

# 特異積分方程式の数値解法

(除く, 固有値問題, Volterra型, 微積分方程式, 純理論)

群馬大, 学芸, 物理

阿曾義之

## § 1 方程式の分類

$$\begin{cases} \text{第 1 種} & \int K(s, t) f(t) dt = g(s). \\ \text{第 2 種} & f + \int K(s, t) f(t) dt = g(s). \\ \text{同 次} & \lambda \int K f = f. \end{cases}$$

$$\begin{cases} \text{regular} & \text{核 } K: \text{自乗積分可能.} \\ \text{singular} & K: \text{自乗積分発散, etc.} \end{cases}$$

例  $K = H(s, t)(s-t)^{-\alpha}$ ,  $K = H(s, t) \cot(s-t)$ ,  $? K = \ln |s-t|$

## § 2 教科書

実用的

- 1) 中田義元, 加藤敏夫, ..., "微分方程式の近似解法 I"  
岩波講座, 現代応用数学.
- 2) 日高孝次, "積分方程式論"
- 3) Bückner, H.F., "Numerical methods for integral equations," in J. Todd, ed., "Survey of numerical analysis," pp.439-484  
McGraw-Hill, N. Y. 1962.
- 4) -----, "Die praktische Behandlung von Integralgleichungen," Ergebnisse der angewandten Math. Bd.1,  
Springer DM 18.60 1952.

- 5) Hamel, G., "Integralgleichungen, Einführung in  
Lehre und Gebrauch," 1937.  
"Integral equations to their study and appli-  
cation,"  
Holden-day \$ 6.50 1964.
- 6) Hildebrand, F.B., "Methods of applied mathemat-  
ics,"  
Prentice-Hall 1952.
- 7) Schmeidler, W. "Integralgleichungen mit Anwend-  
ungen in Phys. und Technik, I. Linear Inte-  
gralgleichungen," Math. und ihre Anwendungen  
in Phys. und Technik, Reihe A, Bd.22  
Akademische Verlag. 38 DM 1960.
- 8) Mikhlin, S.G. "Integral equations and their  
applications to certain problems in mechanics,  
mathematical phys. and technology," 2nd rev.  
ed., International series of monographs in  
pure and applied math.  
Pergamon, London ¥ 4,800 1964.  
(----- "Linear integ. eqs."  
Hindustan ¥ 3,600. 1960.)
- 9) Wiarda, G. "Integralgleichungen unter besondere  
Berücksichtigung der Anwendungen."  
Teubner, Leipzig 1930.  
紀伊国屋, セロックス版, ¥ 3,170.  
理論的
- 10) 吉田耕作 "積分方程式論" 岩波全書 1950.
- 11) Muskhelishvili, N.I. "Singular integral equations"  
Groningen, 28.5 Dutch florin 1953.
- 12) Tricomi, F.G. "Integral eqs."  
Interscience Publisher, N.Y. 1957.
- 13) Vekua, N.P. "Systems of singular integ. eqs. and  
some boundary problems!" Gosdarstv Izdat. Tehn-  
Teor. Lit.  
Moskow-Leningrad 1950.
- 14) Anselone, P.M. "Nonlinear integral equations"  
U. of Wis. \$ 6.50 1964.

- 15) Krasnosel'skii, M.A. "Topological methods in the theory of nonlinear integ. eqs."  
Pergamon ¥ 4,260 1964.
- 16) Khantrovitch, L.V. "Functional analysis and applied math." in G.E. Forsyth ed. "National Bur. Standards Report, 1509 1962.  
-----, and Akhilov. "Functional analysis in normed spaces" International Monogs. on Advanced Math. and Phys.  
Pergamon £ 5-0-0. 1964.
- 17) Sobolev, S.L. "Applications of functional analysis in mathematical phys." Translation of math. monographs, 7  
Amer. Math. Soc. ¥ 3,015 1963.
- 18) Noble, B. "Methods based on the Wiener-Hopf technique for the solution of partial differential eqs."  
Pergamon, N. Y. 1958.

#### 應用

- 19) Gakhov, F.D. "Boundary value problems."  
Pergamon £ 4-0-0. 1963.
- 20) Chandrasekhar, S. "Radiative transfer."  
Oxford \$ 7.00. 1950.
- 21) Kourganoff, V. "Basic method in transfer problems. Radiative equilibrium and neutron diffusion"  
Oxford \$ 7.00. 1952.

