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AN EXAMPLE OF STABLE HIGGS BUNDLES WHICH DO NOT SATISFY THE BOGOMOLOV INEQUALITY

YOICHI MIYAOKA

School of Mathematics, University of Tokyo

INTRODUCTION

The Bogomolov inequality for semistable vector bundles on smooth complex projective n-folds reads

$$c_2(\mathcal{E})A^{n-2} \geq \frac{r-1}{2r}c_1(\mathcal{E})^2A^{n-2},$$

where A is an ample divisor and $\mathcal{E}$ is an A-semistable vector bundle of rank r. In case $\mathcal{E}$ is A-stable with vanishing $c_1(\mathcal{E})$, the lower bound of this inequality $c_2(\mathcal{E})A^{n-2} \geq 0$ is attained if and only if $\mathcal{E}$ is a flat hermitian bundle associated with an irreducible unitary representation of the fundamental group $\pi_1(X)$, thereby establishing the one-to-one *Kobayashi-Hitchin correspondence* between the stable bundles with vanishing Chern classes and the irreducible unitary representations of $\pi_1(X)$ [2]. The inequality is natural enough to have proofs by several different approaches (geometric invariant theory [1]; characteristic p method [3]; the theory of effective cones on ruled surfaces [8]; differential geometry [2]) and generalizes to bigger classes of semistable bundles, including orbibundles and parabolic bundles.

Another important class of generalised vector bundles is that of Higgs bundles (see [9]), and it is a natural question to ask if the Bogomolov inequality extends also to this class. The inequality is indeed true for standard types of Higgs bundles listed in Section 1 as Examples 0, 1, 2, and for bundles of small ranks 2, 3 as well [7]. Unfortunately, however, this is not the case for Higgs bundles of higher rank. In this note, we construct stable Higgs 4-bundles on surfaces of general type for which the inequality breaks down (Proposition 4 in Section 3). Starting from this examaple, we also find a nontrivially deforming families of stable Higgs 4-bundles with trivial Chern classes or, equivalently, non-trival deformations of irreducible SL(4)-representations of the fundamental group (Section 4).

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1. HIGGS BUNDLES: DEFINITION AND STANDARD EXAMPLES

Let $\mathcal{E}$ be a vector bundle on a complex manifold X and $\theta\colon \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ an $\mathcal{O}_X$-linear mapping. The pair $(\mathcal{E}, \theta)$ is said to be a *Higgs bundle* if the natural composite map $\theta \wedge \theta\colon \mathcal{E} \to \Omega^2_X \otimes \mathcal{E}$ identically vanishes. Alternatively, $\mathcal{E}$ is a Higgs bundle if an $\mathcal{O}_X$-linear action of the sheaf of the local vector fields Θ_X on $\mathcal{E}$ is given in such a way that $\xi_1(\xi_2(e)) = \xi_2(\xi_1(e))$, $\forall \xi_i \in \Theta_X, \forall e \in \mathcal{E}$. In other words, a Higgs bundles is a vector bundle equipped with a $\operatorname{Sym}\Theta_X$-module structure, where $\operatorname{Sym}\Theta_X = \bigoplus_{i=0}^{\infty} \operatorname{Sym}^i \Theta_X$ is the symmetric tensor algebra generated by Θ_X.

Higgs subsheaves are, by definition, $\operatorname{Sym}\Theta_X$-submodules. Given an ample divisor A on X, the notion of A-(semi)stability of Higgs bundles is naturally defined.

Historically, Higgs structures were introduced as the moduli of flat connections [5]. Let $\mathcal{E}$ be a vector bundle with an integrable connection $\nabla_0 \colon \mathcal{E} \to \Omega_X^1 \otimes \mathcal{E}$. Given another integrable connection ∇, the difference $\theta = \nabla - \nabla_0 \colon \mathcal{E} \to \Omega_X^1 \otimes \mathcal{E}$ obviously gives a Higgs bundle structure and this correspondence translates the moduli of the flat connections on $\mathcal{E}$ into the moduli of Higgs bundle structures.

We give below several standard examples of Higgs bundles.

Example 0. An ordinary vector bundle is viewed as a Higgs bundle with trivial (zero)action of Θ_X.

