

Picone identities for half-linear elliptic equations with $p(x)$ -Laplacians and applications

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1 Introduction

Since the pioneering work of M. Picone [4], efforts have been made to establish Picone identities (or Picone-type inequalities) for differential equations of various type. Picone identities play an important role in the study of Sturmian comparison theorems (cf. [6]) and oscillation results for ordinary or partial differential equations or systems. In 1909, Picone [4] derived the so-called Picone identity

$$\begin{aligned} & \frac{d}{dt} \left(\frac{u}{v} (a(t)u'v - A(t)v'u) \right) \\ = & (a(t) - A(t))(u')^2 + (C(t) - c(t))u^2 + A(t) \left[v \left(\frac{u}{v} \right)' \right]^2 \\ & + \frac{u}{v} (vq[u] - uQ[v]) \end{aligned}$$

to obtain Sturmian comparison theorems for ordinary differential operators q, Q defined by

$$\begin{aligned} q[u] &= (a(t)u')' + c(t)u, \\ Q[v] &= (A(t)v')' + C(t)v. \end{aligned}$$

Recently, much current interest has been focused on various mathematical problems with variable exponent growth condition (cf. [2, 3]). The study of such problems arise from nonlinear elasticity theory, electrorheological fluids (see [5, 12]).

The operator $\nabla \cdot (|\nabla u|^{p(x)-2} \nabla u)$ ($p(x) > 1$) is said to be $p(x)$ -Laplacian, and becomes p -Laplacian $\nabla \cdot (|\nabla u|^{p-2} \nabla u)$ if $p(x) = p$ (constant), where the dot $\cdot$ denotes the scalar product and $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$.

The paper [11] by Zhang seems to be the first paper dealing with oscillations of solutions of $p(t)$ -Laplacian equations of the form

$$(|u'|^{p(t)-2}u')' + t^{-\theta(t)}g(t, u) = 0, \quad t > 0.$$

In this work we present Picone identity, Picone-type inequality and Riccati inequality (which is reduced from Picone identity) to establish Sturmian comparison theorems and oscillation theorems for quasilinear elliptic operators with $p(x)$ -Laplacians (cf. [1, 7–10]).

2 Half-linear elliptic inequalities

We establish Picone identities for half-linear elliptic inequalities

$$uq[u] \geq 0, \tag{1}$$

$$vQ[v] \leq 0, \tag{2}$$

where q and Q are defined by

$$\begin{aligned} q[u] := & \nabla \cdot (a(x)|\nabla u|^{\alpha(x)-1}\nabla u) - a(x)(\log |u|)|\nabla u|^{\alpha(x)-1}\nabla \alpha(x) \cdot \nabla u \\ & + |\nabla u|^{\alpha(x)-1}b(x) \cdot \nabla u + c(x)|u|^{\alpha(x)-1}u, \end{aligned} \tag{3}$$

$$\begin{aligned} Q[v] := & \nabla \cdot (A(x)|\nabla v|^{\alpha(x)-1}\nabla v) - A(x)(\log |v|)|\nabla v|^{\alpha(x)-1}\nabla \alpha(x) \cdot \nabla v \\ & + |\nabla v|^{\alpha(x)-1}B(x) \cdot \nabla v + C(x)|v|^{\alpha(x)-1}v, \end{aligned} \tag{4}$$

to derive Sturmian comparison theorems for q and Q . Let G be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G . It is assumed that $a(x), A(x) \in C(\overline{G}; (0, \infty))$, $b(x), B(x) \in C(\overline{G}; \mathbb{R}^n)$, $c(x), C(x) \in C(\overline{G}; \mathbb{R})$, and that $\alpha(x) \in C^1(\overline{G}; (0, \infty))$. The domain $\mathcal{D}_q(G)$ of q is defined to be the set of all functions u of class $C^1(\overline{G}; \mathbb{R})$ such that $a(x)|\nabla u|^{\alpha(x)-1}\nabla u \in C^1(G; \mathbb{R}^n) \cap C(\overline{G}; \mathbb{R}^n)$. The domain $\mathcal{D}_Q(G)$ of Q is defined similarly. We note that $\log |u|$ in (3) has singularities at zeros x_0 of $u(x)$, but $u \log |u|$ in (1) is continuous at every zero x_0 if we define $u \log |u| = 0$ at $x = x_0$, in view of $\lim_{\varepsilon \rightarrow +0} \varepsilon \log \varepsilon = 0$. We make the similar remark in (4). By a *solution* u [resp. v] of (1) [resp. (2)] we mean a function $u \in \mathcal{D}_q(G)$ [resp. $v \in \mathcal{D}_Q(G)$] which satisfies (1) [resp. (2)] in G . We note that (1) and (2) are *half-linear* in the sense that a constant multiple of a solution u [resp. v] is also a solution of (1) [resp. (2)] in light of

