

Relations between two operator inequalities via operator means

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Abstract

Let A and B be (not necessarily invertible) positive operators. Recently, the author and Yamazaki discussed relations between

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \quad \text{and} \quad A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$$

for $p \geq 0$ and $r \geq 0$, and also Yamazaki and Yanagida discussed relations between

$$\frac{p}{p+r} I + \frac{r}{p+r} B^{\frac{r}{2}} A^p B^{\frac{r}{2}} \geq B^r \quad \text{and} \quad A^p \geq \frac{A^{\frac{p}{2}} B^r A^{\frac{p}{2}}}{\frac{r}{p+r} A^{\frac{p}{2}} B^r A^{\frac{p}{2}} + \frac{p}{p+r} I}$$

for $p \geq 0$ and $r \geq 0$.

In this report, as a generalization of their results via the representing functions of operator means, we shall show relations between two operator inequalities

$$f(B^{\frac{1}{2}} A B^{\frac{1}{2}}) \geq B \quad \text{and} \quad A \geq g(A^{\frac{1}{2}} B A^{\frac{1}{2}}),$$

where f and g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$.

1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator T is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$. We denote the set of positive operators by $\mathcal{B}(\mathcal{H})_+$.

Kubo-Ando [8] investigated an axiomatic approach for operator means (see also [5]). A binary operation $\sigma : \mathcal{B}(\mathcal{H})_+ \times \mathcal{B}(\mathcal{H})_+ \rightarrow \mathcal{B}(\mathcal{H})_+$ is called an operator connection if it satisfies the following conditions (i), (ii) and (iii) for $A, B, C, D \in \mathcal{B}(\mathcal{H})_+$:

- (i) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$,
- (ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$,
- (iii) $A_n, B_n \in \mathcal{B}(\mathcal{H})_+$, $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$,
where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \cdots$ and A_n converges strongly to A .

An operator connection σ is called an *operator mean* if

$$(iv) \quad I\sigma I = I.$$

There exists a one-to-one correspondence between an operator connection σ and an operator monotone function $f \geq 0$ on $[0, \infty)$. The operator connection σ can be defined via the corresponding function f , which is called the *representing function* of σ , by

$$A\sigma B = A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$

if A is invertible, and σ is an operator mean if and only if $f(1) = 1$.

The following are typical examples of operator means. For positive invertible operators A and B , and for $\alpha \in [0, 1]$,

$$(i) \text{ Arithmetic mean: } A\nabla_{\alpha}B = (1 - \alpha)A + \alpha B,$$

$$(ii) \text{ Geometric mean } (\alpha\text{-power mean}): A\sharp_{\alpha}B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}},$$

$$(iii) \text{ Harmonic mean: } A!_{\alpha}B = \{(1 - \alpha)A^{-1} + \alpha B^{-1}\}^{-1}.$$

The representing functions of ∇_{α} , $\sharp_{\alpha}$ and $!_{\alpha}$ are $(1 - \alpha) + \alpha t$, t^{α} and $\{(1 - \alpha) + \alpha t^{-1}\}^{-1} = \frac{t}{(1 - \alpha)t + \alpha}$, respectively. On these operator means, the following relations are known. We remark that (1.1) was shown in [4], and (1.1) and (1.2) can be proved without using properties of operator means. Let A and B be positive invertible operators. For each $p \geq 0$ and $r \geq 0$,

$$B^{-r}\sharp_{\frac{r}{p+r}}A^p \geq I \iff I \geq A^{-p}\sharp_{\frac{p}{p+r}}B^r \quad (1.1)$$

and

$$B^{-r}\nabla_{\frac{r}{p+r}}A^p \geq I \iff I \geq A^{-p}!_{\frac{p}{p+r}}B^r. \quad (1.2)$$

(1.1) is closely related to Furuta inequality [3], and a mean theoretic approach to Furuta inequality was discussed in [1][7] and others. We remark the following relations on inequalities in (1.1) and (1.2): Let A and B be positive invertible operators. For each $p \geq 0$ and $r \geq 0$,

$$A \geq B \implies \log A \geq \log B \implies \begin{cases} B^{-r}\sharp_{\frac{r}{p+r}}A^p \geq I, \\ I \geq A^{-p}\sharp_{\frac{p}{p+r}}B^r \end{cases} \implies \begin{cases} B^{-r}\nabla_{\frac{r}{p+r}}A^p \geq I, \\ I \geq A^{-p}!_{\frac{p}{p+r}}B^r. \end{cases}$$

