

# 非局所項をもつある半線形楕円型固有値問題について

(Eigenvalue problem of semilinear elliptic equation with non-local term)

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## Abstract

In this paper we consider the Gel'fand problem with non-local term  $\Delta v + \lambda e^v / \int_{\Omega} e^v dx = 0$  on  $n$ -dimensional bounded domain  $\Omega$  with Dirichlet boundary condition. If it is star-shaped, then we have an upper bound of  $\lambda$  for the existence of the solution. We also have infinitely many bendings in  $\lambda$  of the connected component of the solution set in  $\lambda - v$  if  $\Omega$  is a ball and  $3 \leq n \leq 9$ .

## 1 Introduction

We consider the following Gel'fand problem with non-local term:

$$\begin{cases} -\Delta v = \lambda \frac{e^v}{\int_{\Omega} e^v dx} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $\lambda$  is a positive constant and  $\Omega$  is a bounded domain in  $\mathbf{R}^n$  with smooth boundary  $\partial\Omega$ . We define the solution set  $\mathcal{C}$  and the section of  $\mathcal{C}$  cut by  $\lambda > 0$  by

$$\mathcal{C} = \{(\lambda, v) \mid v = v(x) \text{ is a classical solution to (1) for } \lambda > 0\}.$$

and

$$\mathcal{C}^{\lambda} = \{v \in C^2(\Omega) \cap C(\overline{\Omega}) \mid v = v(x) \text{ solves (1)}\},$$

respectively. The first theorem is concerned with the star-shaped domain, so that  $x \cdot \nu > 0$  holds for each  $x \in \partial\Omega$ . The second one is concerned with the unit ball.

**Theorem 1** *If  $\Omega$  is star-shaped with respect to the origin, then there is an upper bound of  $\lambda$  for the existence of the solution to (1). Thus we have  $\bar{\lambda} \in (0, +\infty)$  such that  $C^\lambda \neq \emptyset$  and  $C^\lambda = \emptyset$  for  $0 < \lambda < \bar{\lambda}$  and  $\lambda > \bar{\lambda}$ , respectively. Moreover  $C_0$  is unbounded in  $\lambda - v$  plane, and  $\#C^\lambda = 1$  for  $0 < \lambda \ll 1$ , where  $C_0$  stands for the connected component of  $C$  satisfying  $(0, 0) \in \overline{C_0}$ .*

**Theorem 2** *If  $\Omega$  is the unit ball  $B = \{x \in \mathbf{R}^n \mid |x| < 1\}$ , then  $C$  is a one-dimensional open manifold parametrized as*

$$C = \{(\lambda(s), v(\cdot, s)) \mid 0 < s < +\infty\}$$

*with the endpoints  $(0, 0)$  and the weak solution  $(2\omega_n, 2 \log \frac{1}{|x|})$ , so that*

$$\lim_{s \downarrow 0} (\lambda(s), v(\cdot, s)) = (0, 0)$$

*and*

$$\lim_{s \uparrow +\infty} (\lambda(s), v(\cdot, s)) = \left(2\omega_n, 2 \log \frac{1}{|x|}\right)$$

*in  $\mathbf{R} \times C(\overline{B})$  and  $\mathbf{R} \times W^{2,p}(B)$  for  $p \in [1, n/2)$ , respectively, where  $\omega_n$  denotes the  $(n-1)$  dimensional volume of the unit ball in  $\mathbf{R}^n$ . If  $3 \leq n \leq 9$ , then  $C$  bends infinitely many times in  $\lambda$ . Thus there is a sequence  $\{s_k\}$  labeled by  $k = 1, 2, \dots$  with  $0 < s_1 < s_2 < \dots < s_k < \dots$  such that  $s \in [s_{2k-1}, s_{2k}] \mapsto \lambda(s)$  and  $s \in [s_{2k}, s_{2k+1}] \mapsto \lambda(s)$  decreasing and increasing, respectively. Furthermore, it holds that*

$$\begin{aligned} \lambda(s_2) &< \lambda(s_4) < \dots < \lambda(s_{2k}) < \lambda(s_{2k+2}) < \dots < 2\omega_n \\ &< \dots < \lambda(s_{2k+1}) < \lambda(s_{2k-1}) < \dots < \lambda(s_3) < \lambda(s_1) \end{aligned}$$

*and there are infinitely many solutions to (1) for  $\lambda = 2\omega_n$  in particular. If  $n \geq 10$ , on the other hand, then no bending occurs to  $C$  and hence  $s \in [0, \infty) \mapsto \lambda(s)$  is increasing and each  $\lambda \in (0, 2\omega_n)$  takes a unique solution to (1).*

