

# A Relationship Between Dual-Intuitionistic Logic and Nelson's Constructive Logic

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October 30, 2002

## Abstract

*Dual-intuitionistic logics* are logics proposed by Czermak 1977, Goodman 1981 and Urbas 1996. We show a correspondence between Goodman's dual-intuitionistic logic and Nelson's constructive logic  $N^-$ . Moreover we introduce two types of cut-free sequent calculi for  $N^-$ .

## 1 Introduction

*Dual-intuitionistic logics* are logics proposed by Czermak 1977 [4], Goodman 1981 [6] and Urbas 1996 [12]. Czermak [4] introduces a Gentzen-type sequent calculus in which sequents have the restriction that whose antecedent contains at most one formula. This system is called by Czermak the "dual-intuitionistic calculus DJ". Goodman [6] introduces a logic which is called the "logic of contradiction" or the "anti-intuitionistic logic", and shows the completeness theorem (with respect to complete Brouwerian algebras) for this logic. Goodman's logic has a binary connective  $\div$  named the "pseudo-difference". We call here the logic, GJ and also call a  $\div$ -less its sublogic,  $GJ^-$ . Urbas [12] extends these logics of Czermak and Goodman, and moreover extends the cut-elimination result of Czermak, and discusses decidability, paraconsistency and so on. These logics containing Czermak's, Goodman's and Urbas's logics are called by Urbas the "dual-intuitionistic logics". Also an interesting related work for dual-intuitionistic logics is in Goré 2000 [5].

*Logics with strong negation* are first introduced by Nelson 1949 [9] and independently by Markov 1951. A constructive logic  $N^-$ , which is a variant of logics with strong negation, is posed by Almkudad and Nelson 1984 [1]. The paper [1] shows a relationship among constructive logics  $N^-$ ,  $N^+$  and Cleave's three-valued logic [3]. These logics such as  $N^-$ ,  $N$  and  $N^+$  have been studied by many logicians and computer scientists (e.g., [8, 10, 11, 13, 14]). For example, Wansing [14] provides some cut-free display calculi for  $N^-$  and  $N$ , which calculi have four-placed display sequents, and can derive the subformula property. In addition, Kamide [7] develops various kinds of cut-free sequent calculi for a intuitionistic linear logic with strong negation.

Logics with strong negation and dual-intuitionistic logics have been studied in the completely different fields and motivations. In this paper, we clarify a relationship among these different logics. Moreover, applying the resulting technique in [7] we propose new sequent calculi for  $N^-$ , and prove the cut-elimination theorems for these calculi.

This paper is organized as follows. In section 2, we present the dual-intuitionistic logics: Goodman's GJ and  $GJ^-$ , Czermak's DJ and Urbas's LDJ and  $LDJ^+$ . In section 3, we introduce Nelson's constructive logic  $N^-$  and give a correspondence between  $GJ^-$  and  $N^-$ . Using this correspondence result, the cut-elimination theorem for  $GJ^-$  is proved. In section 4, we introduce a *dual calculus*

DC for  $N^-$ , and prove the cut-elimination theorem for DC. This calculus has two kinds of sequents: a *positive sequent*  $\Gamma \Rightarrow^+ \gamma$  and a *negative sequent*  $\Gamma \Rightarrow^- \gamma$ . This idea of using two sorts of sequents is from Kripke-type semantics for logics with strong negation [11], which semantics have two types of valuations:  $\models^+$  (corresponds to provability) and  $\models^-$  (corresponds to refutability). We then show that  $GJ^-$  corresponds to the negative part (using only negative sequents) of a fragment of DC. In section 5, we introduce a *subformula calculus* SC for  $N^-$ , and prove the cut-elimination theorem and the subformula property for SC <sup>1</sup>.

We assume here that the language of the negation-less fragment of the first-order predicate LJ consists of logical connectives  $\wedge, \vee$  and  $\rightarrow$  and quantifiers  $\forall$  and  $\exists$ . Moreover we sometimes add the following connectives to the language:  $\top$  ("sentential constant" for GJ),  $\div$  ("pseudo-difference" for some dual-intuitionistic logics) and  $\neg$  ("negation" for some dual-intuitionistic logics) and  $\sim$  ("strong negation" for  $N^-$ ). Lower case Greek letters  $\alpha, \beta, \dots$  are used for formulas, and Greek capital letters  $\Gamma, \Delta, \dots$  are used for finite (possibly empty) sequences of formulas.  $\sim \Gamma$  denotes the sequence  $\langle \sim \gamma \mid \gamma \in \Gamma \rangle$ . A *sequent* is an expression of the form  $\Gamma \Rightarrow \gamma$  for  $N^-$  or  $\gamma \Rightarrow \Gamma$  for dual-intuitionistic logics. Since all logics discussed in this paper are formulated as sequent calculi, we will sometimes identify a sequent calculus with the logic determined by it.

## 2 Dual-intuitionistic logics

First, we give a precise definition of Goodman's logic GJ [6]. In the following definitions,  $\gamma$  in expression  $\gamma \Rightarrow \Gamma$  for any  $\Gamma$  means single formula.