The first relation holds since $\log t$ is operator monotone, the second was shown in [2][4], and the third holds since $(1 - \alpha) + \alpha t \geq t^{\alpha} \geq \frac{t}{(1 - \alpha)t + \alpha}$ for $t \geq 0$ and $\alpha \in [0, 1]$. We remark that it was shown in [2][4] that

$$\begin{aligned} \log A \geq \log B &\iff B^{-r}\sharp_{\frac{r}{p+r}}A^p \geq I \quad \text{for all } p \geq 0 \text{ and } r \geq 0 \\ &\iff I \geq A^{-p}\sharp_{\frac{p}{p+r}}B^r \quad \text{for all } p \geq 0 \text{ and } r \geq 0. \end{aligned}$$

In this report, firstly we attempt a mean theoretic approach to (1.1) and (1.2). In other words, we shall state a result corresponding to (1.1) and (1.2) on a general operator mean for invertible operators. Secondly we shall show relations between

$$f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \quad \text{and} \quad A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$$

for (not necessarily invertible) positive operators A and B , where f and g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$. This result is a further generalization of the former argument via the representing functions of operator means. Moreover this result includes the ones by the author and Yamazaki [6] and by Yamazaki and Yanagida [11].

2 A result on a general operator mean

In this section, we shall state a result corresponding to (1.1) and (1.2) on a general operator mean for invertible operators. At first we state definitions and properties of some operator means via a operator mean σ .

Definition ([8]). Let σ be the operator mean with a representing function f .

- (i) σ' is said to be the transpose of σ if σ' is the operator mean with a representing function $tf(t^{-1})$.
- (ii) σ^* is said to be the adjoint of σ if σ^* is the operator mean with a representing function $\{f(t^{-1})\}^{-1}$.
- (iii) $\sigma^\perp$ is said to be the dual of σ if $\sigma^\perp$ is the operator mean with a representing function $\frac{t}{f(t)}$.

We remark that these representing functions can be defined on $[0, \infty)$ by setting the value on 0 by the limit to $+0$ since f is operator monotone.

Proposition 2.A ([8]). Let σ be an operator mean and $A, B \in \mathcal{B}(\mathcal{H})_+$.

- (i) $A\sigma'B = B\sigma A$.
- (ii) $A\sigma^*B = (A^{-1}\sigma B^{-1})^{-1}$ if A and B are invertible.
- (iii) $(\sigma')' = (\sigma^*)^* = (\sigma^\perp)^\perp = \sigma$.
- (iv) $\sigma^\perp = (\sigma')^* = (\sigma^*)'$, $\sigma' = (\sigma^*)^\perp = (\sigma^\perp)^*$ and $\sigma^* = (\sigma^\perp)' = (\sigma')^\perp$.

By using Proposition 2.A, we shall show a generalization of (1.1) and (1.2).

Proposition 2.1. *Let A and B be positive invertible operators. For every operator mean σ ,*

$$B^{-1}\sigma A \geq I \iff I \geq A^{-1}\sigma^\perp B. \quad (2.1)$$

Proof. By (i) of Proposition 2.A,

$$B^{-1}\sigma A = A\sigma'B^{-1} \geq I. \quad (2.2)$$

By (ii) and (iv) of Proposition 2.A, (2.2) is equivalent to

$$I \geq (A\sigma'B^{-1})^{-1} = A^{-1}(\sigma')^*B = A^{-1}\sigma^\perp B.$$

Hence the proof is complete. $\square$

Since $(\sharp_\alpha)^\perp = \sharp_{1-\alpha}$ and $(\nabla_\alpha)^\perp = \nabla_{1-\alpha}$, Proposition 2.1 leads (1.1) (resp. (1.2)) by replacing A and B with A^p and B^r and by putting $\sigma = \sharp_{\frac{r}{p+r}}$ (resp. $\sigma = \nabla_{\frac{r}{p+r}}$). We remark that (2.1) can be rewritten by

$$f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \iff A \geq \frac{A^{\frac{1}{2}}BA^{\frac{1}{2}}}{f(A^{\frac{1}{2}}BA^{\frac{1}{2}})} \quad (2.3)$$

with the representing function f of σ .