Example 1. Let X be a complex manifold. Then

$$\mathbf{E}_X^l = \bigoplus_{i=0}^{l} \operatorname{Sym}^i \Theta_X$$

is a Higgs bundle, where the action of $\operatorname{Sym}^j \Theta$ is defined by the standard multiplication

$$\operatorname{Sym}^j \Theta \otimes \operatorname{Sym}^i \Theta_X \to \begin{cases} \operatorname{Sym}^{i+j} \Theta_X, & \text{for } 0 \le i+j \le l \\ 0, & \text{for } i+j > l. \end{cases}$$

Given $m \le l$, the sheaf

$$\mathbf{E}_X^{l,m} = \bigoplus_{i=m}^{l} \operatorname{Sym}^i \Theta_X$$

is a Higgs subbundle of $\mathbf{E}_X^l$, and the quotient $\mathbf{E}_X^l / \mathbf{E}_X^{l,m}$ is isomorphic to $\mathbf{E}_X^m$. (Actually a more natural definition of $\mathbf{E}_X^l$ is the quotient $\mathbf{E}_X^\infty / \mathbf{E}_X^{\infty,l}$.)

If $K_X A > 0$ and Θ_X is A-semistable as an ordinary vector bundle [resp. If $K_X A \ge 0$ and Θ_X is A-semistable], then $\mathbf{E}_X^l$ is an A-stable [resp. A-semistable] Higgs bundle. If K_X is ample and $A = K_X$, then the Yau inequality [10]

$$(-1)^n c_2(X) c_1(X)^{n-2} \ge (-1)^n \frac{\dim X - 1}{2 \dim X} c_1(X)^n$$

yields the Bogomolov inequality for the Higgs bundle $\mathbf{E}_X^l$.

Example 2. Given a non-negative integer l, we define the Higgs bundle $\mathbf{F}_X^l$ by

$$\mathbf{F}_X^l = \bigoplus_{i=m}^{l} \operatorname{Sym}^i \Omega_X^1,$$

where the action of $\operatorname{Sym}^j \Theta$ is given by the contraction homomorphism

$$\operatorname{Sym}^j \Theta \otimes \operatorname{Sym}^i \Omega_X^1 \to \begin{cases} \operatorname{Sym}^{i-j} \Omega_X^1, & \text{for } j \le i \\ 0, & \text{for } j > i. \end{cases}$$

If $m \le l$, then $\mathbf{F}_X^m \subset \mathbf{F}_X^l$ is naturally a Higgs subbundle. $\mathbf{F}_X^l$ is an A-stable [resp. A-semistable] Higgs bundle if $K_X A > 0$ and $\operatorname{Sym}^l \Omega_X^1$ is A-stable as an ordinary vector bundle [resp. if $K_X A \ge 0$ and Ω_X^1 is A-semistable]. When, in addition, $A = K_X$ is ample, the Bogomolov inequality is satisfied by $\mathbf{F}_X^l$.

Example 3. Let $g\colon X \to Y$ be a morphism between complex manifolds and $\mathcal{E}$ a Higgs bundle on Y. The natural homomorphism $\Theta_X \to g^*\Theta_Y$ defines a canonical Higgs bundle structure on $g^*\mathcal{E}$.

Example 4. Given two Higgs bundles $\mathcal{E}_i$, $i = 1, 2$, the tensor bundle $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a Higgs bundle by defining $\theta(e_1 \otimes e_2) = \theta(e_1) \otimes e_2 + e_1 \otimes \theta(e_2)$, $\theta \in \Theta_X$, $e_i \in E_i$. (the tensor Higgs bundle). The dual bundle $\mathcal{E}^\vee$ is a Higgs bundle (the dual Higgs bundle) by $\langle e|\theta(e^\vee)\rangle = -\langle\theta(e)|e^\vee\rangle$. Here $\theta \in \Theta_X$, $e \in \mathcal{E}$, $e^\vee \in \mathcal{E}^\vee$, while $\langle\cdot|\cdot\rangle$ stands for the canonical bilinear pairing. $\mathbf{F}^l_X$ is the dual Higgs bundle of $\mathbf{E}^l_X$, if we give a nondegenerate pairing between $\mathrm{Sym}^m\Theta_X$ and $\mathrm{Sym}^m\Omega^1_X$ by

$$\langle\theta_1 \otimes \cdots \otimes \theta_m|\omega_1 \otimes \cdots \omega_m\rangle = \frac{(-1)^m}{m!} \sum_{\sigma\in\mathfrak{S}_m} \prod_{i=1}^m \omega_i(\theta_{\sigma(i)}).$$

2. Hirzebruch's Kummer covers $X^{(n)}$ attached to the complete quadrilateral line configuration

We briefly review Hirzebruch's construction of Kummer covers of projective plane branching along a complete quadrilateral [4].