$$\begin{aligned} (ku)q[ku] &= |k|^{\alpha(x)+1}uq[u] \quad (k \in \mathbb{R}), \\ (kv)Q[kv] &= |k|^{\alpha(x)+1}vQ[v] \quad (k \in \mathbb{R}). \end{aligned}$$

3 Picone identity

Lemma 1 (Picone identity for Q) *If $v \in \mathcal{D}_Q(G)$ and v has no zero in G , then we obtain the following Picone identity for any $u \in C^1(G; \mathbb{R})$ which has no zero in G :*

$$\begin{aligned}
& -\nabla \cdot \left(u\varphi(u) \frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)} \right) \\
&= -A(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right|^{\alpha(x)+1} \\
& \quad + C(x)|u|^{\alpha(x)+1} \\
& \quad + A(x) \left[\left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right|^{\alpha(x)+1} \right. \\
& \quad \quad \left. + \alpha(x) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)+1} \right. \\
& \quad \quad \left. - (\alpha(x)+1) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)-1} \left(\nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) \right. \right. \\
& \quad \quad \quad \left. \left. - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right) \cdot \left(\frac{u}{v} \nabla v \right) \right] \\
& \quad - \frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}} (vQ[v]) \quad \text{in } G, \tag{5}
\end{aligned}$$

where $\varphi(u) = |u|^{\alpha(x)-1}u = |u(x)|^{\alpha(x)-1}u(x)$.

Theorem 1 (Picone identity for q and Q) *Let $\alpha(x) \in C^2(G; (0, \infty))$ and $b(x)/a(x) \in C^1(G; \mathbb{R}^n)$. Assume that $u \in C^1(G; \mathbb{R})$, u has no zero in G , and that:*

(H_1) *there is a function $f \in C(\overline{G}; \mathbb{R}) \cap C^1(G; \mathbb{R})$ such that*

$$\nabla f = \frac{\log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{b(x)}{(\alpha(x)+1)a(x)} \quad \text{in } G.$$

If $e^f u \in \mathcal{D}_q(G)$, $v \in \mathcal{D}_Q(G)$ and v has no zero in G , then we obtain the following Picone identity:

$$\begin{aligned}
& \nabla \cdot \left(e^{-(\alpha(x)+1)f} (e^f u) a(x) |\nabla(e^f u)|^{\alpha(x)-1} \nabla(e^f u) - \frac{u\varphi(u)}{\varphi(v)} A(x) |\nabla v|^{\alpha(x)-1} \nabla v \right) \\
&= a(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)a(x)} b(x) \right|^{\alpha(x)+1} \\
& \quad - A(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right|^{\alpha(x)+1}
\end{aligned}$$

$$\begin{aligned}
& +(C(x) - c(x))|u|^{\alpha(x)+1} \\
& +A(x) \left[\left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)A(x)} B(x) \right|^{\alpha(x)+1} \right. \\
& \quad \left. + \alpha(x) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)+1} \right. \\
& \quad \left. - (\alpha(x) + 1) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)-1} \left(\nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) \right. \right. \\
& \quad \quad \left. \left. - \frac{u}{(\alpha(x) + 1)A(x)} B(x) \right) \cdot \left(\frac{u}{v} \nabla v \right) \right] \\
& + e^{-(\alpha(x)+1)f} (e^f u) q[e^f u] - \frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}} (v Q[v]) \quad \text{in } G.
\end{aligned}$$

Theorem 2 (Sturmian comparison theorem) *Let $\alpha(x) \in C^2(G; (0, \infty))$ and $b(x)/a(x), B(x)/A(x) \in C^1(G; \mathbb{R}^n)$. Assume that there exists a function $u \in C^1(\overline{G}; \mathbb{R})$ such that $u = 0$ on ∂G , u has no zero in G , the hypothesis (H_1) of Theorem 1 holds and that:*

(H_2) *there is a function $F \in C(\overline{G}; \mathbb{R}) \cap C^1(G; \mathbb{R})$ such that*

$$\nabla F = \frac{\log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{B(x)}{(\alpha(x) + 1)A(x)} \quad \text{in } G.$$