3 Main results

In this section, we shall show a further generalization of Proposition 2.1 via the representing functions of operator means.

When we rewrite (1.1) and (1.2) for positive invertible operators A and B by

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \iff A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \quad (3.1)$$

and

$$\frac{p}{p+r}I + \frac{r}{p+r}B^{\frac{r}{2}}A^pB^{\frac{r}{2}} \geq B^r \iff A^p \geq \frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{\frac{r}{p+r}A^{\frac{p}{2}}B^rA^{\frac{p}{2}} + \frac{p}{p+r}I} \quad (3.2)$$

with the representing functions, we can consider non-invertible operators on this argument. On relations between two inequalities in (3.1) and (3.2) for (not necessarily invertible) positive operators A and B , the following results were obtained in [6] and [11].

Theorem 3.A ([6]). *Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$, the following assertions hold:*

- (i) *If $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, then $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$.*
- (ii) *If $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subseteq N(B)$, then $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.*

Theorem 3.B ([11]). *Let A and B be positive operators. Then for each $p > 0$ and $r \geq 0$, the following assertions hold:*

- (i) *If $\frac{p}{p+r}I + \frac{r}{p+r}B^{\frac{r}{2}}A^pB^{\frac{r}{2}} \geq B^r$, then $A^p \geq \frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{\frac{r}{p+r}A^{\frac{p}{2}}B^rA^{\frac{p}{2}} + \frac{p}{p+r}I}$.*
- (ii) *If $A^p \geq \frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{\frac{r}{p+r}A^{\frac{p}{2}}B^rA^{\frac{p}{2}} + \frac{p}{p+r}I}$ and $N(A) \subseteq N(B)$, then $\frac{p}{p+r}I + \frac{r}{p+r}B^{\frac{r}{2}}A^pB^{\frac{r}{2}} \geq B^r$.*

Here we shall obtain a generalization of Proposition 2.1 via the form of (2.3). This result is also an extension of Theorems 3.A and 3.B.

Theorem 3.1. *Let A and B be positive operators, and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying*

$$f(t)g(t) = t. \quad (3.3)$$

- (i) *If $g(0) = 0$ or $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$, then $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$ ensures $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$.*
- (ii) *If $N(A) \subseteq N(B)$, then $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$ ensures $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$.*

In Theorem 3.1, f and g are not necessarily operator monotone functions. We also remark that if $f(0) > 0$, then automatically $g(0) = 0$ by (3.3).

If A and B are positive invertible operators and σ is the operator mean with a representing function f , Theorem 3.1 ensures Proposition 2.1 since (2.1) is equivalent to (2.3). Theorem 3.1 also leads Theorem 3.A (resp. Theorem 3.B) by replacing A and B with A^p and B^r and by putting $f(t) = t^{\frac{r}{p+r}}$ and $g(t) = t^{\frac{p}{p+r}}$ (resp. $f(t) = \frac{p}{p+r} + \frac{r}{p+r}t$ and $g(t) = \frac{r}{p+r}t + \frac{p}{p+r}$). We remark that $g(0) = 0$ in these cases.

We need some lemmas in order to prove Theorem 3.1.

Lemma 3.C. *Let T be a positive operator. Then*

$$\lim_{\varepsilon \rightarrow +0} T^{\frac{1}{2}}(T + \varepsilon I)^{-1}T^{\frac{1}{2}} = \lim_{\varepsilon \rightarrow +0} (T + \varepsilon I)^{-1}T = P_{N(T)^\perp},$$

where $P_{\mathcal{M}}$ is a projection onto a closed subspace $\mathcal{M}$.

Lemma 3.C is a well-known result. For example, it was shown in [9] and [6].

