

Classes of non-normal operators defined by inequalities for operator means

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1 Class A - f and A - f -paranormality

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H . An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. Following [12], class A is a class of non-normal operators T such that

$$|T^2| \geq |T|^2.$$

It is also shown in [12] that class A includes p -hyponormal ($(T^*T)^p \geq (TT^*)^p$ for $p > 0$) and log-hyponormal (T is invertible and $\log T^*T \geq \log TT^*$) operators, and is included in the classes of *paranormal* ($\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$) and *normaloid* ($\|T\| = r(T)$ (the spectral radius)) operators. It is shown in [24] that T belongs to class A if and only if

$$(|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2,$$

and in [2] that T is paranormal if and only if $T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$ for all $\lambda > 0$, or equivalently,

$$\frac{1}{2} (I + \lambda^2 |T^*||T|^2|T^*|) \geq \lambda |T^*|^2 \text{ for all } \lambda > 0.$$

From these points of view, we introduced generalizations of class A and paranormality in [29].

Definition 1.A ([29]). Let f be a non-negative continuous function on $[0, \infty)$.

$$(i) \quad T \in \text{class } A\text{-}f \iff f(|T^*||T|^2|T^*|) \geq |T^*|^2.$$

$$(ii) \quad T \text{ is } A\text{-}f\text{-paranormal} \iff \lambda T \in \text{class } A\text{-}f \text{ for all } \lambda > 0.$$

When f is a representing function of an operator connection σ (see [19]), we also call class A - f and A - f -paranormal class A - σ and A - σ -paranormal, respectively.

In fact, class A and paranormality coincide with class $A\sharp$ and $A\nabla$ -paranormality, respectively, where ∇ and $\sharp$ are the arithmetic and geometric means, that is,

$$A \nabla B = \frac{1}{2}(A + B) \quad \text{and} \quad A \sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Hence we can explain the inclusion relation between class A and the class of paranormal operators shown in [12] in terms of class A- f and A- f -paranormality as follows:

$$\begin{aligned} T \in \text{class A} &\iff T \in \text{class A-}\sharp && \text{by Definition 1.A} \\ &\iff T \text{ is A-}\sharp\text{-paranormal} && \text{since } f_{\sharp}(\lambda^2 t) = (\lambda^2 t)^{\frac{1}{2}} = \lambda t^{\frac{1}{2}} = \lambda f_{\sharp}(t) \\ &\implies T \text{ is A-}\nabla\text{-paranormal} && \text{since } f_{\sharp}(t) = t^{\frac{1}{2}} \leq \frac{1}{2}(1+t) = f_{\nabla}(t) \\ &\iff T \text{ is paranormal} && \text{by Definition 1.A.} \end{aligned}$$

Furthermore, in [29], we introduced parametrized generalizations of class A- f and A- f -paranormality.

Definition 1.B ([29]). Let f be a non-negative continuous function on $[0, \infty)$, and $s, t > 0$.

- (i) $T \in \text{class } A(s, t)\text{-}f \iff f(|T^*|^t |T|^{2s} |T^*|^t) \geq |T^*|^{2t}.$
- (ii) $T \text{ is } A(s, t)\text{-}f\text{-paranormal} \iff \lambda T \in \text{class } A(s, t)\text{-}f \text{ for all } \lambda > 0.$

When f is a representing function of an operator connection σ (see [19]), we also call class $A(s, t)\text{-}f$ and $A(s, t)\text{-}f$ -paranormal class $A(s, t)\text{-}\sigma$ and $A(s, t)\text{-}\sigma$ -paranormal, respectively.

We remark that class $A(s, t)\text{-}\sharp_{\frac{t}{s+t}}$ and $A(s, t)\text{-}\nabla_{\frac{t}{s+t}}$ -paranormality, introduced in [8] and [26], coincide with class $A(s, t)$ ($(|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$) and absolute- (s, t) -paranormality ($\frac{s}{s+t}I + \frac{t}{s+t}\lambda^{s+t}|T^*|^t |T|^{2s} |T^*|^t \geq \lambda^t |T^*|^t$ for all $\lambda > 0$), respectively, where

$$A \nabla_{\alpha} B = (1 - \alpha)A + \alpha B \quad \text{and} \quad A \sharp_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} \quad \text{for } \alpha \in [0, 1].$$

Particularly, it is pointed out in [17] that class $A(\frac{1}{2}, \frac{1}{2})$ coincides with the class of *w-hyponormal* ($|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$, where $\tilde{T}$ is the Aluthge transformation of T) operators introduced in [1].