If $e^f u \in \mathcal{D}_q(G)$, $(e^f u)q[e^f u] \geq 0$ in G , and

$$\begin{aligned}
& \int_G \left[a(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)a(x)} b(x) \right|^{\alpha(x)+1} \right. \\
& \quad \left. - A(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)A(x)} B(x) \right|^{\alpha(x)+1} \right. \\
& \quad \left. + (C(x) - c(x))|u|^{\alpha(x)+1} \right] dx \geq 0, \tag{6}
\end{aligned}$$

then every solution $v \in \mathcal{D}_Q(G)$ of (2) must vanish at some point of $\overline{G}$.

Corollary 1 (Sturmian comparison theorem) *Let $\alpha(x) \in C^2(G; (0, \infty))$, $b(x)/a(x), B(x)/A(x) \in C^1(G; \mathbb{R}^n)$. Assume that:*

$$(i) \quad \frac{b(x)}{a(x)} = \frac{B(x)}{A(x)} \quad \text{in } G;$$

$$(ii) \quad a(x) \geq A(x), \quad C(x) \geq c(x) \quad \text{in } G.$$

If there exists a function $u \in C^1(\overline{G}; \mathbb{R})$ such that $u = 0$ on ∂G , u has no zero in G , the hypotheses (H_1) and (H_2) of Theorems 1 and 2 hold, $e^f u \in \mathcal{D}_q(G)$, $(e^f u)q[e^f u] \geq 0$ in G , then every solution $v \in \mathcal{D}_Q(G)$ of (2) must vanish at some point of $\overline{G}$.

4 Picone-type inequality

We derive Picone-type inequality and Sturmian comparison theorem for the half-linear elliptic operator q defined by

$$q[u] := \nabla \cdot (a(x)|\nabla u|^{\alpha(x)-1}\nabla u) - a(x)(\log |u|)|\nabla u|^{\alpha(x)-1}\nabla \alpha(x) \cdot \nabla u \\ + |\nabla u|^{\alpha(x)-1}b(x) \cdot \nabla u + c(x)|u|^{\alpha(x)-1}u,$$

and the quasilinear elliptic operator $\tilde{Q}$ defined by

$$\tilde{Q}[v] := \nabla \cdot (A(x)|\nabla v|^{\alpha(x)-1}\nabla v) - A(x)(\log |v|)|\nabla v|^{\alpha(x)-1}\nabla \alpha(x) \cdot \nabla v \\ + |\nabla v|^{\alpha(x)-1}B(x) \cdot \nabla v + C(x)|v|^{\alpha(x)-1}v \\ + D(x)|v|^{\beta(x)-1}v + E(x)|v|^{\gamma(x)-1}v,$$

where $D(x), E(x) \in C(\bar{G}; [0, \infty))$ and $\alpha(x), \beta(x), \gamma(x) \in C(\bar{G}; (0, \infty))$ with $0 < \gamma(x) < \alpha(x) < \beta(x)$.

Theorem 3 (Picone-type inequality for q and $\tilde{Q}$) *Assume that $\alpha(x) \in C^2(G; (0, \infty))$, $b(x)/a(x) \in C^1(G; \mathbb{R}^n)$, and that $u \in C^1(G; \mathbb{R})$, u has no zero in G , and the hypothesis (H_1) of Theorem 1 holds. If $e^f u \in \mathcal{D}_q(G)$, $v \in \mathcal{D}_{\tilde{Q}}(G)$ and v has no zero in G , then we obtain the Picone-type inequality:*

$$\begin{aligned} & \nabla \cdot \left(e^{-(\alpha(x)+1)f} (e^f u) a(x) |\nabla(e^f u)|^{\alpha(x)-1} \nabla(e^f u) - \frac{u \varphi(u)}{\varphi(v)} A(x) |\nabla v|^{\alpha(x)-1} \nabla v \right) \\ & \geq a(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)a(x)} b(x) \right|^{\alpha(x)+1} \\ & \quad - A(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right|^{\alpha(x)+1} \\ & \quad + (C(x) + \tilde{C}(x) - c(x)) |u|^{\alpha(x)+1} \\ & \quad + A(x) \left[\left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right|^{\alpha(x)+1} \right. \\ & \quad \quad \left. + \alpha(x) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)+1} \right. \\ & \quad \quad \left. - (\alpha(x)+1) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)-1} \left(\nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) \right. \right. \\ & \quad \quad \quad \left. \left. - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right) \cdot \left(\frac{u}{v} \nabla v \right) \right] \\ & \quad + e^{-(\alpha(x)+1)f} (e^f u) q[e^f u] - \frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}} (v \tilde{Q}[v]) \quad \text{in } G, \end{aligned}$$