Next we study the spectral and related properties of the following linearized problem of (1):

$$\begin{cases} \Delta\phi + \lambda \frac{e^v}{\int_{\Omega} e^v dx} \phi - \lambda \frac{\int_{\Omega} e^v \phi dx}{(\int_{\Omega} e^v dx)^2} e^v = -\mu\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Let us denote by  $i = i(\lambda, v)$  and  $i_R = i_R(\lambda, v)$  the number of negative eigenvalues of (3) and that for radially symmetric eigenfunctions to (3), respectively. We call these numbers Morse index and radial Morse index at  $(\lambda, v) \in \mathcal{C}$ , respectively.

**Theorem 3** *Under the circumstances described in the previous theorem, if  $3 \leq n \leq 9$  then it holds that  $i = i_R = k$  on the arc  $T_k T_{k+1}$  of  $\mathcal{C}$  for  $k = 0, 1, \dots$ , where  $T_k = (\lambda(s_k), v(s_k))$  with  $s_0 = 0$ . If  $n \geq 10$ , on the other hand, it always holds that  $i = i_R = 0$ .*

In §2, we treat the star-shaped domain and prove Theorem 1. We omit the proof of Theorems 2 and 3. See [8] and [9] for detail.

## 2 Star-shaped domain

Throughout the present section,  $\Omega$  denotes the general star-shaped domain with respect to the origin in  $\mathbf{R}^n$  for  $n \geq 3$  provided with the smooth boundary  $\partial\Omega$ , and  $\nu$  stands for the outer unit normal vector.

*Proof of Theorem 1:* It follows from McGough [7] that the star-shaped  $\Omega$  takes  $\tilde{\sigma} > 0$  such that the solution of

$$\begin{cases} -\Delta v = \sigma e^v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

with a constant  $\sigma > 0$  is unique for  $0 < \sigma < \tilde{\sigma}$ . However, any solution  $v = v(x)$  to (1) solves (3) with

$$\sigma = \frac{\lambda}{\int_{\Omega} e^v dx} \leq \frac{\lambda}{|\Omega|}$$

because of its positivity, where  $|\Omega|$  denotes the volume of  $\Omega$ . Therefore, the solution to (1) is unique for  $0 < \lambda < \tilde{\lambda} = \tilde{\sigma} |\Omega|$ . Hence we can prove the uniqueness result.

To have an upper bound  $\lambda$  we apply the Pohozaev identity [10].

Unboundedness of the component  $\mathcal{C}_0$  follows from the standard degree argument similarly to [12] and [13].  $\square$

The first eigenvalue of (2), denoted by  $\mu_1(\lambda, v)$ , is positive around the trivial solution  $(\lambda, v) = (0, 0)$  similarly to (3). Therefore, it generates a branch in  $\mathcal{C}$ . This branch continues as far as  $\mu_1(\lambda, v) > 0$  and because we

have an upper bound for  $\mathcal{C}_\lambda \neq \emptyset$  if  $\Omega$  is star-shaped, only two possibilities arise then. That is, there is a one-dimensional manifold contained in  $\mathcal{C}$  starting from  $(\lambda, v) = (0, 0)$  denoted by

$$\underline{\mathcal{C}} = \{(\lambda(s), v(\cdot, s)) \mid 0 < s < s_0\},$$

and we have either that  $\lim_{s \rightarrow s_0} (\lambda(s), v(\cdot, s)) = (\lambda^*, v^*) \in \mathcal{C}$  exists in  $\mathbf{R} \times C(\overline{\Omega})$  with

$$\mu_1(\lambda^*, v^*) = 0,$$

or else that  $\limsup_{s \rightarrow s_0} \|v(\cdot, s)\|_\infty = +\infty$ . For simplicity, we say that  $\underline{\mathcal{C}}$  is closed and open in the former and the latter cases, respectively. Those notions are kept, if there is an upper bound of  $\lambda$  for the existence of the solution to (1), and then the alternatives between openness and closedness of  $\underline{\mathcal{C}}$  given above, arise. In any case, the connected component  $\mathcal{C}_0$  mentioned in Theorem 1 contains this  $\underline{\mathcal{C}}$ . We now describe its spectral properties.