Initial sequents of GJ are of the forms:

$$\alpha \Rightarrow \alpha, \quad \gamma \Rightarrow \top.$$

The cut rule is of the form:

$$\frac{\gamma \Rightarrow \Delta, \alpha \quad \alpha \Rightarrow \Sigma}{\gamma \Rightarrow \Delta, \Sigma} \text{ (cut-d)}.$$

The inference rules of GJ <sup>2</sup> are as follows:

$$\begin{array}{c} \frac{\gamma \Rightarrow \Delta, \alpha, \alpha}{\gamma \Rightarrow \Delta, \alpha} \text{ (co-d)}, \quad \frac{\gamma \Rightarrow \Delta}{\gamma \Rightarrow \Delta, \alpha} \text{ (we-d)}, \quad \frac{\gamma \Rightarrow \Delta, \beta, \alpha, \Gamma}{\gamma \Rightarrow \Delta, \alpha, \beta, \Gamma} \text{ (ex-d)}, \\ \\ \frac{\alpha \Rightarrow \Delta}{\alpha \wedge \beta \Rightarrow \Delta} \text{ (\wedge left1-d)}, \quad \frac{\beta \Rightarrow \Delta}{\alpha \wedge \beta \Rightarrow \Delta} \text{ (\wedge left2-d)}, \\ \frac{\gamma \Rightarrow \Delta, \alpha \quad \gamma \Rightarrow \Delta, \beta}{\gamma \Rightarrow \Delta, \alpha \wedge \beta} \text{ (\wedge right-d)}, \quad \frac{\alpha \Rightarrow \Delta \quad \beta \Rightarrow \Delta}{\alpha \vee \beta \Rightarrow \Delta} \text{ (\vee left-d)}, \\ \frac{\gamma \Rightarrow \Delta, \alpha}{\gamma \Rightarrow \Delta, \alpha \vee \beta} \text{ (\vee right1-d)}, \quad \frac{\gamma \Rightarrow \Delta, \beta}{\gamma \Rightarrow \Delta, \alpha \vee \beta} \text{ (\vee right2-d)}, \\ \frac{\alpha[t/x] \Rightarrow \Gamma}{\forall x \alpha \Rightarrow \Gamma} \text{ (\forall left-d)}, \quad \frac{\gamma \Rightarrow \Gamma, \alpha[z/x]}{\gamma \Rightarrow \Gamma, \forall x \alpha} \text{ (\forall right-d)}, \\ \frac{\alpha[z/x] \Rightarrow \Gamma}{\exists x \alpha \Rightarrow \Gamma} \text{ (\exists left-d)}, \quad \frac{\gamma \Rightarrow \Gamma, \alpha[t/x]}{\gamma \Rightarrow \Gamma, \exists x \alpha} \text{ (\exists right-d)}, \\ \frac{\alpha \Rightarrow \Delta, \beta}{\alpha \div \beta \Rightarrow \Delta} \text{ (\div left)}, \quad \frac{\alpha \div \beta \Rightarrow \Delta}{\alpha \Rightarrow \Delta, \beta} \text{ (\div left}^{-1}\text{)}.$$

Here,  $\alpha[z/x]$  ( $\alpha[t/x]$ ) is the formula obtained from  $\alpha$  by replacing all free occurrences of  $x$  in  $\alpha$  by an individual variable  $z$  (a term  $t$ , respectively), but avoiding the crash of variables. Also, in the

<sup>1</sup>The idea of showing subformula property was first appeared in [14].

<sup>2</sup>Strictly speaking, the original logic of Goodman has the quantifier rules which are slightly different forms.

rules for quantifiers,  $t$  is an arbitrary term and  $z$  is an arbitrary individual variable not occurring in the lower sequent.

Here we define  $GJ^- = GJ - (\div\text{left}) - (\div\text{left}^{-1}) - (\gamma \Rightarrow \top)$ .

In the following, we consider an expression  $\gamma \Rightarrow \Gamma$  for any  $\gamma$  where  $\gamma$  denotes a single formula or an empty sequence.

Czermak's logic DJ [4] is defined as follows:  $DJ = GJ^- - (\exists\text{left-d}) - (\exists\text{right-d}) + (\text{we-left-d}) + (\neg\text{left-d}) + (\neg\text{right-d})$  where

$$\frac{\Rightarrow \Delta}{\alpha \Rightarrow \Delta} (\text{we-left-d}), \quad \frac{\Rightarrow \Delta, \alpha}{\neg \alpha \Rightarrow \Delta} (\neg\text{left-d}), \quad \frac{\alpha \Rightarrow \Delta}{\Rightarrow \Delta, \neg \alpha} (\neg\text{right-d}).$$

Urbas's logics LDJ and LDJ<sup>+</sup> [12] are defined as follows:  $LDJ = DJ + (\exists\text{left-d}) + (\exists\text{right-d}) + (\rightarrow\text{left-d}) + (\rightarrow\text{right1-d}) + (\rightarrow\text{right2-d})$  and  $LDJ^+ = LDJ + (\div\text{left}) + (\div\text{right})$  where

$$\begin{aligned} \frac{\gamma \Rightarrow \Delta, \alpha \quad \beta \Rightarrow \Gamma}{\gamma \Rightarrow \Delta, \Gamma, \alpha \div \beta} (\div\text{right}), \quad & \frac{\Rightarrow \Delta, \alpha \quad \beta \Rightarrow \Gamma}{\alpha \rightarrow \beta \Rightarrow \Delta, \Gamma} (\rightarrow\text{left-d}), \\ \frac{\alpha \Rightarrow \Delta}{\Rightarrow \Delta, \alpha \rightarrow \beta} (\rightarrow\text{right1-d}), \quad & \frac{\gamma \Rightarrow \Delta, \beta}{\gamma \Rightarrow \Delta, \alpha \rightarrow \beta} (\rightarrow\text{right2-d}). \end{aligned}$$

### 3 Nelson's $N^-$ and Goodman's $GJ^-$

We introduce Nelson's constructive logic  $N^-$ . In the following definitions,  $\gamma$  in expression  $\Gamma \Rightarrow \gamma$  for any  $\Gamma$  means single formula.