Take general four points $P_1, \ldots, P_4$ on projective plane $\mathbb{P}^2$, and let $L_{ij} = L_{ji}$ denote the line connecting P_i and P_j $(i \neq j)$. The reduced divisor $D = \bigcup L_{ij}$ is the so-called complete quadrilateral consisting of six lines, and the P_i are the triple points of D. The complete quadrilateral D has extra three double points of the form $L_{i_1,i_2} \cap L_{j_1,j_2}$, where $\{i_1, i_2, j_1, j_2\} = \{1, 2, 3, 4\}$. Exactly three singular points of D lies on each L_{ij}, two of which are triple points and one a double point. Thus the Euler number of the nonsingular locus of D is $6 \times (2 - 3) = -6$, while that of D is $-6 + 4 + 3 = 1$. Therefore the Euler number of the complement of D is given by $\mathrm{e}(X \setminus D) = 3 - 1 = 2$.

Let $\mu\colon X \to \mathbb{P}^2$ be the blowing up at the four triple points $P_1, \ldots, P_4$ of D and $E_i \subset X$ the exceptional curve over P_i. X is a Del Pezzo surface of degree 5, with very ample anticanonical divisor $-K_X \sim 3H - \sum E_i$, where H denotes the pullback of the hyperplane of $\mathbb{P}^2$. The effective divisor μ^*D is supported by a reduced effective divisor

$$\tilde{D} \sim \mu^* \sum L_{ij} - 2 \sum E_i \sim 6\mu^*H - 2 \sum E_i \sim -2K_X$$

with only simple normal crossings. Each E_i contains three singular points of $\tilde{D}$ so that $\tilde{D}$ has exactly $4 \times 3 + 3 = 15$ double points. If $\tilde{L}_{ij} \subset X$ denotes the strict transform of L_{ij}, we have

$$\#\tilde{L}_{ij} \cap \mathrm{Sing}(\tilde{D}) = 3 = \#E_i \cap \mathrm{Sing}(\tilde{D}).$$

Given a positive integer n, there exists a Kummer covering $\pi^{(n)}\colon X^{(n)} \to X$ of degree n^5 branching along $\tilde{D}$ (see Hirzebruch[4]). The function field of $X^{(n)}$ is simply obtained by adjoining the n-th roots $\sqrt[n]{l_{ij}/l_{kl}}$ $(i, j, k, l \in \{1, 2, 3, 4\})$ to $\mathbb{C}(\mathbb{P}^2)$, where l_{ij} is a linear defining equation of the line L_{ij}. $X^{(n)}$ is a smooth projective surface and the local description of X_n is quite simple: if $\tilde{D}$ is locally defined by the equation $x = 0$ or $xy = 0$, then $\pi_n^*\colon \mathcal{O}_X \to \mathcal{O}_{X_n}$ is given by $(x, y) \mapsto (t^n, y)$ or $(x, y) \mapsto (t^n, u^n)$, where (x, y) and (t, u) are local coordinates of X and $X^{(n)}$.