(方程式  $J(\tau) = \frac{1}{2} \int_0^\infty J(t) E_1(|t-\tau|) dt$  について)

#### § 3 積分方程式の解法

1) 厳密解, 解析的方法. (参考: 教科書, 文献 6. 48)-76))

例. Abel's eq.  $\int_0^s \frac{f(t)}{\sqrt{s-t}} dt = g(s).$

解:  $f(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t \frac{g(s)}{\sqrt{t-s}} ds.$

Cauchy 核 をもつ eq.

$$\frac{1}{2\pi} \int_{-1}^1 \frac{f(t)}{\lambda - t} dt = g(\lambda).$$

$$\text{解: } f(t) = \frac{1}{\pi} \frac{1}{\sqrt{1-t^2}} \left[ C - \int_{-1}^1 \frac{g(\lambda)}{t-\lambda} \sqrt{1-\lambda^2} d\lambda \right]$$

Wiener-Hopf の方程式

## 2) 近似解法 (正規核 及び 特異核)

(i) 特異核を正規な核に変換して, 通常の近似法を用いる. (参考: §2 の教科書, 文献 5. の 44) - 47.)

例 Hamel (教科書 5) P140.

Bückner (教科書 3) pp 443 - 444).

(ii) 核を縮退核で近似. (参考: §2 の教科書 1), 3), 4), 文献 29) - 31).)

例 Bateman の方法, 直交関数展開, Galerkin 法.

(iii) 逐次近近似法 iterative methods.

(参考: 教科書 1), 3), 4) etc. 文献 1. の 1) - 23).)

例 第 2 種 方程式 に対して

Neuman-Liouville, Schmidt, Wiarda,

Bückner, Schönberg, Bellman, Wagner, ...

第 1 種 方程式 に対して

Berg, ...

(iv) 変分法

(v) Analog methods, 数値積分公式 または 補間公式 を  
使う. (参考: 教科書, 中田 1) pp 28-32, P40, 日高 2),  
文献 4 の 32) - 43).)

$$\int K(s, t) f(t) dt \doteq \sum K_{ij} f(t_j).$$

特異方程式に対して,

(イ) Kryloff の方法 (中田 1) p40).

$$K(s, t) = \frac{d}{dt} H(s, t) \text{ とすると}$$

$$\int K f dt = \int f dH \doteq \sum f(t_n) \Delta H$$

(ロ) Nystrom の方法 (日高 2)).

補間公式を用いる。煩雑。

(ハ) Sofronov (文献 35) または Math. Rev. 19 (1958) p. 66.)

(ニ) 簡便な方法 (中田 1) p31).

$$\lim_{t \rightarrow s} K(s, t)(t-s) = 0 \text{ のとき}$$

$$\begin{aligned} \int K(s, t) f(t) dt &= \int K(s, t) \{f(t) - f(s)\} dt \\ &\quad + f(s) \int K(s, t) dt. \end{aligned}$$

右辺の初項は通常の数値積分公式を使う。第2項は  $K$  が解析的に積分できないときは,

$$K = K_0 + (K - K_0)$$

とすると ( $K_0$ :  $K$  の principal part, 初等関数)

$$\int K dt = \int K_0 dt + \int (K - K_0) dt$$

(ホ) Young, (Aso),

$$K = K_0 P + (K - K_0 P),$$

$P$ : 多項式 (または正則関数)

数値積分公式  $\int K_0 F(t) dt = \sum_j w_j(s) F(t_j)$ .

$$\int K f dt = \int K_0 P f dt + \int (K - K_0 P) f dt.$$

$K_0 P$  の選択に注意.

- { 特異点近傍で  $K$  のよい近似.
- { 特異点から離れた所でも  $K$  との差は大きくならない.

失敗例 ----- § 4.

(vi) Monte Carlo 法

#### § 4. 計算失敗例.

流れに垂直な平板を過ぎる粘性流 (Oseen 近似)

$$l = R/4, \quad (k = U/2\nu) = 1, \quad R = 2lU/\nu = 4kl.$$

$$\text{Eq. } \int_{-l}^l [K_0(y-\eta)F(\eta) + iH(y-\eta)\bar{F}(\eta)] d\eta = 4\pi U.$$

$$\text{ただし } H(Y) = (1/Y) - K_1(Y),$$

$K_0, K_1$ : 変形ベッセル関数.

$F(y)$  は  $y = \pm l$  で  $1/\sqrt{l^2 - y^2}$  なる形の特異性を持つ。  
よって

$$\text{変換: } y = l \sin t, \quad \eta = l \sin \tau,$$

$$f(t) = l \cos t \cdot F \equiv f_1 + if_2.$$

数値解 — § 3 (v) (ホ) の方法による.