Initial sequents of  $N^-$  are of the form:

$$\alpha \Rightarrow \alpha.$$

The cut rule of  $N^-$  is of the form:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma} (\text{cut}).$$

The inference rules of  $\sim$ -free part of  $N^-$  are as follows:

$$\begin{aligned} \frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} (\text{co-left}), \quad & \frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} (\text{we-left}), \quad & \frac{\Delta, \beta, \alpha, \Gamma \Rightarrow \gamma}{\Delta, \alpha, \beta, \Gamma \Rightarrow \gamma} (\text{ex-left}), \\ \frac{\alpha, \Delta \Rightarrow \gamma}{\alpha \wedge \beta, \Delta \Rightarrow \gamma} (\wedge\text{left1}), \quad & \frac{\beta, \Delta \Rightarrow \gamma}{\alpha \wedge \beta, \Delta \Rightarrow \gamma} (\wedge\text{left2}), \\ \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} (\wedge\text{right}), \quad & \frac{\alpha, \Delta \Rightarrow \gamma \quad \beta, \Delta \Rightarrow \gamma}{\alpha \vee \beta, \Delta \Rightarrow \gamma} (\vee\text{left}), \\ \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} (\vee\text{right1}), \quad & \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} (\vee\text{right2}), \\ \frac{\Gamma \Rightarrow \alpha \quad \beta, \Sigma \Rightarrow \gamma}{\alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \gamma} (\rightarrow\text{left}), \quad & \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} (\rightarrow\text{right}), \\ \frac{\alpha[t/x], \Gamma \Rightarrow \gamma}{\forall x \alpha, \Gamma \Rightarrow \gamma} (\forall\text{left}), \quad & \frac{\Gamma \Rightarrow \alpha[z/x]}{\Gamma \Rightarrow \forall x \alpha} (\forall\text{right}), \\ \frac{\alpha[z/x], \Gamma \Rightarrow \gamma}{\exists x \alpha, \Gamma \Rightarrow \gamma} (\exists\text{left}), \quad & \frac{\Gamma \Rightarrow \alpha[t/x]}{\Gamma \Rightarrow \exists x \alpha} (\exists\text{right}). \end{aligned}$$

The strong negation inference rules of  $N^-$  are as follows:

$$\begin{array}{c}
\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \sim \sim \alpha} (\sim\text{right}), \quad \frac{\alpha, \Delta \Rightarrow \gamma}{\sim \sim \alpha, \Delta \Rightarrow \gamma} (\sim\text{left}), \\
\\
\frac{\sim \alpha, \Delta \Rightarrow \gamma \quad \sim \beta, \Delta \Rightarrow \gamma}{\sim (\alpha \wedge \beta), \Delta \Rightarrow \gamma} (\sim \wedge\text{left}), \\
\\
\frac{\Gamma \Rightarrow \sim \alpha}{\Gamma \Rightarrow \sim (\alpha \wedge \beta)} (\sim \wedge\text{right1}), \quad \frac{\Gamma \Rightarrow \sim \beta}{\Gamma \Rightarrow \sim (\alpha \wedge \beta)} (\sim \wedge\text{right2}), \\
\\
\frac{\sim \alpha, \Delta \Rightarrow \gamma}{\sim (\alpha \vee \beta), \Delta \Rightarrow \gamma} (\sim \vee\text{left1}), \quad \frac{\sim \beta, \Delta \Rightarrow \gamma}{\sim (\alpha \vee \beta), \Delta \Rightarrow \gamma} (\sim \vee\text{left2}), \\
\\
\frac{\Gamma \Rightarrow \sim \alpha \quad \Gamma \Rightarrow \sim \beta}{\Gamma \Rightarrow \sim (\alpha \vee \beta)} (\sim \vee\text{right}), \\
\\
\frac{\sim \beta, \alpha, \Delta \Rightarrow \gamma}{\sim (\alpha \rightarrow \beta), \Delta \Rightarrow \gamma} (\sim \rightarrow\text{left}), \quad \frac{\Gamma \Rightarrow \sim \beta \quad \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \sim (\alpha \rightarrow \beta)} (\sim \rightarrow\text{right}), \\
\\
\frac{\sim \alpha[z/x], \Gamma \Rightarrow \gamma}{\sim \forall x \alpha, \Gamma \Rightarrow \gamma} (\sim \forall\text{left}), \quad \frac{\Gamma \Rightarrow \sim \alpha[t/x]}{\Gamma \Rightarrow \sim \forall x \alpha} (\sim \forall\text{right}), \\
\\
\frac{\sim \alpha[t/x], \Gamma \Rightarrow \gamma}{\sim \exists x \alpha, \Gamma \Rightarrow \gamma} (\sim \exists\text{left}), \quad \frac{\Gamma \Rightarrow \sim \alpha[z/x]}{\Gamma \Rightarrow \sim \exists x \alpha} (\sim \exists\text{right}).
\end{array}$$

Here  $\alpha[z/x]$  and  $\alpha[t/x]$  denote the same meaning presented in section 2. We remark that this system is in [13, 10] and is different from that in [1], but both the systems are essentially the same thing. The system in [1] uses the multiple conclusion sequent, that is, the system is defined as so-called the "LJ'-style" formulation.