In particular, the inverse image $(\pi^{(n)})^{-1}(p) \subset X^{(n)}$ of a closed point $p \in X$ consists of n^5 [resp. n^4, n^3] points when $p \in X \setminus \tilde{D}$ [resp. $p \in \tilde{D} \setminus \mathrm{Sing}(\tilde{D})$, $p \in \mathrm{Sing}(\tilde{D})$]. The topological Euler number $\mathrm{e}(X^{(n)})$ of $X^{(n)}$ is thus given by

$$\begin{aligned} n^{-5}\mathrm{e}(X^{(n)}) &= \mathrm{e}(X \setminus \tilde{D}) + n^{-1}\mathrm{e}\big(\tilde{D} \setminus \mathrm{Sing}(\tilde{D})\big) + n^{-2}\mathrm{e}\big((\mathrm{Sing}(\tilde{D})\big) \\ &= 2 + n^{-1} \times (6+4) \times (2-3) + n^{-2} \times 15 = 2 - 10n^{-1} + 15n^{-2}. \end{aligned}$$

On the other hand, we calculate $K_{X^{(n)}}$ as

$$K_{X^{(n)}} \sim \pi^{(n)*}(K_X + (1 - n^{-1})\tilde{D}) \sim (1 - 2n^{-1})\pi^{(n)*}(-K_X),$$

and hence

$$c_1(X^{(n)})^2 = c_1(\Omega^1_{X^{(n)}})^2 = 5n^5(1 - 2n^{-1})^2.$$

The surface X_n has ample canonical divisor if $n \geq 3$.

When $n = 5$, we have $c_1^2(X_5) = 5^4 \times 9$, $c_2(X_5) = e(X_5) = 5^4 \times 3$, meaning that X_5 is a surface of general type which attains the upper bound of the Miyaoka-Yau inequality $K^2 \leq 3c_2$.

The Del Pezzo surface X carries five linear pencils $|2H - \sum E_i|$, $|H - E_1|$, ... , $|H - E_4|$, defining five surjective morphisms from X onto $\mathbb{P}^1$. Each of these morphisms has exactly three fibres contained in the branch locus $\tilde{D}$. For instance, for the morphism associated with $|2H - \sum E_i|$, the three curves $\tilde{L}_{12} + \tilde{L}_{34}$, $\tilde{L}_{13} + \tilde{L}_{24}$ and $\tilde{L}_{14} + \tilde{L}_{23}$ are such fibres, and so are $\tilde{L}_{1i} + E_i$, $i = 2, 3, 4$ for $|H - E_1|$.

Upstairs on $X^{(n)}$, there are thus five morphisms $f_0^{(n)}, \ldots, f_4^{(n)}$ onto the curve $C^{(n)}$, an n^2-sheeted Kummer cover of $\mathbb{P}^1$ branching at three points, 0, 1 and ∞, say. The pullback line bundle $\mathcal{L}_j^{(n)} = f_j^{(n)*}\omega_{C^{(n)}}$ is an invertible subsheaf of $\Omega^1_{X^{(n)}}$. We easily check that $\mathcal{L}_j^{(n)}$ is saturated in $\Omega^1_{X^{(n)}}$ and that

$$\begin{aligned} \mathcal{L}_0^{(n)} &\sim (1 - 3n^{-1})\pi^{(n)*}(2H - \textstyle\sum E_i), \\ \mathcal{L}_i^{(n)} &\sim (1 - 3n^{-1})\pi^{(n)*}(H - E_i), \quad i = 1, \ldots, n. \end{aligned}$$

Ishida [6] determined the irregularity of $X^{(n)}$ by showing that the natural map $\bigoplus_j f_j^* \mathrm{H}^0(C^{(n)}, \Omega^1_{C^{(n)}}) \to \mathrm{H}^0(X^{(n)}\Omega^1_{X^{(n)}})$ is an isomorphism (for instance, $q(X^{(5)}) = 5g(C^{(5)}) = 30$).

In view of the definitions of $X^{(n)}$ and $C^{(n)}$, the family $(X^{(n)}, C^{(n)}, f_i^{(n)})$ form a partially ordered tower: there are natural Kummer covers $\pi_{(n)}^{(mn)} : X^{(mn)} \to X^{(n)}$ of degree m^5 and $p_{(n)}^{(mn)} : C^{(mn)} \to C^{(n)}$ such that the diagram

$$\begin{tikzcd} X^{(mn)} \arrow[r, "\pi_{(n)}^{(mn)}"] \arrow[d, "f_i^{(mn)}"'] & X^{(n)} \arrow[d, "f_i^{(n)}"'] \\ C^{(mn)} \arrow[r, "p_{(n)}^{(mn)}"] & C^{(n)} \end{tikzcd}$$

commutes.