数値積分公式  $\int_{-1}^1 f(x) \ln |x - \xi_m| dx = \sum_n w_n(\xi_m) f(x_n),$

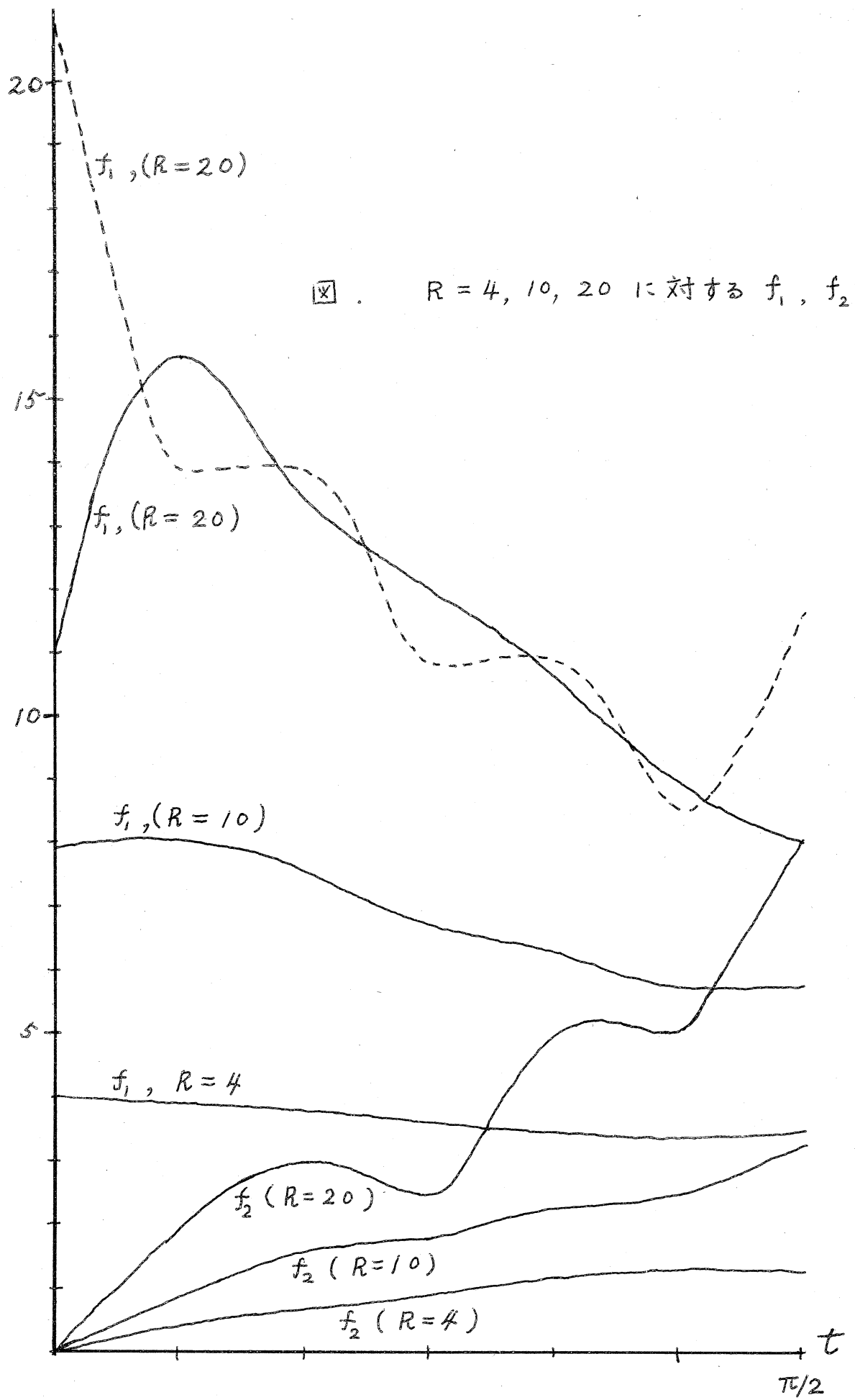
係数  $w_n(\xi_m)$  はあらかじめ求めてある。

$f_1(t), f_2(t)$  はそれぞれ偶, 奇関数であるので積分域を  $[0, \pi/2]$  とする。更に変換  $(t, \eta \rightarrow x, \xi)$  して  $[-1, 1]$  とする。

$$K_0(y-\eta) = \ln 2 - \gamma - \ln\left(\frac{\pi}{4} \ell \cos \tau\right) \\ - \ln|x-\xi| \left[ 1 + \frac{1}{4} \ell^2 \left\{ \cos^2 \tau (t-\tau)^2 - \cos \tau \sin \tau (t-\tau)^3 \right\} \right] \\ + \dots$$

$$H(y-\eta) = \dots \square \dots$$

そこで  $K_0P$  として上式の  $\square$  まで, または終までを採用する。前者による計算結果は図の実線, 後者によるものは点線をもって示されてある。区間  $[-1, 1]$  は 6 等分した。後者の方が, 労多くしてかえって悪い結果に導いた。ただし, 低 Reynolds 数では, 両者は一致した。Y が大なるとき  $K_0(Y) \rightarrow 0$  であるのに,  $\ln|Y| \cdot Y^3$  等はかなり大となり,  $\int (K_0 - K_0P) f dt$  の評価において誤差を大きくしたものと思われる。