**Proposition 1** *If  $(\lambda^*, v^*) \in \mathcal{C}$  satisfies  $\mu_2(\lambda^*, v^*) > \mu_1(\lambda^*, v^*) = 0$ , with  $\mu_1(\lambda^*, v^*) = 0$  admitting the eigenfunction  $\phi^* > 0$ , then  $\mathcal{C}$  is locally one-dimensional manifold parametrized as*

$$\mathcal{C}^* = \{(\lambda(s), v(s)) \mid |s| < \delta\}$$

*with  $(\lambda(0), v(0)) = (\lambda^*, v^*)$ . Here  $\mu_2(\lambda^*, v^*)$  denotes the second eigenvalue of (2) at  $(\lambda, v) = (\lambda^*, v^*)$ . Furthermore,  $\mathcal{C}^*$  bends to the left with respect to  $\lambda$  at  $(\lambda^*, v^*)$ , so that  $\lambda(s) < \lambda^*$  holds for  $0 < |s| < \delta$  and the mappings  $s \in (-\delta, 0] \mapsto \lambda(s)$  and  $s \in [0, \delta) \mapsto \lambda(s)$  are increasing and decreasing, respectively. Finally,  $\mu_1(\lambda(s), v(s))$  changes sign at  $s = 0$ , say,  $\pm\mu_1(\lambda(s), v(s)) > 0$  according as  $-\delta < \pm s < 0$ .*

*Proof:* Given  $(\lambda^*, v^*) \in \mathcal{C}$  with  $\mu_1(\lambda^*, v^*) = 0$ , let the linearized operator, the left-hand side of (2) with  $(\lambda, v) = (\lambda^*, v^*)$  be  $A^*$ . Then, from the assumption we have  $\text{Ker}(A^*) = \langle \phi^* \rangle$  with  $\phi^* = \phi^*(x) \in H_0^1(\Omega) \setminus \{0\}$  positive in  $\Omega$ . Now, we take the nonlinear operator

$$\Phi(s, \sigma, w) = \Delta(v^* + s\phi^* + w) + (\lambda^* + \sigma) \frac{e^{v^* + s\phi^* + w}}{\int_\Omega e^{v^* + s\phi^* + w} dx},$$

defined for  $s \in \mathbf{R}$ ,  $\sigma \in \mathbf{R}$ , and  $w \in Y$ , where

$$Y = \left\{ w \in C^2(\overline{\Omega}) \mid w|_{\partial\Omega} = 0, \int_\Omega w\phi^* dx = 0 \right\}.$$

It is obvious that  $\Phi(0, 0, 0) = 0$  and the linearized operator

$$\Phi_{\sigma,w}(0, 0, 0) = \begin{pmatrix} e^{v^*} / \int_{\Omega} e^{v^*} dx \\ -A^* \end{pmatrix} : \begin{matrix} \mathbf{R} \\ \times \\ Y \end{matrix} \rightarrow C(\bar{\Omega})$$

is an isomorphism by  $\phi^* > 0$ . Because classical solution to (1) near  $(\lambda^*, v^*)$  is identified with zero of  $\Phi$ , the implicit function theorem then guarantees a  $C^2$ -family  $\{(\lambda(s), v(s)) \mid |s| < s_0\}$  of classical solutions satisfying  $(\lambda(0), v(0)) = (\lambda^*, v^*)$ , where  $s_0 > 0$ . It also follows from the standard perturbation theory ([4]) that the linearized operator around this  $(\lambda(s), v(s))$  takes the simple eigenvalue  $\mu(s)$  and the eigenfunction  $\phi(s)$  with  $C^2$  dependence in  $s$  such that  $(\mu(0), \phi(0)) = (0, \phi^*)$  so that (2) is valid to

$$(\lambda, v, \mu, \phi) = (\lambda(s), v(s), \mu(s), \phi(s))$$

for  $|s| < s_0$ .

Differentiating with respect to  $s$ , we have from (1) that

$$\begin{cases} \Delta \dot{v} + \dot{\lambda} \frac{e^v}{\int_{\Omega} e^v dx} + \lambda \frac{e^v}{\int_{\Omega} e^v dx} \dot{v} - \lambda \frac{\int_{\Omega} e^v \dot{v} dx}{(\int_{\Omega} e^v dx)^2} e^v = 0 & \text{in } \Omega \\ \dot{v} = 0 & \text{on } \partial\Omega. \end{cases} \quad (4)$$

Then, subtracting (2) from (4) with  $s = 0$  multiplied by  $\dot{v}$  and  $\phi^*$ , respectively, we get that

$$\dot{\lambda}(0) \frac{\int_{\Omega} e^{v^*} \phi^* dx}{\int_{\Omega} e^{v^*} dx} = 0,$$

and hence  $\dot{\lambda}(0) = 0$  holds true. This implies  $\dot{v}(0) \in \text{Ker } A^*$  by (4), and we can assume that  $\dot{v}(0) = \phi^*$  without loss of generality, because  $(\dot{\lambda}(0), \dot{v}(0))$  does not vanish from the implicit function theorem.