where

$$\tilde{C}(x) = \left(\frac{\beta(x) - \gamma(x)}{\alpha(x) - \gamma(x)} \right) \left(\frac{\beta(x) - \alpha(x)}{\alpha(x) - \gamma(x)} \right)^{\frac{\alpha(x) - \beta(x)}{\beta(x) - \gamma(x)}} D(x)^{\frac{\alpha(x) - \gamma(x)}{\beta(x) - \gamma(x)}} E(x)^{\frac{\beta(x) - \alpha(x)}{\beta(x) - \gamma(x)}}.$$

Theorem 4 (Sturmian comparison theorem) *Under the same assumptions of Theorem 2 with $C(x)$ in (6) replaced by $C(x) + \tilde{C}(x)$, every solution $v \in \mathcal{D}_{\tilde{Q}}(G)$ of $v\tilde{Q}[v] \leq 0$ must vanish at some point of $\overline{G}$.*

Corollary 2 (Sturmian comparison theorem) *Let $\alpha(x) \in C^2(G; (0, \infty))$, $b(x)/a(x), B(x)/A(x) \in C^1(G; \mathbb{R}^n)$. Assume that:*

- (i) $\frac{b(x)}{a(x)} = \frac{B(x)}{A(x)}$ in G ;
- (ii) $a(x) \geq A(x), C(x) + \tilde{C}(x) \geq c(x)$ in G .

If there exists a function $u \in C^1(\overline{G}; \mathbb{R})$ such that $u = 0$ on ∂G , u has no zero in G , the hypotheses (H_1) and (H_2) of Theorems 1 and 2 hold, $e^f u \in \mathcal{D}_q(G)$, $(e^f u)q[e^f u] \geq 0$ in G , then every solution $v \in \mathcal{D}_{\tilde{Q}}(G)$ of $v\tilde{Q}[v] \leq 0$ must vanish at some point of $\overline{G}$.

5 Riccati inequality

Let Ω be an exterior domain in $\mathbb{R}^n$, that is, Ω includes the domain $\{x \in \mathbb{R}^n; |x| \geq r_0\}$ for some $r_0 > 0$. It is assumed that $A(x) \in C(\Omega; (0, \infty))$, $B(x) \in C(\Omega; \mathbb{R}^n)$, $C(x) \in C(\Omega; \mathbb{R})$, and that $\alpha(x) \in C^1(\Omega; (0, \infty))$. The domain $\mathcal{D}_Q(\Omega)$ of Q is defined to be the set of all functions v of class $C^1(\Omega; \mathbb{R})$ such that $A(x)|\nabla v|^{\alpha(x)-1}\nabla v \in C^1(\Omega; \mathbb{R}^n)$.

A solution $v \in \mathcal{D}_Q(\Omega)$ of (2) is said to be *oscillatory* in Ω if it has a zero in Ω_r for any $r > 0$, where

$$\Omega_r = \Omega \cap \{x \in \mathbb{R}^n; |x| > r\}.$$

We use the notation $A[r, \infty) = \{x \in \mathbb{R}^n; |x| \geq r\}$, and find that $\Omega_{r_1} = A(r_1, \infty)$ for some large $r_1 \geq r_0$. Noting Picone identity (5) holds in any domain of $\mathbb{R}$ and letting $u = 1$ in (5), we obtain the following lemma.

Lemma 2 If $v \in \mathcal{D}_Q(\Omega)$ and v has no zero in $A[r_2, \infty)$ for some $r_2 > r_1$, then we obtain the following:

$$\begin{aligned} & -\nabla \cdot \left(\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{|v|^{\alpha(x)-1}v} \right) \\ &= C(x) + \alpha(x)A(x) \left| \frac{\nabla v}{v} \right|^{\alpha(x)+1} + B(x) \cdot \left(\frac{|\nabla v|^{\alpha(x)-1}\nabla v}{|v|^{\alpha(x)-1}v} \right) \\ & \quad - \frac{vQ[v]}{|v|^{\alpha(x)+1}} \quad \text{in } A[r_2, \infty). \end{aligned}$$

Based on Lemma 2 we obtain the following.