## 5. 文献

- (i) 主として Mathematical Review (vol.13, 1950 以降) より抄録. MR. と略記する。
  - (ii) 次に関するものはできるだけ除いたが、厳密ではない。  
固有値問題, Volterra 型, 微積分方程式, 純理論, ロシア語の論文。
  - (iii) 分類は Math. Review により見当をつけたが、誤りも多いと思われる。
  - (iv) 英訳ソビエト雑誌には次のようなものがある。(筆者の手許には(a)があるが純数学的で非常に読みにくい。)
- a) Dokl. Akad. Nauk SSSR, 1960 -----  
Soviet Mathematics, Amer. Math. Soc.
  - b) Mathematical series, Bulletin of the USSR Acad. of Sci., vol.24 (1960) -----. Hindustan.
  - c) Math. Symposium, USSR Acad. of Sci. and the Moskow Math. Soc. vol.51 (1961) -----. Hindustan.
1. Iterative method.  
(See References in Bückner's text-book.)
- 1) Schönberg, M. Sur la méthode d'iteration de Viarda et Bückner pour la resolution de l'equation de Fredholm, I. Acad. Roy. Belg. cl.Sci., Ser.37, vol.5(1951)1141-1156.
  - 2) Bellman, R. Notes on summability of formal soluts. of linear int. integ. eqs.  
Duke Math. J. 17 (1950) 53-55.
  - 3) Frey, T. Über ein neues Iterationsverfahren zur Lösung von Integral-Integrodifferentialgleichungen,  
Symposium on the numerical treatment of ordinary diff. eqs integral and integro-diff. eqs., (Rohm, 1960) 530-548.

- 4)) Samuelson, P.A. Rapidly converging soluts. to integ. eqs.  
J. Math. Phys., 31(1953) 276-286

$$\left\{ \begin{aligned} f_{n+1} &= f_n - E(g, f_n, K) - j E(g, f_n, K), \quad E(g, f, K) \equiv g - f - \int K f. \\ j &= 0 \quad \text{ととる}, \quad \text{Neuman series.} \\ j &= \text{const.} \neq 0, \quad \text{Wiarda, Bückner.} \\ j &= -\bar{K}(s) / [1 + (b-a)\bar{K}], \quad \text{C. Wagner の方法,} \\ \text{また } \bar{K} &\equiv \int_a^b K(s, t) dt / (b-a), \quad \bar{\bar{K}} \equiv \int \bar{K}(s) ds / (b-a). \end{aligned} \right.$$

- 5)) Muller, C. A new method of solving Fredholm integ. eqs.  
Comm. Pure Appl. Math. 8(1955) 635-640. (MR17, p1215)  
-----
- 6)) Sokoloff, G.D. Sur la methode de moyennage des corrections fonctionnelles, (French summary).  
Ukrain. Mat. Z. 9(1957) 82-100. (MR19, 687)
- 7)) Sokolov, Yu.D. On a method of approximate soluts. of linear integ. and differential eqs.  
Dopovidi Akad. Nauk Ukrain. RSR (1955) 107-111. (MR17, p196)
- 8)) Gernitsenko, E.A.  
Ukrain. Mat. Z. 6(1954) 305-313. (MR17, p665)
- 9)) -----, On a variant of the method of averaging.  
Dopovidi Akad. Nauk Ukrain RSR 1956, 10-12. (MR17, p901)
- 10)) Hurka, A.Yu. A sufficient conditions for the convergence of the procedure for averaging functional corrections.  
Dokl. Akad. Nauk SSSR 122(1958) 179-182. (MR20, 4922)

---

11))-13)): eqs. of the first kind.

- 11) Fridman, V.M. Method of successive approximations for a Fredholm integ. eq. of the first kind.  
Uspehi Mat. Nauk (N.S.) 11(1956) no.1(67)233-234.  
eq.  $\int_a^b K f = g$ ,  $K$ : 対称, positive definite.  
solut.  $f_n = f_{n-1} + \lambda [g - \int K f_{n-1}]$ ,  $0 < \lambda < 2\lambda_1$ , ( $\lambda_1$ : 最小固有値)
- 12) Landweber, L. An iterative formula for Fredholm integ. eqs of the first kind.  
Amer. J. Math. 73(1951) 615-624. (MR13,p247).
- 13) Berg, L. Lösungsverfahren für singulare Integralgleichungen, I.  
Math. Machr. 14(1955) 193-212. (MR19,p66).  
(solut.  $f_n = f_{n-1} + \alpha_{n-1}(\lambda)(g - \int K f_{n-1})$ ).