3. Construction of a stable Higgs 4-bundle $\mathcal{H}$

Let the notation be as in the previous section.

We construct a Higgs 4-bundle $\mathcal{H}$ on $X^{(15)}$ as a subsheaf of $\pi_{(5)}^{(15)*}\mathbf{F}^2_{X^{(5)}}$ (see Section 1, Examples 2 and 3).

Let $C_{ij}^{(n)}, F_k^{(n)} \subset X^{(n)}$ ($1 \le i < j \le 4$, $1 \le k \le 4$) be the inverse images of $\tilde{L}_{ij}, E_k \subset X$ with reduced scheme structures. Each of them is a union of n^3 copies of a curve isomorphic to $C^{(n)}$. $C_{ij}^{(n)}$ is contained in n^3 fibres of $f_i^{(n)}, f_j^{(n)}$ and $f_0^{(n)}$, whereas it is union of sections of $f_l^{(n)}$ for $l \in \{1,2,3,4\} \setminus \{i,j\}$. Similarly, $F_i^{(n)}$ is a union of sections of $f_i^{(n)}$, $f_0^{(n)}$ and contained in fibres of the other three projections to $C^{(n)}$.

Let $n = 3$. Then $C^{(3)}$ is an elliptic curve. Fix a basis η of $H^0(C^{(3)}, \Omega^1_{C^{(3)}}) \simeq \mathbb{C}$, and put $\eta_i = f_i^{(3)*}\eta \in H^0(X^{(3)}, \Omega^1_{X^{(3)}})$.

Proposition 1. *The 1-forms η_0 and $\omega = \eta_0 - \eta_1 - \eta_2 - \eta_3 - \eta_4$ are non-zero and sit in the subsheaves*

$$\operatorname{Ker}(\Omega^1_{X^{(3)}} \to \bigoplus_{ij} \Omega^1_{C_{ij}^{(3)}})$$

and

$$\operatorname{Ker}(\Omega^1_{X^{(3)}} \to \bigoplus_{i=1}^{4} \Omega^1_{F_i^{(3)}}),$$

respectively.

Proof. This immediately follows from the following two facts.

(1) f_0 maps each $\tilde{L}_{ij}$ to a single point on $\mathbb{P}^1$ or, equivalently, $f_0^{(3)}$ maps each $C_{ij}^{(3)}$ to finitely many points of $C^{(3)}$;

(2) $f_i : E_j \to \mathbb{P}^1$ is either an isomorphism ($i = 0$ or $i = j$) or a constant map ($j \neq i = 1,2,3,4$) or, equivalently, $f_i^{(3)} : F_j^{(3)} \to C^{(3)}$ restricted to each irreducible component is either an isomorphism or a constant map.

Indeed, viewed as 1-forms on $C_{ij}^{(3)} \simeq C^{(3)}$, $\eta_0 = \eta_i = \eta_j = 0$, $\eta_l = \eta$, $l \neq 0, i, j$, so that $\eta_0|_{C_{ij}^{(3)}} = 0$, $\omega|_{C_{ij}^{(3)}} = -2\eta \neq 0$. On $F_i^{(3)}$, we have $\eta_0 = \eta_i = \eta$, $\eta_j = 0$, $j \neq 0, i$, so that $\eta_0|_{F_i^{(3)}} = \eta \neq 0$, $\omega|_{F_i^{(3)}} = 0$.

Corollary 2. *In the same notation as above,*

$$\pi_{(3)}^{(3n)*}\eta_0 \in \Omega^1_{X^{(3n)}}(-(n-1)\sum_{1 \le i < j \le 4} C_{ij}^{(3n)}),$$

$$\pi_{(3)}^{(3n)*}\omega \in \Omega^1_{X^{(3n)}}(-(n-1)\sum_{i=1}^{4} F_i^{(3n)}).$$

Proof. Let (x, y) be a local coordinate at a general point of $C_{ij}^{(3)}$ such that $C_{ij}^{(3)}$ is defined by $x = 0$. $\pi_{(3)}^{(3n)}$ is then given by $(t, u) \mapsto (x, y) = (t^n, u)$. Proposition 1 asserts that η_0 is of the form $\alpha dx + x\beta dy$, $\alpha, \beta \in \mathcal{O}$, so that $\pi_{(3)}^{(3n)*}\eta_0$ is of the form $nt^{n-1}\alpha dt + t^n\beta du$. The second statement for ω is similarly proved.