Lemma 3.2. *Let f be a non-negative continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$. Then $N(f(T)) = N(T)$ for every positive operator T .*

Proof. Let $T = \int_0^{\|T\|} t dE_t$ be the spectral decomposition of a positive operator T . Then

$$(f(T)x, y) = \int_0^{\|T\|} f(t) d(E_t x, y) \quad \text{for } x, y \in \mathcal{H}. \quad (3.4)$$

We remark that $E_{-0} = 0$.

Assume that $x \in N(T)$. Then $E_0 x = (E_0 - E_{-0})x = P_{N(T)}x = x$, and $(f(T)x, y) = f(0)(x, y) = 0$ for any $y \in \mathcal{H}$ by (3.4). Therefore $f(T)x = 0$, so that $x \in N(f(T))$.

Conversely, assume that $x \in N(f(T))$. Then for $\varepsilon > 0$,

$$0 = (f(T)x, x) = \int_0^\varepsilon f(t) d(E_t x, x) + \int_\varepsilon^{\|T\|} f(t) d(E_t x, x)$$

by (3.4). Since $f(t) > 0$ for $t > 0$, $E_\varepsilon x = x$ for $\varepsilon > 0$. By tending $\varepsilon \rightarrow +0$, we have $P_{N(T)}x = E_0 x = x$, so that $x \in N(T)$. $\square$

Lemma 3.3. *Let $T = U|T|$ be the polar decomposition of an operator T , and let f be a continuous function on $[0, \infty)$. Then*

$$Uf(|T|)U^* = f(|T^*|) - f(0)(I - UU^*).$$

Proof. First we shall show the case $f(0) = 0$ by the same way to [10, Lemma]. Since $U|T|^n U^* = |T^*|^n$ for each positive integer n , $Up(|T|)U^* = p(|T^*|)$ holds for any polynomials p such that $p(0) = 0$. By taking a sequence $\{p_n\}$ of polynomials with $p_n(0) = 0$ which converges uniformly to f on $[0, \|T\|]$, we obtain $Uf(|T|)U^* = f(|T^*|)$ for general f with $f(0) = 0$.

Next, let $g(t) = f(t) - f(0)$. Then $g(0) = 0$, so that

$$\begin{aligned} Uf(|T|)U^* &= U\{g(|T|) + f(0)I\}U^* = Ug(|T|)U^* + f(0)UU^* \\ &= g(|T^*|) + f(0)I - f(0)(I - UU^*) = f(|T^*|) - f(0)(I - UU^*). \end{aligned}$$

Hence the proof is complete. $\square$

Proof of Theorem 3.1. Let $\varepsilon > 0$.

Proof of (i). Since $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$, we obtain

$$(B + \varepsilon I)^{-1} \geq \{f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2) + \varepsilon I\}^{-1}.$$

Let $A^{\frac{1}{2}}B^{\frac{1}{2}} = U|A^{\frac{1}{2}}B^{\frac{1}{2}}|$ be the polar decomposition of $A^{\frac{1}{2}}B^{\frac{1}{2}}$. Then we have

$$\begin{aligned}
& A^{\frac{1}{2}}B^{\frac{1}{2}}(B + \varepsilon I)^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \\
& \geq A^{\frac{1}{2}}B^{\frac{1}{2}}\{f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2) + \varepsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \\
& = U|A^{\frac{1}{2}}B^{\frac{1}{2}}|\{f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2) + \varepsilon I\}^{-1}|A^{\frac{1}{2}}B^{\frac{1}{2}}|U^* \\
& = U\{f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2) + \varepsilon I\}^{-1}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2U^* \\
& = U\{f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2) + \varepsilon I\}^{-1}f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)g(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)U^* \quad \text{by (3.3).}
\end{aligned} \tag{3.5}$$

In (3.5), by tending $\varepsilon \rightarrow +0$ and Lemma 3.C, we obtain

$$A^{\frac{1}{2}}P_{N(B)^{\perp}}A^{\frac{1}{2}} \geq UP_{N(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2))^{\perp}}g(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)U^* = Ug(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)U^* \tag{3.6}$$

by the following: If $f(0) > 0$, then $f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)$ is invertible and $P_{N(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2))^{\perp}} = I$. If $f(0) = 0$, then $UP_{N(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2))^{\perp}} = UP_{N(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)^{\perp}} = UP_{N(A^{\frac{1}{2}}B^{\frac{1}{2}})^{\perp}} = U$ by Lemma 3.2.

