

CONFIGURATION SPACES OF DIVISORS AND HOLOMORPHIC MAPS

MARTIN GUEST
ANDRZEJ KOZŁOWSKI
KOHHEI YAMAGUCHI

The University of Rochester
Toyama International University
The University of Electro-Communications

§1. Introduction.

Spaces of holomorphic maps between complex manifolds have played a major role in such diverse branches of mathematics as analysis, differential geometry, topology and mathematical physics.

If $X \subset \mathbb{C}P^n$ is a projective variety we denote by $\text{Hol}_d^*(S^2, X)$ ($\text{Hol}_d(S^2, X)$) the space of all based (non-based) holomorphic maps $S^2 \rightarrow X$ of degree d , where S^2 is the Riemann sphere, $S^2 = \mathbb{C} \cup \{\infty\}$. For simplicity we shall assume that the degree d is a non-negative integer. The corresponding space of based (non-based) continuous maps of degree d is denoted by $\text{Map}_d^*(S^2, X)$ ($\text{Map}_d(S^2, X)$).

In [S] Segal studied the inclusion map

$$I_d : \text{Hol}_d^*(S^2, \mathbb{C}P^n) \rightarrow \text{Map}_d^*(S^2, \mathbb{C}P^n)$$

and showed that this inclusion map is a homotopy equivalence up to dimension $d(2n-1)$.

For any projective variety X it is natural to consider the following

Problem. When is the map $I_d : \text{Hol}_d^*(S^2, X) \rightarrow \text{Map}_d^*(S^2, X)$ a homotopy equivalence up to some dimension $n(d)$, such that $n(d) \rightarrow \infty$ as $d \rightarrow \infty$?

Segal's results for $X = \mathbb{C}P^n$ have been generalized to the case when X is a Grassmanian, or more generally, a flag manifold (see [G], [Gu], [K], [M²]). In this note we consider the case where X is the complement of a union of linear subspaces in $\mathbb{C}P^n$: $X = \mathbb{C}P^n \setminus \bigcup_{\alpha \in \Lambda} H_\alpha$, where $\{H_\alpha : \alpha \in \Lambda\}$ is a family of linear subspaces of $\mathbb{C}P^n$. Our purpose is to announce the main results of [GKY], which extend Segal's results [S] to the case $X = X_n = \mathbb{C}P^n \setminus \bigcup_{0 \leq i < j \leq n} H_{i,j}$, where $H_{i,j} = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{C}P^n : z_i = z_j = 0\}$.

The precise statements of our results are as follows:

Theorem 1. *The inclusion maps*

$$I_d : \text{Hol}_d^*(S^2, X_n) \rightarrow \text{Map}_d^*(S^2, X_n)$$

and

$$J_d : \text{Hol}_d(S^2, X_n) \rightarrow \text{Map}_d(S^2, X_n)$$

are homology equivalences up to dimension d .

Theorem 2. *If $2d > n$ the two maps above are homotopy equivalences up to dimension d .*

Here we call an inclusion map $X \rightarrow Y$ a *homotopy equivalence* (*homology equivalence*) up to dimension m if $\pi_j(Y, X) = 0$ when $j \leq m$ (if $H_j(Y, X) = 0$ when $j \leq m$).

Remark.

- (1) For $n = 1$ the above results were obtained in [S].
- (2) We expect that similar methods can be used to obtain analogous results when $X = \mathbb{CP}^n \setminus \bigcup_I P(I)$, where $P(I) = \{[z_0 : \dots : z_n] \in \mathbb{CP}^n : p_j = 0 \text{ if } j \in I\}$, and the union is over a collection of subsets I of $\{0, 1, 2, \dots, n\}$.

§2. Configuration Spaces of Divisors.

Definition 2.1. For a connected pair of CW-complexes (X, Y) , let $Sp^d(X, Y)$ denote the d -fold symmetric product of X/Y . Adding a base point gives rise to a natural inclusion $Sp^d(X, Y) \rightarrow Sp^{d+1}(X, Y)$ and we put $Sp^\infty(X, Y) = \bigcup_{d \geq 1} Sp^d(X, Y)$. We define a space $Q_d^{(n)}(X, Y)$ by

$$Q_d^{(n)}(X, Y) = \{(\xi_0, \dots, \xi_n) \in (Sp^d(X, Y))^{n+1} : \xi_i \cap \xi_j = \emptyset \text{ if } i \neq j\}.$$

If $Y = \emptyset$, we write $Sp^d(X) = Sp^d(X, \emptyset)$ and $Q_d^{(n)}(X) = Q_d^{(n)}(X, \emptyset)$.