The following cut-elimination result is already presented in [10, 13].

**Theorem 3.1 (Cut-Elimination Theorem for  $N^-$ )** *The rule (cut) is admissible in cut-free  $N^-$ .*

The following theorem is a main result of this paper.

**Theorem 3.2 (Correspondence Between  $GJ^-$  and  $N^-$ )** *Let  $\Gamma$  be a sequence of  $\sim$ -free formulas,  $\gamma$  be a  $\sim$ -free formula. (1)  $\gamma \Rightarrow \Gamma$  is provable in  $GJ^-$  if and only if  $\sim \Gamma \Rightarrow \sim \gamma$  is provable in the  $\rightarrow$ -free fragment of  $N^-$ . (2) If  $\sim \Gamma \Rightarrow \sim \gamma$  is cut-free provable in the  $\rightarrow$ -free fragment of  $N^-$  then  $\gamma \Rightarrow \Gamma$  is cut-free provable in  $GJ^-$ .*

Using this theorem, we can obtain the following <sup>3</sup>.

**Theorem 3.3 (Cut-Elimination Theorem for  $GJ^-$ )** *The rule (cut-d) is admissible in cut-free  $GJ^-$ .*

**Proof** Suppose that  $\gamma \Rightarrow \Gamma$  is provable in  $GJ^-$ . Then  $\sim \Gamma \Rightarrow \sim \gamma$  is provable in the  $\rightarrow$ -free fragment of  $N^-$  by Theorem 3.2 (1). By Theorem 3.1 and its subformula-property-like corollary <sup>4</sup>,  $\sim \Gamma \Rightarrow \sim \gamma$  is cut-free provable in the  $\rightarrow$ -free fragment of  $N^-$ . Hence  $\gamma \Rightarrow \Gamma$  is cut-free provable in  $GJ^-$  by Theorem 3.2 (2). ■

<sup>3</sup>Goodman [6] does not discuss the cut-elimination theorems for  $GJ$  and  $GJ^-$ . Maybe the cut-elimination theorem does not hold for  $GJ$ .

<sup>4</sup>We remark that  $N^-$  has no subformula property, but similar property holds for  $N^-$ .

## 4 Dual calculus for $N^-$

In this section, we introduce a dual calculus DC for  $N^-$ . In the following, a sequent of the form  $\Gamma \Rightarrow^+ \gamma$  is called a positive sequent, and a sequent of the form  $\Gamma \Rightarrow^- \gamma$  is called a negative sequent. In the following definitions,  $\gamma$  in expression  $\Gamma \Rightarrow^+ \gamma$  or  $\Gamma \Rightarrow^- \gamma$  for any  $\Gamma$  means single formula.

The initial sequents of DC are of the forms:

$$\alpha \Rightarrow^+ \alpha, \quad \alpha \Rightarrow^- \alpha.$$

The specific inference rules of DC are as follows:

$$\begin{array}{cc} \frac{\sim \Gamma, \Delta \Rightarrow^- \gamma}{\Gamma, \sim \Delta \Rightarrow^+ \sim \gamma} (-/+1), & \frac{\sim \Gamma, \Delta \Rightarrow^- \sim \gamma}{\Gamma, \sim \Delta \Rightarrow^+ \gamma} (-/+2), \\ \frac{\sim \Gamma, \Delta \Rightarrow^+ \gamma}{\Gamma, \sim \Delta \Rightarrow^- \sim \gamma} (+/-1), & \frac{\sim \Gamma, \Delta \Rightarrow^+ \sim \gamma}{\Gamma, \sim \Delta \Rightarrow^- \gamma} (+/-2). \end{array}$$

The cut rules of DC are as follows:

$$\frac{\Gamma \Rightarrow^+ \alpha \quad \alpha, \Sigma \Rightarrow^+ \gamma}{\Gamma, \Sigma \Rightarrow^+ \gamma} (+\text{cut}), \quad \frac{\Gamma \Rightarrow^- \alpha \quad \alpha, \Sigma \Rightarrow^- \gamma}{\Gamma, \Sigma \Rightarrow^- \gamma} (-\text{cut}).$$

The positive inference rules of DC are the same as that of  $\sim$ -free  $N^-$  where we use  $\Rightarrow^+$  instead of  $\Rightarrow$ .