14)-17): applications of iterative methods.

- 14) Bismas, S.N. Solutions of Heitler's integ. eqs. by iteration meth.  
Phys. Review (2) 94(1954) 1767-1772. (MR16,p79)
- 15) Stiefel, E. On solving Fredholm integ. eqs. Applications to conformal mapping and variational problems of potential theory.  
J. Soc. Indust. Appl. Math. 4(1956) 63-85. (MR19,p324)
- 16) Warschawski, S.E. Recent results in numerical methods of conformal mapping.  
"Proceedings of Symposia in Appl. Math. vol VI Numerical Analysis" (1956) 219-250. (MR19,p980)
- 17) Todd, J. and Warschawski, S.E. On the solut. of the Lichtenstein-Gersgorin integ. eq. in conformal mapping, I, II,  
in J. Todd, ed., "Experiments in the computation of conformal mappings." National Bur. Standards Appl. Math. Series, vol.42(1955) 31-44. (MR17,p54)

追加

- 18)? Karush, W. Convergence of a method of solving linear problems.  
Proc. Amer. Math. Soc. 3(1952) 831-851.
- 19)? Collatz, L. Fehlerabschätzungen zum Iterationsverfahren bei linearen und nichtlinearen Randwertaufgaben.  
Z. Angew. Math. Mech. 33(1953) 116-127  
(MR13, p67)
- 20)? ----- . Einige Anwendungen funktionalanalytischer Methoden in der praktischen Analysis.  
Z. Angew. Math. Phys. 4(1953) 327-357.
- 21)? Newmark, N.M. Bounds and convergence of relaxation and iteration procedures. "Proceedings of the first U.S. National Congress of Applied Mechanics," Chicago, 1951, 9-14. Amer. Soc. Mechanical Engineers, N.Y. 1952.
- 22)? Kikuta, Takashi. Convergence of iterative methods. Progress Theoret. Phys. 10(1953). (MR15, p77)
- 23) Bauer, F.L. Der Newton-Prozess als quadratisch konvergente Abkürzung des allgemeinen linearen stationären Iterationsverfahrens I.  
Z. Angew. Math. Mech. 35(1955) 469-470 (MR17, p666)

## 2. 変分法, 他.

- 24) Huang, Su-Schu. The variational method for problems of radiative transfer, I, II, III  
Astrophys. J. 117(1953) 211-214, 215-220, 221-224.  
(eq.  $f(s) = \frac{1}{2} \int_0^\infty f(t) E_1(|t-s|) dt$ . (MR14, p765)
- 25) Bellman, R. and Lattès, R. On the integ. eq.  
 $\lambda f(x) = \int_0^a K(x-y) f(y) dy$ .  
Proc. Amer. Math. Soc. 3(1952) 884-891 (MR14, p764)
- 26) Noble, B. The numerical solution of the singular integ. eq. for the charge distribution on a flat rectangular lamina. Symposium on the numerical treatment of ordin. differ. eqs., integ. and integro-differ. eqs. (Rohm) 1960, 530-548 (MR23, no.4B)

$$\text{eq. } \int_a^b \int_a^b f(\xi, \eta) [(x-\xi)^2 + (y-\eta)^2]^{-1/2} d\xi d\eta = g(x, y),$$

maximum of

$$I(F) = 2 \iint F dx dy \\ - \iiint F(\xi, \eta) F(x, y) [(x-\xi)^2 + (y-\eta)^2]^{-1/2} dx dy d\xi d\eta,$$

where  $F = \sum \sum \alpha_{rs} f_{rs}(x, y)$ .

- 27) Grout, P.D. At least square procedure of solving homogeneous integ. eqs.  
J. Math. Phys. 40(1961) 103-115 (MR26, 3220)

$$\text{eq. 同次式 } f = \lambda \int K f. \quad \text{Minimize } E = \int_a^b [f - \lambda \int K f]^2.$$

- 28) Kantrovitch, L.V.  
Uspehi Mat. Nauk (N.S.) 3(1948) no.6(28) 87-185  
(MR10, p380, MR14, p766)

### 3. 核の展開, 縮退核, 直交級数, Galerkin 法.

- 29) Bateman, H. Numerical solutions of linear integ. eqs.  
Proc. Roy. Soc. London, ser.A, 100(1922) 441-449
- 30) Thomson, G.T. On Bateman's method for solving linear integ. eqs.  
J. Assoc. Comput. Machine 4(1957) 314-328 (MR20, 1423)
- $$\begin{cases} K(s, t) \rightarrow H_n(s, t) \\ H_n(s, t) = \sum \sum \alpha_{ij} K(s, t_j) K(s_i, t) / W_n, \quad W_n = \det(K(s_i, t_j)) \end{cases}$$
- 31) Shinbrot, M. A class of difference kernels.  
Proc. Amer. Math. Soc. 13(1962) 399-406 (MR25, 1414)

$$\text{eq. } \int_{-1}^1 K(|s-t|) f(t) dt = g(s). \quad K \text{ を Fourier 展開.}$$