In [29], we showed several properties of these classes introduced above, which are generalizations of the results on class $A(s, t)$ and absolute- (s, t) -paranormal operators shown in [8][15][17][20][24][25][26][28].

Theorem 1.A ([29]). Let $s_0, t_0 > 0$ and $\{f_{s,t} \mid s \geq s_0, t \geq t_0\}$ be a family of non-negative operator monotone functions on $[0, \infty)$ satisfying $f_{s,t}(x^t g(x)^s) = x^t$, where g is a continuous function. If T is invertible and $T \in \text{class } A(s_0, t_0)\text{-}f_{s_0, t_0}$, then $T \in \text{class } A(s, t)\text{-}f_{s,t}$ for all $s > s_0$ and $t > t_0$.

Theorem 1.B ([29]). Let f be a non-negative, continuously differentiable and convex (or concave) function on $[0, \infty)$ satisfying $f(1) \leq 1$ and $0 < f'(1) < 1$, and $p_0 > 0$. If T is invertible and $T \in \text{class } A(\theta'p, \theta p)$ - f for all $p \in (0, p_0)$, then T is log-hyponormal, where $\theta = f'(1)$ and $\theta + \theta' = 1$.

Theorem 1.C ([29]). Let f be a non-negative operator monotone function on $[0, \infty)$, and $s, t \in (0, 1]$. If $T \in \text{class } A(s, t)$ - f and $T \in \text{class } A$, then $T^n \in \text{class } A(\frac{s}{n}, \frac{t}{n})$ - f for every positive integer n .

Proposition 1.D ([29]). Let f be a non-negative operator monotone function on $[0, \infty)$, and $s, t \in (0, 1]$. If $T \in \text{class } A(s, t)$ - f , then $T|_{\mathcal{M}} \in \text{class } A(s, t)$ - f , where $T|_{\mathcal{M}}$ is the restriction of T onto an invariant subspace $\mathcal{M}$.

Theorem 1.E ([29]). Let f and g be non-negative continuous increasing functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ and $g(0) = 0$, and $s, t > 0$. If $T \in \text{class } A(s, t)$ - f , then the following hold, where $T = U|T|$ is the polar decomposition and $\tilde{T}_{s,t} = |T|^s U |T|^t$:

- (i) $\tilde{T}_{s,t}$ is f -hyponormal if $f \circ g^{-1}$ is operator monotone and $x^t \geq (f \circ g^{-1})(x^s)$.
- (ii) $\tilde{T}_{s,t}$ is g -hyponormal if $g \circ f^{-1}$ is operator monotone and $(g \circ f^{-1})(x^t) \geq x^s$.

2 Furuta inequality and its generalizations

The following result is essential for the study of class $A(s, t)$ operators.

Theorem F (Furuta inequality [9]).

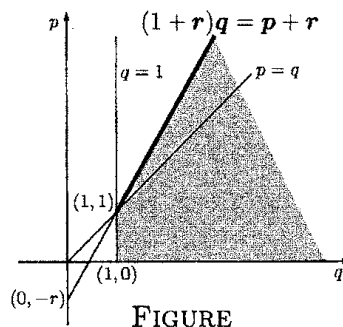
If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(i) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(ii) \quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



FIGURE

We remark that Theorem F yields Löwner-Heinz theorem " $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$ " when we put $r = 0$ in (i) or (ii) stated above. Other proofs are given in [5][18] and also an elementary one-page proof in [10]. It is shown in [21] that the domain of p, q and r is the best possible in Theorem F.

The chaotic order defined by $\log A \geq \log B$ for $A, B > 0$ is weaker than the usual order since $\log t$ is operator monotone. The following extension of a result in [3] can be obtained as an application of Theorem F. Other proofs are given in [7][22], and the best possibility is shown in [27].