The negative inference rules of DC are as follows:

$$\begin{array}{ccc} \frac{\Gamma \Rightarrow^- \gamma}{\alpha, \Gamma \Rightarrow^- \gamma} (-\text{we}), & \frac{\alpha, \alpha, \Gamma \Rightarrow^- \gamma}{\alpha, \Gamma \Rightarrow^- \gamma} (-\text{co}), & \frac{\Delta, \beta, \alpha, \Gamma \Rightarrow^- \gamma}{\Delta, \alpha, \beta, \Gamma \Rightarrow^- \gamma} (-\text{ex}), \\ \frac{\alpha, \Delta \Rightarrow^- \gamma \quad \beta, \Delta \Rightarrow^- \gamma}{\alpha \wedge \beta, \Delta \Rightarrow^- \gamma} (-\wedge \text{left}), & & \\ \frac{\Gamma \Rightarrow^- \alpha}{\Gamma \Rightarrow^- \alpha \wedge \beta} (-\wedge \text{right1}), & \frac{\Gamma \Rightarrow^- \beta}{\Gamma \Rightarrow^- \alpha \wedge \beta} (-\wedge \text{right2}), & \\ \frac{\alpha, \Delta \Rightarrow^- \gamma}{\alpha \vee \beta, \Delta \Rightarrow^- \gamma} (-\vee \text{left1}), & \frac{\beta, \Delta \Rightarrow^- \gamma}{\alpha \vee \beta, \Delta \Rightarrow^- \gamma} (-\vee \text{left2}), & \\ \frac{\Gamma \Rightarrow^- \alpha \quad \Gamma \Rightarrow^- \beta}{\Gamma \Rightarrow^- \alpha \vee \beta} (-\vee \text{right}), & & \\ \frac{\beta, \sim \alpha, \Delta \Rightarrow^- \gamma}{\alpha \rightarrow \beta, \Delta \Rightarrow^- \gamma} (-\rightarrow \text{left}), & \frac{\Gamma \Rightarrow^- \beta \quad \Delta \Rightarrow^- \sim \alpha}{\Gamma, \Delta \Rightarrow^- \alpha \rightarrow \beta} (-\rightarrow \text{right}), & \\ \frac{\alpha[z/x], \Gamma \Rightarrow^- \gamma}{\forall x \alpha, \Gamma \Rightarrow^- \gamma} (-\forall \text{left}), & \frac{\Gamma \Rightarrow^- \alpha[t/x]}{\Gamma \Rightarrow^- \forall x \alpha} (-\forall \text{right}), & \\ \frac{\alpha[t/x], \Gamma \Rightarrow^- \gamma}{\exists x \alpha, \Gamma \Rightarrow^- \gamma} (-\exists \text{left}), & \frac{\Gamma \Rightarrow^- \alpha[z/x]}{\Gamma \Rightarrow^- \exists x \alpha} (-\exists \text{right}). & \end{array}$$

Here  $\alpha[z/x]$  and  $\alpha[t/x]$  denote the same meaning presented in section 2.

**Theorem 4.1 (Equivalence Between DC and  $N^-$ )** *Let  $\Gamma$  be a sequence of formulas,  $\gamma$  be a formula. (1) If  $\Gamma \Rightarrow^* \gamma$  ( $*$   $\in \{+, -\}$ ) is provable in DC, then the sequent  $\Gamma \Rightarrow \gamma$  is provable in  $N^-$  if  $*$  = +, or the sequent  $\sim \Gamma \Rightarrow \sim \gamma$  is provable in  $N^-$  if  $*$  = -. (2) If  $\Gamma \Rightarrow \gamma$  is cut-free provable in  $N^-$  then the sequent  $\Gamma \Rightarrow^+ \gamma$  is cut-free provable in DC.*

**Theorem 4.2 (Cut-Elimination Theorem for DC)** *The rules (+cut) and (-cut) are admissible in cut-free DC.*

**Proof** Suppose that a sequent  $\Gamma \Rightarrow^* \gamma$  ( $* \in \{+, -\}$ ) is provable in DC. Then, by Theorem 4.1 (1), the sequent  $\Gamma \Rightarrow \gamma$  is provable in  $N^-$  if  $* = +$ , or  $\sim \Gamma \Rightarrow \sim \gamma$  is provable in  $N^-$  if  $* = -$ . Hence the sequent  $\Gamma \Rightarrow \gamma$  or  $\sim \Gamma \Rightarrow \sim \gamma$  is cut-free provable in  $N^-$  by Theorem 3.1. If  $\Gamma \Rightarrow \gamma$  is cut-free provable in  $N^-$  then  $\Gamma \Rightarrow^+ \gamma$  is cut-free provable in DC by Theorem 4.1 (2). If  $\sim \Gamma \Rightarrow \sim \gamma$  is cut-free provable in  $N^-$  then  $\sim \Gamma \Rightarrow^+ \sim \gamma$  is cut-free provable in DC by Theorem 4.1 (2), and hence  $\Gamma \Rightarrow^- \gamma$  is cut-free provable in DC. ■

**Theorem 4.3 (Correspondence Between  $GJ^-$  and DC)** *Let  $\Gamma$  be a sequence of  $\sim$ -free formulas,  $\gamma$  be a  $\sim$ -free formula.  $\gamma \Rightarrow \Gamma$  is provable in  $GJ^-$  if and only if  $\Gamma \Rightarrow^- \gamma$  is provable in the  $\rightarrow$ -free fragment of DC.*