### 4. Analog method (数値積分公式, 補間公式等の応用)

- 32) Misovskih, I.P. Estimate of the error involved in the numerical solut. of a linear integ. eq.  
Soviet. Math. 2(1961)

$$\begin{aligned}
& f(s) - \lambda \int_a^b K f dt = g(s), \quad \text{求積法による} \int_a^b F(t) dt \approx \sum_{j=1}^n A_j F(t_j) \\
& \therefore \int K(s_i, t) f(t) dt \approx \sum L_{ij} f_j, \quad \text{近似解のベクトル} \tilde{f} = (f_j) \text{ とする.} \\
& \text{ベクトル } p \equiv (I - \lambda L) \tilde{f} - g, \quad (I: \text{単位行列}) \\
& \text{Put } \varepsilon_f(s) \equiv \int K_f - \sum A_j K(s, t_j) g(t_j), \quad E(s, t) \equiv \int K K - \sum A_j K(s, t_j) K(t_j, t) \\
& \quad \varepsilon_i \equiv \max_{a \leq s \leq b} \sum_{j=1}^n |A_j E(s, t_j)|, \quad \varepsilon \equiv \max_{1 \leq i \leq n} \int_a^b |E(t_i, t)| dt, \\
& \quad K_i \equiv \max_{a \leq s \leq b} \sum_{j=1}^n |A_j K(s, t_j)|, \quad R(s, t, \lambda) \equiv K(s, t) \text{ の} \\
& \quad \max \int |R(s, t, \lambda)| dt \leq P, \quad B \equiv 1 + |\lambda| P, \quad q \equiv \lambda^2 \varepsilon, B \leq 1, \\
& \quad \|f - \tilde{f}\| \leq (|\lambda| \|E_g\| + |\lambda|^2 B \|f\| \varepsilon + \|P\|) (1 + |\lambda| K_i B) / (1 - q).
\end{aligned}$$

従来の誤差評価よりも精密である。ただし煩雑。

- 33) Good, I.J. On the numerical soluts. of integ. eqs.  
Math. Tables Aids, Comput. 11(1957) 82-83  
(MR19, p180)

- 34) Brakhage, H. Über die numerische Behandlung von  
Integ.gleichungen nach der Quadratureformeltheorie.  
Numer. Math. 2(1960) 183-196

-----  
singular eqs.

- 35) Sofronov, I.D. On approx. solut. of singular integ. eqs.  
Dokl. Akad. Nauk SSSR(N.S.) 110(1956) (MR18, p906)  
111(1956) (MR19, p66)

$$[\text{eq. } a(s)f(s) + \frac{1}{2\pi} \int_0^{2\pi} n(t, s) \cot \frac{1}{2}(t-s) f(t) dt = g(s).]$$

- 36) Nyström, E.J. Zur numerischen Lösung von Randwertaufgaben bei gewöhnlichen Differentialgleichungen.  
Acta. Math. 76(1944) 157-184  
(See 日高教科書 (2).)

- 37) Yong, A. Approximate product -integrations.  
Proc. Roy. Soc. London Ser. A 224(1954) 552-561  
(MR16, p179)

- 38) Yong, A. The application of approx. product integration to the numerical solution of integ. eqs.  
ibid. 561-571 (MR16, p17)
- 39) Aso, Y. A numerical method of solving singular integ. eqs. Sci. Reports, Gunma Univ. 13(1965)
- 40) See Todd, Warschawski 17)
- 41) Symm, G.T. Integ. eq. methods in potential theory, II  
Proc. Roy. Soc. London Ser. A 275(1963) 33-46
- 42) Kantrovitch, L.V., Krylov, V.I. and iCernin, K.E.  
"Tables for numerical solut. of boundary value problems in the theory of harmonic functions," 1956  
Constable, 1964, L5-0-0. Unger, S 12.50 (MR19, p687)
- Dirichlet problem:  
domain 1) rectangle 2) half-plane 3) strip 4) half-s  
5) angle 6) circle 7) half-circle
- 43) Vzeroval, A.I. "Tables for solut. of the Laplace eq. in elliptic domains"  
Izdat. Akad. Nauk. SSSR, Moskow, 1957, 30 rubles (MR19, p637)
5. Regularization (Singular Regular)
- 44) Mihlin,  
Uspehi Matem. Nauk (N.S.) 3no. 3(25)(1948) 29-112 (MR10, p)
- 45) Ganin, M.P. The equivalent regularizing operator for a system of singular integ. eqs.  
Doklady Akad. Nauk SSSR (N.S.) 79(1951) 385-387 (MR13,
- 46) Hvdeldidze, B.V. Regularization problem in the theory integ. eqs. with Cauchy kernel.  
Dokl. Akad. Nauk SSSR 140(1961) 66-68 (MR26, 563)
- 47) Elliot, J. On a class of integ. eqs.  
Proc. Amer. Math. Soc. 3(1952) 566-572 (MR14, p180)