Theorem C ([6][11]). *Let $A, B > 0$. The following are mutually equivalent:*

- (i) $\log A \geq \log B$.
- (ii) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ for all $p \geq 0$ and $r \geq 0$.
- (iii) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ for all $p \geq 0$ and $r \geq 0$.

A lot of related studies to Theorem F and Theorem C have been done. Among others, we here introduce the following result.

Theorem 2.A ([13] et al.). *Let $A, B > 0$ and $\alpha_0, \beta_0 > 0$. If*

$$(B^{\frac{\beta_0}{2}} A^{\alpha_0} B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0 + \beta_0}} \geq B^{\beta_0} \quad \text{or} \quad A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0 + \beta_0}}, \quad (2.1)$$

then for each real number δ ,

$$B^{-\frac{\beta}{2}} (B^{\frac{\beta}{2}} A^{\alpha} B^{\frac{\beta}{2}})^{\frac{\delta + \beta}{\alpha + \beta}} B^{-\frac{\beta}{2}} \quad \text{and} \quad A^{-\frac{\alpha}{2}} (A^{\frac{\alpha}{2}} B^{\beta} A^{\frac{\alpha}{2}})^{\frac{-\delta + \alpha}{\alpha + \beta}} A^{-\frac{\alpha}{2}} \quad (2.2)$$

is increasing and decreasing, respectively, for $\alpha \geq \max\{\alpha_0, \delta\}$ and $\beta \geq \max\{\beta_0, -\delta\}$.

The “order-like” relations between $A, B \geq 0$ defined by the inequalities in (2.1) for some fixed $\alpha_0, \beta_0 > 0$ are weaker than the usual and chaotic orders by Theorem F and Theorem C. For $A, B > 0$, the inequalities in (2.1) are mutually equivalent and each function in (2.2) is the inverse of the other since

$$S^{\frac{1}{2}} (S^{-\frac{1}{2}} T S^{-\frac{1}{2}})^{\alpha} S^{\frac{1}{2}} = S \sharp_{\alpha} T = T \sharp_{1-\alpha} S = T^{\frac{1}{2}} (T^{-\frac{1}{2}} S T^{-\frac{1}{2}})^{1-\alpha} T^{\frac{1}{2}}$$

for $S, T > 0$ and $\alpha \in [0, 1]$. Hence Theorem 2.A can be summarized as follows: for each $p, a > 0$ and $\delta \in [-a, p]$,

$$\begin{aligned} (B^{\frac{a}{2}} A B^{\frac{a}{2}})^{\frac{a}{p+a}} \geq B^a &\implies B^{-\frac{r}{2}} (B^{\frac{r}{2}} A B^{\frac{r}{2}})^{\frac{\delta + r}{p+r}} B^{-\frac{r}{2}} \text{ is increasing for } r \geq a, \\ A^a \geq (A^{\frac{a}{2}} B A^{\frac{a}{2}})^{\frac{a}{p+a}} &\implies A^{-\frac{r}{2}} (A^{\frac{r}{2}} B A^{\frac{r}{2}})^{\frac{\delta + r}{p+r}} A^{-\frac{r}{2}} \text{ is decreasing for } r \geq a, \end{aligned} \quad (2.3)$$

and it turns out by scrutinizing the proof of Theorem 2.A that (2.3) is still valid even if the hypotheses are weakened to

$$\log(B^{\frac{a}{2}} A B^{\frac{a}{2}})^{\frac{a}{p+a}} \geq \log B^a \quad \text{and} \quad \log A^a \geq \log(A^{\frac{a}{2}} B A^{\frac{a}{2}})^{\frac{a}{p+a}}.$$

The following generalizations of Theorem F, Theorem C and Theorem 2.A are shown in the recent paper [23] by M. Uchiyama. In fact, Theorem 2.B yields Theorem F and Theorem C by putting $\psi_r(x) = x^{\frac{r}{p+r}}$, $\phi_r(x) = x^{\frac{1+r}{p+r}}$, $g(x) = x^p$ and $h(x) = x$. Theorem 2.B also yields (2.3) by putting $\psi_r(x) = x^{\frac{r}{p+r}}$, $\phi_r(x) = x^{\frac{\delta+r}{p+r}}$, $g(x) = x^p$ and $h(x) = x^{\delta}$.