Differentiating (4) once more and putting  $s = 0$ , we have

$$\begin{aligned} \Delta \ddot{v} + \ddot{\lambda} \frac{e^v}{\int_{\Omega} e^v dx} - \lambda \frac{\int_{\Omega} e^v \phi^* dx}{(\int_{\Omega} e^v dx)^2} e^v + \lambda \frac{e^v \phi^{*2}}{\int_{\Omega} e^v dx} \\ + \lambda \frac{e^v \ddot{v}}{\int_{\Omega} e^v dx} - \lambda \frac{\int_{\Omega} e^v \phi^* dx}{(\int_{\Omega} e^v dx)^2} e^v \phi^* = 0 \quad \text{in } \Omega \end{aligned} \quad (5)$$

with  $\ddot{v} = 0$  on  $\partial\Omega$ . Then, subtracting (5) from (2) multiplied by  $\phi^*$  and  $\ddot{v}$ , respectively, we obtain that

$$\begin{aligned} \ddot{\lambda}(0) \frac{\int_{\Omega} e^{v^*} \phi^* dx}{\int_{\Omega} e^{v^*} dx} = \\ \lambda^* \left\{ 3 \frac{\int_{\Omega} e^{v^*} \phi^* dx \int_{\Omega} e^{v^*} \phi^{*2} dx}{(\int_{\Omega} e^{v^*} dx)^2} - 2 \frac{\left( \int_{\Omega} e^{v^*} \phi^* dx \right)^3}{(\int_{\Omega} e^{v^*} dx)^3} - \frac{\int_{\Omega} e^{v^*} \phi^{*3} dx}{\int_{\Omega} e^{v^*} dx} \right\}. \end{aligned}$$

Letting  $\frac{e^v \phi^* dx}{\int_{\Omega} e^v \phi^* dx} = d\mu$ , we have

$$\begin{aligned} \frac{\lambda(0)}{\lambda^*} \int_{\Omega} \phi^* d\mu &= 3 \int_{\Omega} \phi^* d\mu \int_{\Omega} \phi^{*2} d\mu - 2 \left( \int_{\Omega} \phi^* d\mu \right)^3 - \int_{\Omega} \phi^{*3} d\mu \\ &= 3 \int_{\Omega} \phi^* d\mu \cdot \left\{ \int_{\Omega} \phi^{*2} d\mu - \left( \int_{\Omega} \phi^* d\mu \right)^2 \right\} + \left( \int_{\Omega} \phi^* d\mu \right)^3 - \int_{\Omega} \phi^{*3} d\mu \leq 0 \end{aligned}$$

with the equality only when  $\phi^*$  is a constant. This is impossible, and we get that  $\bar{\lambda}(0) < 0$ .

To complete the proof, we differentiate (2) and obtain

$$\begin{aligned} \Delta \dot{\phi} + \lambda \frac{e^v \phi^{*2}}{\int_{\Omega} e^v dx} - \lambda \frac{\int_{\Omega} e^v \phi^* dx}{\left( \int_{\Omega} e^v dx \right)^2} e^v \phi^* + \lambda \frac{e^v \dot{\phi}}{\int_{\Omega} e^v dx} - \lambda \frac{\int_{\Omega} e^v \phi^{*2} dx}{\left( \int_{\Omega} e^v dx \right)^2} e^v \\ - \lambda \frac{\int_{\Omega} e^v \dot{\phi} dx}{\left( \int_{\Omega} e^v dx \right)^2} e^v + 2\lambda \frac{\left( \int_{\Omega} e^v \phi^* dx \right)^2}{\left( \int_{\Omega} e^v dx \right)^2} e^v - \lambda \frac{\int_{\Omega} e^v \phi^* dx}{\left( \int_{\Omega} e^v dx \right)^2} e^v \phi^* = -\dot{\mu} \phi^* \quad \text{in } \Omega \end{aligned} \quad (6)$$

with  $\dot{\phi} = 0$  on  $\partial\Omega$  by putting  $s = 0$ . Integrating (6) multiplied by  $\phi^*$  we have

$$-\dot{\mu}(0) \frac{\|\phi^*\|_2^2}{\lambda^*} = \int_{\Omega} \phi^{*3} d\mu - 3 \int_{\Omega} \phi^* d\mu \cdot \int_{\Omega} \phi^{*2} d\mu + 2 \left( \int_{\Omega} \phi^* d\mu \right)^3,$$

similarly. The proof is complete.  $\square$

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