## 6. 嚴密解, 解析的方法.

48), 49) ---

- 48) Peters, A.S. A note on the integ. eq. of the first kind with a Cauchy kernel.  
Comm. Pure Appl. Math. 16(1963) 57-61 (MR26,5372)
- 49) Byrd, P.E. An integ. eq. occurring in the theory of a slender quasi-axisymmetric body.  
J. Aeronaut. Sci. 21(1954) 351. (MR15,p716)
- 50) Tricomi, F.G. The airfoil equation for a double interval.  
Z. Angew. Math. Phys. 2(1951) 402-406 (MR13, )

$$\left\{ \begin{array}{l} \text{eq. } \frac{1}{\pi} \left[ \int_{-1}^{-k} + \int_1^k \right] \frac{\varphi(y)}{y-x} dy = f(x), \quad k < x < 1. \\ \text{sol. } \varphi = g - \left\{ \frac{1}{\pi} \int_{-k}^k [(1-y^2)(k^2-y^2)]^{\frac{1}{2}} \frac{g(y)}{y-x} dy + C_1 x + C_2 \right\}, \\ \text{where } C_1, C_2: \text{ arbit. consts.} \\ g(x) = -\frac{1}{\pi} \left( \int_{-1}^{-k} + \int_k^1 \right) \left( \frac{1-y^2}{1-x^2} \right)^{\frac{1}{2}} \frac{f(y)}{y-x} dx. \end{array} \right.$$

- 51) -----, Equazioni integrali singolari del tipo Carleman.  
Ann. Mat. Pure Appl. (4) 39(1955) 229-244 (MR18,p49)

$$\text{eq. } a(\lambda)f(\lambda) - \lambda \int_{-1}^1 \frac{f(t)}{t-\lambda} dt = g(\lambda).$$

- 52) Leonov, M.Ya. Solutions of an integ.eq. of the theory of the Newtonian potential.  
Ukrain Mat. Z. 5(1953) 50-57

$$\text{eq. } u(x, y) = \iint_S p(\xi, \eta) r^{-1} d\xi d\eta, \quad S = \{x^2 + y^2 \leq a^2\}.$$

- 53) Jackel, K. Über die Eigenlösungen gewisser Integ. gleichungen der Potentialtheorie.  
J. Reine Angew. Math. 189(1951) 141-147 (MR13, ).

- 54) -----, Herleitung von Umkehrformeln und Identitäten zur Behandlung besonderer Integ.gleich.  
Z. Angew. Math. Mech. 35(1955) 474-475 (MR17,p861)

$$\text{eq. } 53) \int_{-1}^1 f(t)(1-t^2)^{-\frac{1}{2}} \ln |t-\lambda| dt = \lambda g(\lambda), \quad \lambda \in \mathbb{C}^+ \\ \frac{d^2}{d\lambda^2} \int_{-1}^1 f(t)(1-t^2)^{\frac{1}{2}} \ln |t-\lambda| dt = \lambda g(\lambda).$$

$$54) \quad f(\lambda) = -\frac{1}{2\pi} \int_0^\pi g(t) \ln \frac{1-\cos(\lambda+t)}{1-\cos(\lambda-t)} dt \\ g(t) = \frac{1}{2\pi} \int_0^\pi f''(\lambda) \ln \frac{1-\cos(\lambda+t)}{1-\cos(\lambda-t)} d\lambda.$$

- 55) Dörr, J. Über zwei mit der Tragflügeltheorie in Zusammenhang stehende Integ.gleich.  
Ing-Arch. 20(1952) 88-90 (MR13, p54)

- 56) -----, Zwei Integralgleich. erster Art, die sich mit Hilfe Mathieuscher Funktionen lösen lassen.  
Z. Angew. Math. Phys. 3(1952) 427-439 (MR14, p19562)

$$\left. \begin{aligned} \text{eq. } g(x) &= \int_{-1}^1 f(y) Z_0(k|x-y|) dy, \\ g(x) &= \int_{-1}^1 f(y) \frac{|x-y|}{x-y} Z_1(k|x-y|) dy, \\ \text{f. f. L. } Z_n(x) &= A J_n(x) + B N_n(x). \end{aligned} \right\}$$

- 57) Busbridge, I.W. On soluts. of the non-homogeneous form of Milne's first integ. eq.  
Quart.J.Math.Oxford ser.(2) 6(1955) 218-231 (MR18, p49)

$$\text{eq. } f(s) = \frac{1}{2} s \int_0^\infty E_1(1s-t) f(t) dt + g(s),$$

$E_n$ : the n-th expon. integ.