Therefore, noting that $UU^* = P_{N(B^{\frac{1}{2}}A^{\frac{1}{2}})^{\perp}} = P_{N(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\perp}} = I$ if $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$, we have

$$\begin{aligned}
A & \geq A^{\frac{1}{2}}P_{N(B)^{\perp}}A^{\frac{1}{2}} \\
& \geq Ug(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)U^* \quad \text{by (3.6)} \\
& = g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2) - g(0)(I - UU^*) \quad \text{by Lemma 3.3} \\
& = g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2) \quad \text{since } g(0) = 0 \text{ or } N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\} \\
& = g(A^{\frac{1}{2}}BA^{\frac{1}{2}}).
\end{aligned}$$

Proof of (ii). Since $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$, we obtain

$$(A + \varepsilon I)^{-1} \leq \{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2) + \varepsilon I\}^{-1}.$$

Let $B^{\frac{1}{2}}A^{\frac{1}{2}} = V|B^{\frac{1}{2}}A^{\frac{1}{2}}|$ be the polar decomposition of $B^{\frac{1}{2}}A^{\frac{1}{2}}$. Then we have

$$\begin{aligned}
& B^{\frac{1}{2}}A^{\frac{1}{2}}(A + \varepsilon I)^{-1}A^{\frac{1}{2}}B^{\frac{1}{2}} \\
& \leq B^{\frac{1}{2}}A^{\frac{1}{2}}\{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2) + \varepsilon I\}^{-1}A^{\frac{1}{2}}B^{\frac{1}{2}} \\
& = V|B^{\frac{1}{2}}A^{\frac{1}{2}}|\{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2) + \varepsilon I\}^{-1}|B^{\frac{1}{2}}A^{\frac{1}{2}}|V^* \\
& = V\{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2) + \varepsilon I\}^{-1}|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2V^* \\
& = V\{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2) + \varepsilon I\}^{-1}g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)f(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)V^* \quad \text{by (3.3).}
\end{aligned} \tag{3.7}$$

In (3.7), by tending $\varepsilon \rightarrow +0$ and Lemma 3.C, we obtain

$$B^{\frac{1}{2}}P_{N(A)^{\perp}}B^{\frac{1}{2}} \leq VP_{N(g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2))^{\perp}}f(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)V^* = Vf(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)V^* \tag{3.8}$$

by the following: If $g(0) > 0$, then $g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)$ is invertible and $P_{N(g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2))^\perp} = I$. If $g(0) = 0$, then $VP_{N(g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2))^\perp} = VP_{N(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)^\perp} = VP_{N(B^{\frac{1}{2}}A^{\frac{1}{2}})^\perp} = V$ by Lemma 3.2.

Therefore, noting that $N(A) \subseteq N(B)$ is equivalent to $P_{N(A)^\perp} \geq P_{N(B)^\perp}$, we have

$$\begin{aligned}
 B &= B^{\frac{1}{2}}P_{N(B)^\perp}B^{\frac{1}{2}} \\
 &\leq B^{\frac{1}{2}}P_{N(A)^\perp}B^{\frac{1}{2}} && \text{since } N(A) \subseteq N(B) \\
 &\leq Vf(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)V^* && \text{by (3.8)} \\
 &= f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2) - f(0)(I - VV^*) && \text{by Lemma 3.3} \\
 &\leq f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2) \\
 &= f(B^{\frac{1}{2}}AB^{\frac{1}{2}}).
 \end{aligned}$$

Hence the proof is complete. $\square$

Corollary 3.4. *Let A and B be positive operators, and let f and g be positive continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$. If $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$, then $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$ is equivalent to $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$.*

Proof. Since $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$ ensures $\{0\} = N(A) \subseteq N(B)$, $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$ is equivalent to $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$ by Theorem 3.1. $\square$

Of course $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$ if A and B are invertible.

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