$\Omega^1_{X^{(3n)}}$ contains $\pi^{(3n)*}_{(n)}\Omega^1_{X^{(n)}}$ as well as $\pi^{(3n)*}_{(3)}\Omega^1_{X^{(3)}}$. At a general point p of each component of the ramification locus, the former subsheaf is generated by t^2dt, du, while the latter by $t^{n-1}dt, du$. In particular, when $n > 3$,

$$\pi^{(3n)*}_{(3)}\Omega^1_{X^{(3)}} \subset \pi^{(3n)*}_{(n)}\Omega^1_{X^{(n)}} \subset \Omega^1_{X^{(3n)}}.$$

The following assertion follows from the above local description of $\pi^{(3n)*}_{(n)}\Omega^1_{X^{(n)}}$ together with the proof of Corollary 2.

Corollary 3. *Fix $n \geq 4$. Then*

$$\pi^{(3n)*}_{(3)}\eta_0 \in (\pi^{(3n)*}_{(n)}\Omega^1_{X^{(n)}})(-(n-3)\sum_{1\leq i<j\leq 4} C^{(3n)}_{ij}),$$

$$\pi^{(3n)*}_{(3)}\omega \in (\pi^{(3n)*}_{(n)}\Omega^1_{X^{(n)}})(-(n-3)\sum_{i=1}^{4} F^{(3n)}_{i}).$$

Hence

$$\begin{aligned}\pi^{(3n)*}_{(3)}\eta_0\omega &\in \pi^{(3n)*}_{(n)}\mathrm{Sym}^2\Omega^1_{X^{(n)}}(-(n-3)(\sum_{1\leq i<j\leq 4} C^{(3n)}_{ij} + \sum_{i=1}^{4} F^{(3n)}_{i}))\\ &= (\pi^{(3n)*}_{(n)}\mathrm{Sym}^2\Omega^1_{X^{(n)}})(-\frac{2(n-3)}{3n}\pi^{(3n)*}(3H - \sum E_i))\\ &= \pi^{(3n)*}_{(n)}\mathrm{Sym}^2\Omega^1_{X^{(n)}}(-\frac{2(n-3)}{3(n-2)}K_{X^{(n)}}).\end{aligned}$$

Put $n = 5$. Then we get the following

Proposition 4. (1) $\pi^{(15)*}_{(5)}\mathrm{Sym}^2\Omega^1_{X^{(5)}} \supset \mathcal{L} = \mathcal{O}(\frac{4}{9}\pi^{(15)*}_{(5)}K_{X^{(5)}})$.

(2) *$\mathcal{H} = \mathcal{L} \oplus \pi^{(15)*}_{(5)}\Omega^1_{X^{(5)}} \oplus \mathcal{O}$ is a Higgs subsheaf of*

$$\pi^{(15)*}_{(5)}\mathbf{F}^2_{X^{(5)}} = \pi^{(15)*}_{(5)}(\mathrm{Sym}^2\Omega^1_{X^{(5)}} \oplus \Omega^1_{X^{(5)}} \oplus \mathcal{O}_{X^{(5)}}).$$

(3) $c_1(\mathcal{H}) = \frac{13}{9}\pi^{(15)*}_{(5)}K_{X^{(5)}}$, $c_2(\mathcal{H}) = \pi^{(15)*}_{(5)}(c_2(\Omega^1_{X^{(5)}})+\frac{4}{9}K^2_{X^{(5)}})^2 = \frac{7}{9}\pi^{(15)*}_{(5)}K^2_{X^{(5)}}$, *so that*

$$\frac{c_1^2(\mathcal{H})}{c_2(\mathcal{H})} = \frac{169}{63} = \frac{8}{3} \times \frac{169}{168} > \frac{8}{3}.$$