Theorem 2.B ([23]). Let $\{\psi_r \mid r > 0\}$ and $\{\phi_r \mid r > 0\}$ be families of non-negative operator monotone functions satisfying

$$\psi_r(x^r g(x)) = x^r \quad \text{and} \quad \phi_r(x^r g(x)) = x^r h(x),$$

where g and h are non-negative continuous functions. If $A \geq B \geq 0$ or if $A, B > 0$ and $\log A \geq \log B$, then for $r > 0$,

$$\begin{aligned} \psi_r(B^{\frac{r}{2}} g(A) B^{\frac{r}{2}}) &\geq B^r, & A^r &\geq \psi_r(A^{\frac{r}{2}} g(B) A^{\frac{r}{2}}), \\ \phi_r(B^{\frac{r}{2}} g(A) B^{\frac{r}{2}}) &\geq B^{\frac{r}{2}} h(A) B^{\frac{r}{2}}, & A^{\frac{r}{2}} h(B) A^{\frac{r}{2}} &\geq \phi_r(A^{\frac{r}{2}} g(B) A^{\frac{r}{2}}). \end{aligned}$$

Theorem 2.C ([23]). Let $A, B \geq 0$ and $a > 0$, and let $\{\psi_r \mid r \geq a\}$ and $\{\phi_r \mid r \geq a\}$ be families of non-negative operator monotone functions satisfying

$$\psi_r(x^r g(x)) = x^r \quad \text{and} \quad \phi_r(x^r g(x)) = x^r h(x),$$

where g and h are non-negative continuous functions. Then the following hold:

- (i) If $A^a \sigma_{\psi_a} B \geq I$, then $A^r \sigma_{\phi_r} B$ is increasing for $r \geq a$.
- (ii) If $A, B > 0$ and $A^a \sigma_{\psi_a} B \leq I$, then $A^r \sigma_{\phi_r} B$ is decreasing for $r \geq a$.

Here σ_f denotes the operator mean whose representing function is f .

Theorem 2.B and Theorem 2.C play important roles for the study of class $A(s, t)$ - f and $A(s, t)$ - f -paranormal operators. Particularly, the proof of Theorem 1.A is based on Theorem 2.C. In this report, we shall give modifications of Theorem 2.C and Theorem 1.A.

3 Results

The following is a modification of Theorem 2.C.

Theorem 3.1. Let $A, B \geq 0$ and $a > 0$, and let $\{\psi_r \mid r \geq a\}$ and $\{\phi_r \mid r \geq a\}$ be families of non-negative operator monotone functions satisfying

$$\psi_r(x^r g(x)) = x^r \quad \text{and} \quad \phi_r(x^r g(x)) = x^r h(x), \quad (3.1)$$

where g and h are non-negative continuous functions. Then the following hold for $a \leq s \leq t$:

- (i) If $\psi_a(B^{\frac{a}{2}} AB^{\frac{a}{2}}) \geq B^a$, or if $A, B > 0$ and $\log \psi_a(B^{\frac{a}{2}} AB^{\frac{a}{2}}) \geq \log B^a$, then

$$B^{\frac{t-s}{2}} \phi_s(B^{\frac{s}{2}} AB^{\frac{s}{2}}) B^{\frac{t-s}{2}} \leq \phi_t(B^{\frac{t}{2}} AB^{\frac{t}{2}}).$$

(ii) If $A^a \geq \psi_a(A^{\frac{a}{2}}BA^{\frac{a}{2}})$ and $\overline{R(A)} \cap N(B) = \{0\}$, or if $A, B > 0$ and $\log A^a \geq \log \psi_a(A^{\frac{a}{2}}BA^{\frac{a}{2}})$, then

$$A^{\frac{t-s}{2}} \phi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})A^{\frac{t-s}{2}} \geq P \phi_t(A^{\frac{t}{2}}BA^{\frac{t}{2}})P,$$

where P is the projection onto $N(A)^\perp$.

The following is a modification of Theorem 1.A.