In the following, we assume the language of  $N$  by deleting  $\exists$ . Let GBL be the predicate version of Arieli and Avron's logic [2] which is called the "logic of logical bilattices" <sup>5</sup>, and  $N_r^-$  and  $N_r^+$  be the multiple conclusion sequent calculi for the  $\rightarrow$ -free parts of Nelson's logics  $N^-$  and  $N^+$  <sup>6</sup> presented in [1], and  $C$  be Cleave's logic [3] <sup>7</sup>. We can conclude the following fact:

$$GJ^- \subseteq^- \rightarrow\text{-free-DC} \doteq \rightarrow\text{-free-}N^- \subseteq N_r^- \subseteq \otimes\oplus\text{-free-GBL} \subseteq C \subseteq N_r^+$$

where  $\subseteq^-$  denotes the result of Theorem 4.3,  $\doteq$  denotes the results of Theorems 3.1, 4.1 and 4.2, and  $\subseteq$  denotes the inclusion between the sets of provable sequents. We remark that the fact  $N_r^- \subseteq C \subseteq N_r^+$  above is established by Almkudad and Nelson [1].

## 5 Subformula calculus for $N^-$

In this section, we introduce a subformula calculus SC for  $N^-$ . The sequent of SC is of the forms  $\Gamma : \Delta \Rightarrow \emptyset : \gamma$  and  $\Gamma : \Delta \Rightarrow \gamma : \emptyset$  where  $\gamma$  is a formula, and  $\Gamma$  and  $\Delta$  are sequences of formulas. The sequents

$$\gamma_1, \dots, \gamma_m : \delta_1, \dots, \delta_n \Rightarrow \emptyset : \gamma, \quad \gamma_1, \dots, \gamma_m : \delta_1, \dots, \delta_n \Rightarrow \gamma : \emptyset$$

( $0 \leq m, n$ ) in SC intuitively mean that

$$\sim \gamma_1, \dots, \sim \gamma_m, \delta_1, \dots, \delta_n \Rightarrow \gamma, \quad \sim \gamma_1, \dots, \sim \gamma_m, \delta_1, \dots, \delta_n \Rightarrow \sim \gamma$$

in  $N^-$ . In the following definitions,  $C$  means  $\emptyset : \gamma$  or  $\gamma : \emptyset$ .

The initial sequents of SC are of the forms:

$$\emptyset : \alpha \Rightarrow \emptyset : \alpha, \quad \alpha : \emptyset \Rightarrow \alpha : \emptyset.$$

The specific inference rules of SC are as follows:

$$\frac{\Gamma : \Delta \Rightarrow \alpha : \emptyset}{\Gamma : \Delta \Rightarrow \emptyset : \sim \alpha} (\sim r+), \quad \frac{\Gamma : \Delta \Rightarrow \emptyset : \alpha}{\Gamma : \Delta \Rightarrow \sim \alpha : \emptyset} (\sim r-),$$

$$\frac{\alpha, \Gamma : \Delta \Rightarrow C}{\Gamma : \sim \alpha, \Delta \Rightarrow C} (\sim l+), \quad \frac{\Gamma : \alpha, \Delta \Rightarrow C}{\sim \alpha, \Gamma : \Delta \Rightarrow C} (\sim l-).$$

<sup>5</sup>Strictly speaking, the original GBL is a propositional logic, but section 3.5 in [2] proposes the quantifier rules for GBL. We consider here the predicate extension. GBL has fusion and fission connectives  $\otimes$  and  $\oplus$ . Also GBL is regarded as a logic with strong negation.

<sup>6</sup> $N^+$  is first introduced by Thomason [11]. This logic is obtained from (a Hilbert-style system)  $N$  by adding the constant domain axiom scheme:  $\forall x(\alpha(x) \vee \beta) \rightarrow \forall \alpha(x) \vee \beta$  where  $x$  is not free in  $\beta$ .  $N$  is obtained from (a Hilbert-style system)  $N^-$  by adding the axiom scheme:  $\alpha \wedge \sim \alpha \rightarrow \beta$ .

<sup>7</sup>The original logic of Cleave has no structural rules, and hence we must add the appropriate structural rules. The logic with the structural rules is called  $C$  in the present paper.

The cut rules of SC are as follows:

$$\frac{\Gamma_1 : \Delta_1 \Rightarrow \alpha : \emptyset \quad \alpha, \Gamma_2 : \Delta_2 \Rightarrow C}{\Gamma_1, \Gamma_2 : \Delta_1, \Delta_2 \Rightarrow C} (\text{cut-}),$$

$$\frac{\Gamma_1 : \Delta_1 \Rightarrow \emptyset : \alpha \quad \Gamma_2 : \alpha, \Delta_2 \Rightarrow C}{\Gamma_1, \Gamma_2 : \Delta_1, \Delta_2 \Rightarrow C} (\text{cut+}).$$