(4) *The Higgs 4-bundle $\mathcal{H}$ is $\pi^{(15)*}_{(5)}K_{X^{(5)}}$-stable.*

Proof. Corollary 3 is rephrased into (1), which in turn yields (2). (3) follows from direct computation. In order to show (4), we check that the avarage degree of a saturated Higgs subsheaf of $\mathcal{H}$ is strictly smaller than $(13/36)\pi^{(15)*}_{(5)}K^2_{X^{(5)}}$, the average degree of $\mathcal{H}$. At a general point q of $C^{(3)}_{ij}$, the product $\eta_0\omega$ is of the form $\alpha dxdy$, where $\alpha \in \mathcal{O}^\times$ and (x, y) is a local coordinate. Hence $\Theta_{X^{(3)}}\mathcal{O}\eta_0\omega = \Omega^1_{X^{(3)}}$ around q. Then it is obvious that, at a general point $p \in X^{(15)}$, $\Theta_{X^{(15)},p} = (\pi^{(15)*}_{(5)}\Theta_{X^{(5)}})_p = (\pi^{(15)*}\Theta_X)_p$, $\Theta_{X^{(15)}}\mathcal{L}_p = \pi^{(15)*}_{(5)}\Omega^1_{X^{(5)},p}$. This shows that a proper Higgs subsheaf of $\mathcal{H}$ must be contained in $\pi^{(15)*}_{(5)}\mathbf{F}^1_{X^{(5)}} = \pi^{(15)*}_{(5)}(\Omega^1_{X^{(5)}} \oplus \mathcal{O}_{X^{(5)}})$, and the assertion follows from the semistability of $\mathbf{F}^1_{X^{(5)}}$ of average degree $(1/3)\pi^{(15)*}_{(5)}K^2_{X^{(5)}}$.

4. Stable Higgs 4-bundles with vanishing Chern classes

Starting from the 4-bundle $\mathcal{H}$ described in the previous section, we can construct many stable Higgs 4-bundles with trivial Chern classes.

Recall that $\mathcal{H}$ is the direct sum $\mathcal{L} \oplus \pi_{(5)}^{(15)*}\Omega^1_{X^{(5)}} \oplus \mathcal{O}$. Take line bundles $\mathcal{L}_1, \mathcal{L}_2$ such that $\mathcal{L}_1 \subset \mathcal{L}$, $\mathcal{L}_2 \supset \mathcal{O}$. Then $\mathcal{H}' = \mathcal{L}_1 \oplus \pi_{(5)}^{(15)*}\Omega^1_{X^{(5)}} \oplus \mathcal{L}_2$ is naturally a Higgs bundle. $\mathcal{H}'$ is stable if

(1) $c_1(\mathcal{L}_1)\pi_{(5)}^{(15)*}K_{X^{(5)}} > \frac{1}{3}(\pi_{(5)}^{(15)*}K_{X^{(5)}} + c_1(\mathcal{L}_2))\pi_{(5)}^{(15)*}K_{X^{(5)}}$, and

(2) $c_1(\mathcal{L}_2)\pi_{(5)}^{(15)*}K_{X^{(5)}} < \frac{1}{2}(\pi_{(5)}^{(15)*}K_{X^{(5)}})^2$.

Choose $\mathcal{L}_i$ to be of the form $\mathcal{O}(t_i\pi_{(5)}^{(15)*}K_{X^{(5)}})$, where $t_i \in \mathbb{Q}$, $t_1 < 4/9, t_2 > 0$. Such L_i's make sense if we replace $X^{(15)}$ by a suitable ramified cover $Y = X^{(15l)}$, where l is a sufficiently divisible positive integer. Thus we consider the vector bundle $\mathcal{H}' = \mathcal{L}_1 \oplus \pi_{(5)}^{Y*}\Omega^1_{X^{(5)}} \oplus \mathcal{L}_2$ on Y, where $\pi_{(5)}^{Y} : Y \to X^{(5)}$ is the projection.