If M is a connected open manifold, adding $(n+1)$ distinct points “from infinity” (c. f. [Mc]) gives a natural stabilization map $i_d : Q_d^{(n)}(M) \rightarrow Q_{d+1}^{(n)}(M)$ and we define $\hat{Q}^{(n)}(M)$ to be the “identity component” of $\lim_{d \rightarrow \infty} Q_d^{(n)}(M)$. Let $F(X, m)$ be the configuration space of m -tuples of distinct points in X . In particular, $Q_1^{(n)}(X) = F(X, n+1)$, and it is well-known that $\pi_1(F(\mathbb{C}, m)) = I(m)$, where $I(m)$ denotes the group of pure braids of m strings. Then we have

Proposition 2.2.

- (1) $\text{Hol}_d^*(S^2, X_n) = Q_d^{(n)}(\mathbb{C})$.
- (2) $\pi_1(\text{Hol}_d^*(S^2, X_n)) = \begin{cases} I(n+1) & \text{if } d = 1 \\ \mathbb{Z}^{n(n+1)/2} & \text{if } d \geq 2. \end{cases}$

(Part (2) is proved in [E].)

§3 The Stabilization Theorem.

Theorem 3.1. ([GKY],[Ko]). *The stabilization map $i_d : \text{Hol}_d^*(S^2, X_n) \rightarrow \text{Hol}_{d+1}^*(S^2, X_n)$ is a homology equivalence up to dimension d .*

Using the McDuff-Segal transfer ([Mc],[S]) we obtain

Proposition 3.2. *For any commutative ring R , the induced homomorphism $i_{d*} : H_*(\text{Hol}_d^*(S^2, X_n), R) \rightarrow H_*(\text{Hol}_{d+1}^*(S^2, X_n), R)$ is a split monomorphism. More precisely, there is a family of graded R -modules $\{R_m : m \geq 0\}$ such that*

- (a) $H_*(\text{Hol}_d^*(S^2, X_n), R) = \bigoplus_{0 \leq m \leq d} R_m$.
- (b) *The above isomorphism is compatible with the splitting monomorphism.*

These results lead us to expect

Conjecture 3.3. *There is a stable splitting*

$$\text{Hol}_d^*(S^2, X_n) \underset{\cong}{\simeq} \bigvee_{1 \leq j \leq d} D_j(n)$$

such that

$$D_j(n) \underset{\cong}{\simeq} \text{Hol}_j^*(S^2, X_n) / \text{Hol}_{j-1}^*(S^2, X_n).$$

Remark 3.4. ($[C^2M^2]$) *This is true for $n = 1$.*

§4. The Scanning Map.

Definition 4.1. *Let $\varepsilon > 0$ be any positive real number, and let $D_z(\varepsilon)$ denote the open disk of radius ε with centre at $z \in \mathbb{C}$. Define the map $S_d : Q_d^{(n)}(\mathbb{C}) \times \mathbb{C} \rightarrow Q^{(n)}(S^2, \infty)$ by*

$$\begin{aligned} ((\xi_0, \dots, \xi_n), z) &\mapsto (\xi_0 \cap D_z(\varepsilon), \dots, \xi_n \cap D_z(\varepsilon)) \in Q^{(n)}(\bar{D}_z(\varepsilon), \partial \bar{D}_z(\varepsilon)) \\ &\simeq Q^{(n)}(S^2, \infty). \end{aligned}$$

Since $\lim_{z \rightarrow \infty} S_d(\Xi, z) = (\emptyset, \emptyset, \dots, \emptyset)$ for any $\Xi \in Q_d^{(n)}(\mathbb{C})$, we define $S_d(\Xi, \infty) = (\emptyset, \emptyset, \dots, \emptyset)$ and obtain a map

$$S_d : Q_d^{(n)}(\mathbb{C}) \times S^2 \rightarrow Q^{(n)}(S^2, \infty).$$

Taking the adjoint we obtain a map

$$S_d : Q_d^{(n)}(\mathbb{C}) \rightarrow \text{Map}_d^*(S^2, Q^{(n)}(S^2, \infty)).$$

Its homotopy class is independent of the choice of ε . We call S_d the scanning map.

