad (t \in [0, 1)), \\ (2) \quad D_0(A \natural_v B|A \natural_{v+s} B) &= s\mathcal{R}(v) \{T_s(A|B) - S(A|B)\}, \\ (3) \quad D_1(A \natural_v B|A \natural_{v+s} B) &= s\mathcal{R}(v) \{S_s(A|B) - T_s(A|B)\} \end{aligned}$$

hold for $s, v \in \mathbf{R}$.

Proof. (1) Lemma 2.2, Proposition 4.6, and Theorem 2.9, we have

$$\begin{aligned}
 D_t(A \natural_v B | A \natural_{v+s} B) &= D_t(A \natural_v B | (A \natural_v B) \natural_s (A \natural_{v+1} B)) \\
 &= s \frac{T_s(A \natural_v B | A \natural_{v+1} B) - T_{st}(A \natural_v B | A \natural_{v+1} B)}{1-t} \\
 &= s\mathcal{R}(v) \frac{T_s(A|B) - T_{st}(A|B)}{1-t}.
 \end{aligned}$$

(2) We can get this result by putting $t = 0$ in (1).

(3) By (1) and (2) in Theorem 2.9,

$$\begin{aligned}
 D_1(A \natural_v B | A \natural_{v+s} B) &= A \natural_v B - A \natural_{v+s} B + S_1(A \natural_v B | A \natural_{v+s} B) \\
 &= S_1(A \natural_v B | A \natural_{v+s} B) - T_1(A \natural_v B | A \natural_{v+s} B) = s\mathcal{R}(v)S_s(A|B) - s\mathcal{R}(v)T_s(A|B) \\
 &= s\mathcal{R}(v) \{S_s(A|B) - T_s(A|B)\}.
 \end{aligned}$$

□

References

- [1] S. Amari, Differential Geometrical Methods in Statistics, Springer Lecture Notes in Statistics, **28**(1985).
- [2] J. I. Fujii, On the relative operator entropy (in Japanese), RIMS Kokyuroku, **903**(1995), 49–56.
- [3] J. I. Fujii and E. Kamei, Relative operator entropy in noncommutative information theory, Math. Japon., **34**(1989), 341–348.
- [4] J. I. Fujii and E. Kamei, Interpolational paths and their derivatives, Math. Japon., **39**(1994), 557–560.
- [5] J. I. Fujii and E. Kamei, Path of Bregman-Petz operator divergence, Sci. Math. Jpn., **70**(2009), 329–333.
- [6] J. I. Fujii, J. Mičić, J. Pečarić and Y. Seo, Comparison of operator mean geodesics, J. Math. Inequal., **2**(2008), 287–298.
- [7] M. Fujii, J. Mičić, J. Pečarić and Y. Seo, Recent Development of Mond-Pečarić Method in Operator Inequalities, Monographs in Inequalities 4, Element, Zagreb, (2012).
- [8] T. Furuta, Parametric extensions of Shannon inequality and its reverse one in Hilbert space operators, Linear Algebra Appl., **381**(2004), 219–235.
- [9] H. Isa, M. Ito, E. Kamei, H. Tohyama and M. Watanabe, Extensions of Tsallis relative operator entropy and operator valued distance, Sci. Math. Jpn., **76**(2013), 427–435. (online: e-2013 (2013), 427–435.)
- [10] H. Isa, M. Ito, E. Kamei, H. Tohyama and M. Watanabe, On relations between operator valued α -divergence and relative operator entropies, to appear in Sci. Math..
- [11] E. Kamei, Paths of operators parametrized by operator means, Math. Japon., **39**(1994), 395–400.
- [12] D. Petz, Bregman divergence as relative operator entropy, Acta Math. Hungar., **116**(2007), 127–131.
- [13] K. Yanagi, K. Kuriyama, S. Furuichi, Generalized Shannon inequalities based on Tsallis relative operator entropy, Linear Algebra Appl., **394**(2005), 109–118.

- (1) MAEBASHI INSTITUTE OF TECHNOLOGY, 460-1, KAMISADORI-MACHI, MAEBASHI, GUNMA, JAPAN, 371-0816, tohyama@maebashi-it.ac.jp
- (2) MAEBASHI INSTITUTE OF TECHNOLOGY, 460-1, KAMISADORI-MACHI, MAEBASHI, GUNMA, JAPAN, 371-0816, isa@maebashi-it.ac.jp
- (3) MAEBASHI INSTITUTE OF TECHNOLOGY, 460-1, KAMISADORI-MACHI, MAEBASHI, GUNMA, JAPAN, 371-0816, m-ito@maebashi-it.ac.jp
- (4) 1-1-3, SAKURAGAOKA, KANMAKICHO, KITAKATURAGI-GUN, NARA, JAPAN, 639-0202, ekamei1947@yahoo.co.jp
- (5) MAEBASHI INSTITUTE OF TECHNOLOGY, 460-1, KAMISADORI-MACHI, MAEBASHI, GUNMA, JAPAN, 371-0816, masayukiwatanabe@maebashi-it.ac.jp