The positive inference rules of SC are as follows:

$$\frac{\Gamma : \Delta \Rightarrow C}{\Gamma : \alpha, \Delta \Rightarrow C} (\text{w+}), \quad \frac{\Gamma : \alpha, \alpha, \Delta \Rightarrow C}{\Gamma : \alpha, \Delta \Rightarrow C} (\text{c+}), \quad \frac{\Gamma : \Delta_1, \beta, \alpha, \Delta_2 \Rightarrow C}{\Gamma : \Delta_1, \alpha, \beta, \Delta_2 \Rightarrow C} (\text{e+}),$$

$$\frac{\Gamma_1 : \Delta_1 \Rightarrow \emptyset : \alpha \quad \Gamma_2 : \beta, \Delta_2 \Rightarrow C}{\Gamma_1, \Gamma_2 : \alpha \rightarrow \beta, \Delta_1, \Delta_2 \Rightarrow C} (\rightarrow \text{l+}), \quad \frac{\Gamma : \alpha, \Delta \Rightarrow \emptyset : \beta}{\Gamma : \Delta \Rightarrow \emptyset : \alpha \rightarrow \beta} (\rightarrow \text{r+}),$$

$$\frac{\Gamma : \alpha, \Delta \Rightarrow C}{\Gamma : \alpha \wedge \beta, \Delta \Rightarrow C} (\wedge \text{l1+}), \quad \frac{\Gamma : \beta, \Delta \Rightarrow C}{\Gamma : \alpha \wedge \beta, \Delta \Rightarrow C} (\wedge \text{l2+}),$$

$$\frac{\Gamma : \Delta \Rightarrow \emptyset : \alpha \quad \Gamma : \Delta \Rightarrow \emptyset : \beta}{\Gamma : \Delta \Rightarrow \emptyset : \alpha \wedge \beta} (\wedge \text{r1+}), \quad \frac{\Gamma : \alpha, \Delta \Rightarrow C \quad \Gamma : \beta, \Delta \Rightarrow C}{\Gamma : \alpha \vee \beta, \Delta \Rightarrow C} (\vee \text{l1+}),$$

$$\frac{\Gamma : \Delta \Rightarrow \emptyset : \alpha}{\Gamma : \Delta \Rightarrow \emptyset : \alpha \vee \beta} (\vee \text{r1+}), \quad \frac{\Gamma : \Delta \Rightarrow \emptyset : \beta}{\Gamma : \Delta \Rightarrow \emptyset : \alpha \vee \beta} (\vee \text{r2+}),$$

$$\frac{\Gamma : \alpha[t/x], \Delta \Rightarrow C}{\Gamma : \forall x \alpha, \Delta \Rightarrow C} (\forall \text{l+}), \quad \frac{\Gamma : \Delta \Rightarrow \emptyset : \alpha[z/x]}{\Gamma : \Delta \Rightarrow \emptyset : \forall x \alpha} (\forall \text{r+}),$$

$$\frac{\Gamma : \alpha[z/x], \Delta \Rightarrow C}{\Gamma : \exists x \alpha, \Delta \Rightarrow C} (\exists \text{l+}), \quad \frac{\Gamma : \Delta \Rightarrow \emptyset : \alpha[t/x]}{\Gamma : \Delta \Rightarrow \emptyset : \exists x \alpha} (\exists \text{r+}).$$

The negative inference rules of SC are as follows:

$$\frac{\Gamma : \Delta \Rightarrow C}{\alpha, \Gamma : \Delta \Rightarrow C} (\text{w-}), \quad \frac{\alpha, \alpha, \Gamma : \Delta \Rightarrow C}{\alpha, \Gamma : \Delta \Rightarrow C} (\text{c-}), \quad \frac{\Gamma_1, \beta, \alpha, \Gamma_2 : \Delta \Rightarrow C}{\Gamma_1, \alpha, \beta, \Gamma_2 : \Delta \Rightarrow C} (\text{e-}),$$

$$\frac{\beta, \Gamma : \alpha, \Delta \Rightarrow C}{\alpha \rightarrow \beta, \Gamma : \Delta \Rightarrow C} (\rightarrow \text{l-}), \quad \frac{\Gamma_1 : \Delta_1 \Rightarrow \beta : \emptyset \quad \Gamma_2 : \Delta_2 \Rightarrow \emptyset : \alpha}{\Gamma_1, \Gamma_2 : \Delta_1, \Delta_2 \Rightarrow \alpha \rightarrow \beta : \emptyset} (\rightarrow \text{r-}),$$

$$\frac{\alpha, \Gamma : \Delta \Rightarrow C \quad \beta, \Gamma : \Delta \Rightarrow C}{\alpha \wedge \beta, \Gamma : \Delta \Rightarrow C} (\wedge \text{l-}),$$

$$\frac{\Gamma : \Delta \Rightarrow \alpha : \emptyset}{\Gamma : \Delta \Rightarrow \alpha \wedge \beta : \emptyset} (\wedge \text{r1-}), \quad \frac{\Gamma : \Delta \Rightarrow \beta : \emptyset}{\Gamma : \Delta \Rightarrow \alpha \wedge \beta : \emptyset} (\wedge \text{r2-}),$$

$$\frac{\alpha, \Gamma : \Delta \Rightarrow C}{\alpha \vee \beta, \Gamma : \Delta \Rightarrow C} (\vee \text{l1-}), \quad \frac{\beta, \Gamma : \Delta \Rightarrow C}{\alpha \vee \beta, \Gamma : \Delta \Rightarrow C} (\vee \text{l2-}),$$

$$\frac{\Gamma : \Delta \Rightarrow \alpha : \emptyset \quad \Gamma : \Delta \Rightarrow \beta : \emptyset}{\Gamma : \Delta \Rightarrow \alpha \vee \beta : \emptyset} (\vee \text{r-}),$$

$$\frac{\alpha[z/x], \Gamma : \Delta \Rightarrow C}{\forall x \alpha, \Gamma : \Delta \Rightarrow C} (\forall \text{l-}), \quad \frac{\Gamma : \Delta \Rightarrow \alpha[t/x] : \emptyset}{\Gamma : \Delta \Rightarrow \forall x \alpha : \emptyset} (\forall \text{r-}),$$

$$\frac{\alpha[t/x], \Gamma : \Delta \Rightarrow C}{\exists x \alpha, \Gamma : \Delta \Rightarrow C} (\exists \text{l-}), \quad \frac{\Gamma : \Delta \Rightarrow \alpha[z/x] : \emptyset}{\Gamma : \Delta \Rightarrow \exists x \alpha : \emptyset} (\exists \text{r-}).$$

Here  $\alpha[z/x]$  and  $\alpha[t/x]$  denote the same meaning presented in section 2.