It can be shown that $Q^{(n)}(S^2, \infty) \simeq \bigvee^{n+1} \mathbb{C}P^\infty$. It is also easy to see that there is a homotopy equivalence $\alpha_d : \Omega_d^2(\bigvee^{n+1} \mathbb{C}P^n) \simeq \Omega_{d+1}^2(\bigvee^{n+1} \mathbb{C}P^n)$ such that the following diagram is commutative up to homotopy

$$\begin{array}{ccc} Q_d^{(n)}(\mathbb{C}) & \xrightarrow{S_d} & \Omega_d^2(\bigvee^{n+1} \mathbb{C}P^\infty) \\ \downarrow i_d & & \simeq \downarrow \alpha_d \\ Q_{d+1}^{(n)}(\mathbb{C}) & \xrightarrow{S_{d+1}} & \Omega_{d+1}^2(\bigvee^{n+1} \mathbb{C}P^\infty) \end{array}$$

Consider the mapping telescope of the maps

$$Q_1^{(n)}(\mathbb{C}) \xrightarrow{i_1} Q_2^{(n)}(\mathbb{C}) \xrightarrow{i_2} Q_3^{(n)}(\mathbb{C}) \xrightarrow{i_3} Q_4^{(n)}(\mathbb{C}) \rightarrow \dots$$

It is easy to see that this mapping telescope is homotopy equivalent to $\hat{Q}^{(n)}$. Hence we obtain a stabilized scanning map

$$\hat{S} : \hat{Q}^{(n)} \rightarrow \Omega_0^2(\bigvee^{n+1} \mathbb{C}P^\infty).$$

By arguing exactly as in [S], we obtain

Proposition 4.3. *The scanning map $\hat{S}$ is a homotopy equivalence.*

Sketch proofs of Theorems 1 and 2. Let $G = (\mathbb{C}^*)^n$ and define a G -action on X_n by

$$((t_1, \dots, t_n), [p_0 : \dots : p_n]) \mapsto [p_0 : t_1 p_1 : \dots : t_n p_n].$$

Then there is a fibre sequence

$$T^n \rightarrow X_n \xrightarrow{q} \bigvee^{n+1} \mathbb{C}P^\infty.$$

(This follows from the fact that $EG \times_G X_n \simeq \bigvee^{n+1} \mathbb{C}P^\infty$). There is a homotopy commutative diagram:

$$\begin{array}{ccc} \text{Hol}_d^*(S^2, X_n) & \xrightarrow{I_d} & \text{Map}_d^*(S^2, X_n) = \Omega_d^2 X_n \\ \simeq \downarrow & & \simeq \downarrow \Omega^2 q \\ Q_d^{(n)}(\mathbb{C}) & \xrightarrow{S_d} & \Omega_d^2(\bigvee^{n+1} \mathbb{C}P^\infty) \end{array}$$

It follows that $\lim_{d \rightarrow \infty} I_d$ is a homotopy equivalence. Hence Theorem 1 follows from the stabilization theorem.

Finally, an argument analogous to the one given by Segal in [S] shows that the space $Q_d^{(n)}(\mathbb{C})$ is nilpotent up to dimension d if $2d > n$. Theorem 2 follows from the Whitehead Theorem [HR] $\square$

REFERENCES

- [C²M²] Cohen, F. R., Cohen, R. L., Mann, B. M. and Milgram, R. J., *The topology of rational functions and divisors of surfaces*, Acta Math. **166** (1991), 161–221.
- [CLM] F R Cohen, Lada, T. Y., May, J. P., *The homology of iterated loop spaces*, vol. 533, Springer Lecture Notes In Math., 1976.
- [C-S] R L Cohen and D H Shimamoto, *Rational functions, labelled configurations and Hilbert schemes*, J. London Math. Soc. **43** (1991), 505–528.
- [E] Epshtein, S. I., *Fundamental groups of spaces of coprime polynomials*, Functional Analysis and Its Applications **7** (1973), 82–83.
- [G] Gravesen, J., *On the topology of spaces of holomorphic maps*, Acta Math. **162** (1989), 247–286.
- [Gu] Guest, M. A., *Topology of the space of absolute minima of the energy functional*, Amer. Jour. Math. **106** (1984), 21–42.
- [GKY] Guest, M. A., Kozłowski, A. and Yamaguchi, K., *The topology of spaces of coprime polynomials*, Preprint.
- [HR] Hilton, P. and Roitberg, J., *On the Zeeman comparison theorem for the homology of quasi-nilpotent fibrations*, Quarterly J. Math. **27** (1976), 433–444.
- [K] Kirwan, F., *On spaces of maps from Riemann spheres to Grassmannians and applications to the cohomology of moduli of vector bundles*, Ark. M. **24** (1986), 221–275.
- [Ko] Kozłowski, A., *Stabilization of homology groups of spaces of mutually disjoint divisors*, In these Proceedings.
- [M²] Mann, B. M. and Milgram, R. J., *Some spaces of holomorphic maps to complex Grassman manifolds*, J. Differential Geometry **33** (1991), 301–324.
- [Mc] McDuff, D., *Configuration spaces of positive and negative particles*, Topology **14** (1975), 91–107.
- [S] Segal, G. B., *The topology of spaces of rational functions*, Acta Math. **143** (1979), 39–72.