Theorem 3.2. Let $s_0, t_0 > 0$ and $\{f_{s,t} \mid s \geq s_0, t \geq t_0\}$ be a family of non-negative operator monotone functions on $[0, \infty)$ satisfying $f_{s,t}(x^t g(x)^s) = x^t$, where g is a continuous function. If $T \in \text{class } A(s_0, t_0)\text{-}f_{s_0, t_0}$, then $T \in \text{class } A(s, t)\text{-}f_{s, t}$ for all $s > s_0$ and $t > t_0$.

4 Proofs

We use the following well-known results in order to give a proof of Theorem 3.1.

Theorem 4.A ([14]). Let X and A be bounded linear operators on a Hilbert space H . We suppose that $X \geq 0$ and $\|A\| \leq 1$. If f is an operator convex function defined on $[0, \infty)$ such that $f(0) \leq 0$, then

$$A^* f(X) A \geq f(A^* X A).$$

Theorem 4.B ([4]). Let A and B be bounded linear operators on a Hilbert space H . The following statements are equivalent;

- (1) $R(A) \subseteq R(B)$;
- (2) $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$; and
- (3) there exists a bounded linear operator C on H so that $A = BC$.

Moreover, if (1), (2) and (3) are valid, then there exists a unique operator C so that

- (a) $\|C\|^2 = \inf\{\mu \mid AA^* \leq \mu BB^*\}$;
- (b) $N(A) = N(C)$; and
- (c) $R(C) \subseteq \overline{R(B^*)}$.

We consider when the operator C , determined uniquely in Theorem 4.B, satisfies the equality of (c).

Lemma 4.1. Let A and B be operators which satisfy (1), (2) and (3) of Theorem 4.B, and C be the operator which is given in (3) and determined uniquely by (a), (b) and (c) of Theorem 4.B. Then $\overline{R(C)} = \overline{R(B^*)}$ if and only if $N(A^*) = N(B^*)$.

Proof. $N(C^*) \supseteq N(B)$ by (c) of Theorem 4.B, so that $N(C^*) = N(B) \oplus (N(C^*) \cap \overline{R(B^*)})$. Hence $\overline{R(C)} = \overline{R(B^*)}$ is equivalent to $N(C^*) \cap R(B^*) = \{0\}$, which is equivalent to $N(A^*) \subseteq N(B^*)$ since $N(C^*) \cap R(B^*) = \{B^*x \mid x \in N(A^*)\}$ by (3) of Theorem 4.B. $N(A^*) \supseteq N(B^*)$ follows from (2) of Theorem 4.B, hence the proof of complete. $\square$

Proof of Theorem 3.1. (i-1) In case $\psi_a(B^{\frac{a}{2}}AB^{\frac{a}{2}}) \geq B^a$, it suffices to show that

$$\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}) \geq B^s \implies B^{\frac{t-s}{2}}\phi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})B^{\frac{t-s}{2}} \leq \phi_t(B^{\frac{t}{2}}AB^{\frac{t}{2}}) \quad (4.1)$$

holds for $a \leq s \leq t \leq 2s$ since we obtain

$$\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}) \geq B^s \implies \psi_t(B^{\frac{t}{2}}AB^{\frac{t}{2}}) \geq B^{\frac{t-s}{2}}\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})B^{\frac{t-s}{2}} \geq B^t$$

by choosing $\{\psi_r\}$ as $\{\phi_r\}$ in (4.1). If $\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}) \geq B^s$, then there exists a contraction X such that

$$X^*(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}))^{\frac{t-s}{2s}} = (\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}))^{\frac{t-s}{2s}}X = B^{\frac{t-s}{2}} \quad (4.2)$$

by Löwner-Heinz theorem and Theorem 4.B. Hence we have

$$\begin{aligned} \phi_t(B^{\frac{t}{2}}AB^{\frac{t}{2}}) &= \phi_t\left(X^*(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}))^{\frac{t-s}{2s}}B^{\frac{s}{2}}AB^{\frac{s}{2}}(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}))^{\frac{t-s}{2s}}X\right) \quad \text{by (4.2)} \\ &\geq X^*\phi_t\left((B^{\frac{s}{2}}AB^{\frac{s}{2}})(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}))^{\frac{t-s}{s}}\right)X \quad \text{by Theorem 4.A} \\ &= X^*\phi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}))^{\frac{t-s}{s}}X \quad \text{by (4.3)} \\ &= B^{\frac{t-s}{2}}\phi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})B^{\frac{t-s}{2}} \quad \text{by (4.2)}. \end{aligned}$$