Lemma 3 If $v \in \mathcal{D}_Q(\Omega)$, $vQ[v] \leq 0$ in Ω and v has no zero in $A[r_2, \infty)$ for some $r_2 > r_1$, then we derive the Riccati inequality:

$$\nabla \cdot (\psi(x)W(x)) + d(x) + \frac{\alpha(x)}{\alpha(x)+1}e(x)|W(x)|^{1+(1/\alpha(x))} \leq 0$$

in $A[r_2, \infty)$ for any $\psi(x) \in C^1(A[r_2, \infty); (0, \infty))$, where

$$\begin{aligned} e(x) &= \frac{\alpha(x)+1}{2}\psi(x)A(x)^{-1/\alpha(x)}, \\ d(x) &= \psi(x)C(x) - \frac{1}{\alpha(x)+1}e(x)^{-\alpha(x)}\psi(x)^{\alpha(x)+1} \left| \frac{B(x)}{A(x)} - \frac{\nabla\psi(x)}{\psi(x)} \right|^{\alpha(x)+1}. \end{aligned}$$

Lemma 4 Assume that the following hypothesis holds:

(H) $\alpha(x) \equiv \alpha(|x|)$ in $A[r_0, \infty)$.

If $v \in \mathcal{D}_Q(\Omega)$, $vQ[v] \leq 0$ in Ω and v has no zero in $A[r_2, \infty)$ for some $r_2 > r_1$, then we have the Riccati inequality:

$$Y'(r) + \int_{S_r} d(x) dS + \frac{\alpha(r)}{\alpha(r)+1}\Psi(r)^{-1/\alpha(r)}|Y(r)|^{1+(1/\alpha(r))} \leq 0 \quad (7)$$

for $r \geq r_2$, where

$$\begin{aligned} S_r &= \{x \in \mathbb{R}^n; |x| = r\}, \\ \Psi(r) &= \int_{S_r} e(x)^{-\alpha(r)}\psi(x)^{\alpha(r)+1} dS, \\ Y(r) &= \int_{S_r} \psi(x)\langle W(x), \nu(x) \rangle dS, \end{aligned}$$

$\nu(x)$ being the unit exterior normal vector x/r on S_r .

Theorem 5 *Assume that the hypothesis (H) of Lemma 4 holds. If there exists a function $\psi(x) \in C^1(A[r_1, \infty); (0, \infty))$ such that the Riccati inequality (7) has no solution on $[r, \infty)$ for all large r , then every solution $v \in \mathcal{D}_Q(\Omega)$ of $vQ[v] \leq 0$ is oscillatory in Ω .*

We can obtain oscillation results for $vQ[v] \leq 0$ by analyzing one-dimensional Riccati inequalities with variable exponent of the form

$$y'(r) + \frac{1}{\beta(r)} \frac{1}{p(r)} |y(r)|^{\beta(r)} \leq -q(r),$$

where $\beta(r) > 1$, $p(r) \in C([r_1, \infty); (0, \infty))$ and $q(r) \in C([r_1, \infty); \mathbb{R})$.

For example, we obtain the following.

Corollary 3 *Assume that the hypothesis (H) of Lemma 4 holds. Let $\mu > 1$ and ν be a real number. If there exists a function $\psi(x) \in C^1(A[r_1, \infty); (0, \infty))$ such that*

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\mu} \int_{r_1}^r \left[\omega_n s^{\nu+n-1} (r-s)^\mu \bar{d}(s) - \frac{1}{\alpha(s)+1} s^{\nu-\alpha(s)+1} |\nu r - (\mu + \nu)s|^{\alpha(s)+1} (r-s)^{\mu-\alpha(s)-1} \Psi(s) \right] ds = \infty,$$

then every solution $v \in \mathcal{D}_Q(\Omega)$ of $vQ[v] \leq 0$ is oscillatory in Ω , where ω_n denotes the surface area of the unit sphere S_1 and $\bar{d}(r)$ denotes the spherical mean of $d(x)$ over the sphere S_r .

6 Forced oscillations

We study oscillation criteria for $v(\tilde{Q}[v] - f(x)) \leq 0$ with a forcing term $f(x)$. Under some hypotheses we can establish Riccati inequality which is similar to that obtained in Lemma 3. Utilizing the Riccati method as were used for $vQ[v] \leq 0$, we can obtain oscillation results for $v(\tilde{Q}[v] - f(x)) \leq 0$ (see [9]).

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