The Chern classes of the vector bundle $\mathcal{H}'$ are:

$$c_1(\mathcal{H}') = (t_1 + t_2 + 1)\pi_{(5)}^{Y*}K_{X^{(5)}},$$
$$c_2(\mathcal{H}') = \left(t_1t_2 + t_1 + t_2 + \frac{1}{3}\right)\left(\pi_{(5)}^{Y*}K_{X^{(5)}}\right)^2.$$

Thus the condition

$$3c_1^2(\mathcal{H}') = 8c_2(\mathcal{H}')$$

is given by the quadratic equation

$$3t_1^2 - 2t_1t_2 + 3t_2^2 - 2t_1 - 2t_2 + \frac{1}{3} = 0,$$

a solution of which is $(t_1, t_2) = (\frac{1}{3}, 0)$. Hence there are infinitely many rational solutions of the quadratic equation, and the stability condition is a non-empty open condition on those solutions (the rational point $(1/3, 0)$ lies on the boundary of the region given by the stability condition (1) and in the interior of the one given by (2)). For instance,

$$(t_1, t_2) = \left(\frac{25}{3 \cdot 19}, \frac{2}{3^2 \cdot 19}\right)$$

is a solution with

$$(c_1(\mathcal{H}'), c_2(\mathcal{H}')) = \left(\frac{8 \cdot 31}{9 \cdot 19}\pi_{(5)}^{Y*}K_{X^{(5)}}, \frac{3 \cdot 8 \cdot 31^2}{9^2 \cdot 19^2}(\pi_{(5)}^{Y*}K_{X^{(5)}})^2\right).$$

We thus conclude that there are 4-bundles $\mathcal{H}'$ such that the normalized bundles $\mathcal{G} = \mathcal{H}'(-\frac{1}{4}c_1(\mathcal{H}'))$ are stable Higgs 4-bundles with trivial Chern classes. By a theorem of Simpson [9], $\mathcal{G}$ is a flat vector bundle induced by an irreducible representation $\pi_1(Y) \to \mathrm{SL}(4)$.

On $Y = X^{(15l)}$, the (integral) divisor $(\frac{4}{9} - t_1)(\pi_{(5)}^{Y*}K_{X^{(5)}})$ is linerly equivalent to a sum of the fibres $f_i^{Y*}p_{i\alpha}$, where $f_i^Y : Y \to C^{(15l)}$ is the projection and $p_{i\alpha} \in C^{(15l)}$ (indeed, $-2K_X \sim \sum_{i=0}^4 f_i^*\mathcal{O}_{\mathbb{P}^1}(1)$ on the Del Pezzo surface X, and the divisor in question is a rational multiple of the pullback of $-2K_X$). If we replace $p_{i\alpha}$ by another point $q_{i\alpha} \sim p_{i\alpha} + \tau_{i\alpha}$, $\tau_{i\alpha} \in \mathrm{Pic}^0(C^{(15l)})$, we get an effective invertible sheaf

$$\mathcal{O}_Y(\sum_{i,\alpha} f_i^{Y*} q_{i\alpha}) \simeq \mathcal{O}_Y((\frac{4}{9} - t_1)\pi_{(5)}^{Y*} K_{X^{(5)}} + \sum_{i,\alpha} f_i^{Y*} \tau_{i\alpha})) = \mathcal{L} \otimes \mathcal{L}_1^{-1}(\sum f_i^{Y*} \tau_{i\alpha}).$$

This isomorphism induces an injection

$$\mathcal{L}_1(-\tau) = \mathcal{L}_1(-\sum_{i,\alpha} f_i^{Y*} \tau_{i\alpha}) \hookrightarrow \mathcal{L} \subset \pi_{(5)}^{Y*} \mathrm{Sym}^2 \Omega^1_{X^{(5)}},$$

where $\tau = \sum f_i^{Y*} \tau_{i\alpha}$ nontrivially moves in $\mathrm{Pic}^0(Y)$. Putting

$$\mathcal{H}'_\tau = \mathcal{L}_1(-\tau) \oplus \pi_{(5)}^{(15)*} \Omega^1_{X^{(5)}} \oplus \mathcal{L}_2, \quad \mathcal{G}_\tau = \mathcal{H}'_\tau(-\frac{1}{4} c_1(\mathcal{H}'_\tau)),$$

we obtain a deforming family of stable Higgs bundles $\mathcal{G}_\tau$ with $c_1 = 0 \in \mathrm{Pic}(Y)$ and $c_2 = 0 \in \mathrm{H}^4(Y, \mathbb{Z})$. By Simpson's theorem [9], it gives rise to a deformation of irreducible representations $\pi_1(Y) \to \mathrm{SL}(4)$ parametrized by a product of several copies of $C^{(15l)}$.

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