The equality on the third line of the above formula can be shown by (3.1) as follows:

$$\phi_t\left(x(\psi_s(x))^{\frac{t-s}{s}}\right) = \phi_t(y^t g(y)) = y^{t-s}\phi_s(y^s g(y)) = (\psi_s(x))^{\frac{t-s}{s}}\phi_s(x), \quad (4.3)$$

where $x = y^s g(y)$, or equivalently, $y = (\psi_s(x))^{\frac{1}{s}}$.

(i-2) In case $A, B > 0$ and $\log \psi_a(B^{\frac{a}{2}}AB^{\frac{a}{2}}) \geq \log B^a$, put $A_1 = \psi_a(B^{\frac{a}{2}}AB^{\frac{a}{2}})$, $B_1 = B^a$ and $r_1 = \frac{s}{a} - 1 \geq 0$, then we have

$$\Psi_{r_1}(B_1^{\frac{r_1}{2}}G(A_1)B_1^{\frac{r_1}{2}}) \geq B_1^{r_1}, \quad (4.4)$$

where $G(x) = \psi_a^{-1}(x) = xg(x^{\frac{1}{a}})$ and $\Psi_r(x) = (\psi_{a(1+r)}(x))^{\frac{r}{1+r}}$, which satisfy

$$\Psi_r(x^r G(x)) = \left(\psi_{a(1+r)}(x^{1+r} g(x^{\frac{1}{a}}))\right)^{\frac{r}{1+r}} = x^r.$$

(4.4) can be rewritten as $(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}))^{\frac{s-a}{s}} \geq B^{s-a}$, so that

$$(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}))^{\frac{t-s}{s}} \geq B^{t-s}$$

holds for $a \leq s \leq t \leq 2s - a$ by Löwner-Heinz theorem. The rest of the proof can be done in the same way as (i-1).

(ii-1) In case $A^a \geq \psi_a(A^{\frac{a}{2}}BA^{\frac{a}{2}})$ and $\overline{R(A)} \cap N(B) = \{0\}$, it suffices to show that

$$A^s \geq \psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}}) \implies A^{\frac{t-s}{2}}\phi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})A^{\frac{t-s}{2}} \geq \phi_t(A^{\frac{t}{2}}BA^{\frac{t}{2}}) \quad (4.5)$$

holds for $a \leq s \leq t \leq 2s$ since we obtain

$$A^s \geq \psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}}) \implies \psi_t(A^{\frac{t}{2}}BA^{\frac{t}{2}}) \leq A^{\frac{t-s}{2}}\phi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})A^{\frac{t-s}{2}} \leq A^t$$

by choosing $\{\psi_r\}$ as $\{\phi_r\}$ in (4.5). If $A^s \geq \psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})$, then there exists a contraction X such that

$$X^*A^{\frac{t-s}{2}} = A^{\frac{t-s}{2}}X = P(\psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}}))^{\frac{t-s}{2s}}P \quad (4.6)$$

by Löwner-Heinz theorem and Theorem 4.B, where P is the projection onto $N(A)^\perp$. Hence we have

$$\begin{aligned} X^*\phi_t(A^{\frac{t}{2}}BA^{\frac{t}{2}})X &\leq \phi_t\left(X^*A^{\frac{t-s}{2}}A^{\frac{s}{2}}BA^{\frac{s}{2}}A^{\frac{t-s}{2}}X\right) \quad \text{by Theorem 4.A} \\ &= \phi_t\left((A^{\frac{s}{2}}BA^{\frac{s}{2}})(\psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}}))^{\frac{t-s}{s}}\right) \quad \text{by (4.6)} \\ &= \phi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})(\psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}}))^{\frac{t-s}{s}} \quad \text{by (4.3)} \\ &= X^*A^{\frac{t-s}{2}}\phi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})A^{\frac{t-s}{2}}X \quad \text{by (4.6),} \end{aligned}$$

and the proof is complete since $\overline{R(A)} \cap N(B) = \{0\}$ implies $\overline{R(X)} = \overline{R(A)}$ by Lemma 4.1.