**Theorem 5.1 (Equivalence Between SC and  $N^-$ )** Let  $\Gamma$  and  $\Delta$  be sequences of formulas,  $\gamma$  be a formula. (1) If  $\Gamma : \Delta \Rightarrow C$  ( $C$  is either  $\emptyset : \gamma$  or  $\gamma : \emptyset$ ) is provable in SC, then the sequent  $\sim \Gamma, \Delta \Rightarrow C'$  is provable in  $N^-$  where  $C' \equiv \gamma$  if  $C \equiv \emptyset : \gamma$  and  $C' \equiv \sim \gamma$  if  $C \equiv \gamma : \emptyset$ . (2) If  $\sim \Gamma, \Delta \Rightarrow C'$  ( $C'$  is either  $\gamma$  or  $\sim \gamma$ ) is cut-free provable in  $N^-$ , then the sequent  $\Gamma : \Delta \Rightarrow C$  is cut-free provable in SC where  $C \equiv \emptyset : \gamma$  if  $C' \equiv \gamma$  and  $C \equiv \gamma : \emptyset$  if  $C' \equiv \sim \gamma$ .

**Theorem 5.2 (Cut-Elimination Theorem for SC)** The rules (cut+) and (cut-) are admissible in cut-free SC.

**Proof** Suppose that a sequent  $\Gamma : \Delta \Rightarrow C$  is provable in SC. Then the sequent  $\sim \Gamma, \Delta \Rightarrow C'$  is provable in  $N^-$  by Theorem 5.1 (1), and hence the sequent  $\sim \Gamma, \Delta \Rightarrow C'$  is cut-free provable in  $N^-$  by Theorem 3.1. Therefore  $\Gamma : \Delta \Rightarrow C$  is cut-free provable in SC by Theorem 5.1 (2). ■

**Corollary 5.3 (Subformula Property for SC)** The calculus SC has the subformula property, that is, if a sequent  $S$  is provable in SC, then there is a proof of  $S$  such that any formula appearing in it is a subformula of some formula in  $S$ .

**Theorem 5.4 (Correspondence Between  $GJ^-$  and SC)** Let  $\Gamma$  be a sequence of  $\sim$ -free formulas,  $\gamma$  be a  $\sim$ -free formula.  $\gamma \Rightarrow \Gamma$  is provable in  $GJ^-$  if and only if  $\Gamma : \emptyset \Rightarrow \gamma : \emptyset$  is provable in the  $\rightarrow$ -free fragment of SC.

## 6 Notes

Russell's paradox in some naive set theories based on dual-intuitionistic logics was discussed in [6, 12]. In addition, the paper [12] remarks that Curry's paradox cannot be reproduced in these set theories. We mention, in the following, Russell's paradox over  $N^-$ .

Before the discussion for the case of  $N^-$ , we consider a derivation in LJ. Let  $\alpha \equiv t \in t$  where  $t \equiv \{x | \neg(x \in x)\}$  and assume that the sequents  $\neg\alpha \Rightarrow \alpha$  and  $\alpha \Rightarrow \neg\alpha$  are provable. Then we have the following derivation:

$$\frac{\frac{\frac{\neg\alpha \Rightarrow \alpha}{\neg\alpha, \neg\alpha \Rightarrow} (\neg\text{-left})}{\neg\alpha \Rightarrow} (\text{co-left})}{\frac{\alpha \Rightarrow \neg\alpha}{\Rightarrow \neg\alpha} (\neg\text{-right})} (\text{cut}) \quad \frac{\frac{\neg\alpha \Rightarrow \alpha}{\neg\alpha, \neg\alpha \Rightarrow} (\neg\text{-left})}{\neg\alpha \Rightarrow} (\text{co-left}) (\text{cut}).$$

This means that Russell's paradox derives contradiction. It is well-known that, in this derivation, the applications of the contraction rule (co-left) are the causes of the contradiction. However, in this derivation, we feel that the applications of the rules ( $\neg$ -left) and ( $\neg$ -right) are the causes. Then, in order to avoid the contradiction, we use  $N^-$  as a basis of naive set theory. In other words, we adopt the strong negation  $\sim$  instead of the usual (intuitionistic or classical) negation  $\neg$ . Here we remark that the language of  $N^-$  has no usual negation  $\neg$  and falsum constant  $\perp$ . Hence we cannot define  $\neg\alpha := \alpha \rightarrow \perp$ .

We then have the following conjecture: the naive set theory (with unrestricted comprehension) based on  $N^-$  is consistent (i.e., Russell's paradox does not derive the fact that the empty sequent  $\Rightarrow$  is provable).

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