(ii-2) In case $A, B > 0$ and $\log A^a \geq \log \psi_a(A^{\frac{a}{2}}BA^{\frac{a}{2}})$, put $A_1 = A^a$, $B_1 = \psi_a(A^{\frac{a}{2}}BA^{\frac{a}{2}})$ and $r_1 = \frac{s}{a} - 1 \geq 0$, then we have

$$A_1^{r_1} \geq \Psi_{r_1}(A_1^{\frac{r_1}{2}}G(B_1)A_1^{\frac{r_1}{2}}), \quad (4.7)$$

where $G(x)$ and $\Psi_r(x)$ are as defined in (i-2). (4.7) can be rewritten as $A^{s-a} \geq (\psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}}))^{\frac{s-a}{s}}$, so that

$$A^{t-s} \geq (\psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}}))^{\frac{t-s}{s}}$$

holds for $a \leq s \leq t \leq 2s - a$ by Löwner-Heinz theorem. The rest of the proof can be done in the same way as (ii-1). $\square$

We use the following result in order to give a proof of Theorem 3.2.

Theorem 4.C ([16]). *Let A and B be positive operators, and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(x)g(x) = x$. Then the following hold:*

- (i) $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$ ensures $A - g(A^{\frac{1}{2}}BA^{\frac{1}{2}}) \geq A^{\frac{1}{2}}E_BA^{\frac{1}{2}} - g(0)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}$.

(ii) $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$ ensures $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) - B \geq f(0)E_{B^{\frac{1}{2}}AB^{\frac{1}{2}}} - B^{\frac{1}{2}}E_AB^{\frac{1}{2}}$.

Here E_X denotes the projection onto $N(X)$.

Proof of Theorem 3.2. T belongs to class $A(s_0, t_0)$ - f_{s_0, t_0} if and only if

$$f_{s_0, t_0}(|T^*|^{t_0}|T|^{2s_0}|T^*|^{t_0}) \geq |T^*|^{2t_0}.$$

By (i) of Theorem 3.1, we have

$$f_{s_0, t}(|T^*|^t|T|^{2s_0}|T^*|^t) \geq |T^*|^{t-t_0} f_{s_0, t_0}(|T^*|^{t_0}|T|^{2s_0}|T^*|^{t_0})|T^*|^{t-t_0} \geq |T^*|^{2t} \quad (4.8)$$

holds for $t \geq t_0$. Put $f_{s, t}^\perp(x) = \frac{x}{f_{s, t}(x)}$, then (4.8) implies

$$|T|^{2s_0} \geq f_{s_0, t}^\perp(|T|^{s_0}|T^*|^{2t}|T|^{s_0}) \quad (4.9)$$

by (i) of Theorem 4.C. Since

$$f_{s_0, t}(x) = f_{s, t}(xg(y)^{s-s_0}) = f_{s, t}\left(xf_{s_0, t}^\perp(x)^{\frac{s-s_0}{s_0}}\right) \quad (4.10)$$

holds where $x = y^t g(y)^{s_0}$, we have

$$\begin{aligned} f_{s_0, t}(|T^*|^t|T|^{2s_0}|T^*|^t) &= f_{s, t}\left(|T^*|^t|T|^{2s_0}|T^*|^t f_{s_0, t}^\perp(|T^*|^{t_0}|T|^{2s_0}|T^*|^{t_0})^{\frac{s-s_0}{s_0}}\right) \quad \text{by (4.10)} \\ &= f_{s, t}\left(|T^*|^t|T|^{s_0} f_{s_0, t}^\perp(|T|^{s_0}|T^*|^{2t}|T|^{s_0})^{\frac{s-s_0}{s_0}}|T|^{s_0}|T^*|^t\right) \\ &\leq f_{s, t}(|T^*|^t|T|^{2s}|T^*|^t) \quad \text{by (4.9) and Löwner-Heinz theorem,} \end{aligned}$$

so that $f_{s, t}(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t}$ holds for $s_0 \leq s \leq 2s_0$. We obtain the desired conclusion by repeating this process. $\square$

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