58)-72) Dual eqs. triple eqs.

- 58) Heins, A.E. and Mactamy, R.C. A function theoretic solution of certain integ. eqs., I, II.  
Quart.J.Math. Oxford ser(2) 9(1958) 132-143 (MR26, 5382)  
ibid. 10(1959) 280-293 (MR21, 1508)

$$\text{eq. } \begin{cases} f(s) + \int_0^\infty K(s-t) f(t) dt = g(s) & s > 0 \\ f(s) + \int_0^\infty K(s+t) f(t) dt = g(s) & s < 0 \end{cases}$$

$$\text{f. f. L. } K(x) = O(\ln|x|) \quad (x \rightarrow 0).$$

- 59) Gahov, F.D. and Cerskii Yu I. Singular int. eqs. of convol. type.

Izv.Akad.Nauk SSSR Ser.Mat.20(1956) 33-52 (MR

$$\text{eq. } f(s) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty k_1(s-t) f(t) dt = g(s) \quad (0 < s < \infty)$$

$$f(s) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty k_2(s-t) f(t) dt = g(s) \quad (-\infty < s < 0).$$

$$\text{solut. } f(s) = \frac{1}{\sqrt{2\pi}} \int_{ib_1-\infty}^{ib_1+\infty} \phi^+(\zeta) e^{-is\zeta} d\zeta - \frac{1}{\sqrt{2\pi}} \int_{ia_2-\infty}^{ia_2+\infty} \phi^-(\zeta) e^{-is\zeta} d\zeta.$$

f. f. L.  $\phi(\zeta)$ : Riemann 境界値問題の適当な解.

- 60) Gahov, F.D. and Smagina, V.I. Exceptional cases of integ. eq. of convolution type and eqs. of the first kind.

Soviet Math. 2(1961) 184-187

$$\text{eq. } \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_1(\lambda-t)f(t)dt = g(\lambda) & 0 < x < \infty \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_2(\lambda-t)f(t)dt = g(\lambda) & -\infty < x < 0. \end{cases}$$

61)-72) Dual integ. eqs. 核はベッセル関数を含む.

- 61) Gordon, A.N. Dual integ. eqs.

J. London Math. Soc. 29(1954) 360-363

- 62) Copson

Proc. Edinburgh Math. Soc. (2) 8(1947) 14-19  
(MR9, p318)

- 63) -----,

Proc. Glasgow Math. Assoc. 5(1961) 21-24

- 64) Sneddon, I.N. ibid

ibid. 4(1960) 108-110 (MR22, 9825)

- 65) -----, Dual integ. eqs. with trigonometrical kernels.

ibid. 5(1962) 147-152 (MR25, 1412)

- 66) Erdelyi, H.C. and Sneddon, I.N. Fractional integration and dual integral eqs.

Canad. J. Math. 14(1962) 685-693 (MR26, 564)

- 67) Lowengrub, M. and Sneddon, I.N. The solution of a pair of dual integ. eqs.

Proc. Glasgow Math. Assoc. 6(1963) 14-18 (MR26, 6715)

- 68) Burlack, J. Proc. Edinburgh Math. Soc. (2) 13(1962/63) 179-187

(MR26, 5384)

- 69) -----, On the solution of certain dual integ. eqs.

Proc. Glasgow Math. Assoc. 6(1963) 39-44 (MR26, 6716)

- 70) Noble, B. J. Math. Phys. 37(1958) 128-136 (MR20, 4754)

- 71) -----, The solut. of Bessel funct. dual integ. eqs. by a multiplying factor method.

Proc. Cambridge Philos. Soc. 59(1963) 351-362

(MR26, 2842)

72) Williams, W.E. A class of integ. eqs.  
 ibid. 59(1963) 589-597 (MR26,6710)

73) Cooke, J.C. Triple integ. eqs.  
 Quart. J. Mech. Appl. Math. 16(1963) 193-203 (MR27,553)

$$\left[ \begin{array}{l} \text{eq. } \left\{ \int_0^{\infty} J_{\nu}(\lambda t) f(t) dt = 0 \quad (0 < \lambda < a, \text{ and } b < \lambda < \infty) \right. \\ \left. \int_0^{\infty} J_{\nu}(\lambda t) t^p f(t) dt = g(\lambda) \quad (a < \lambda < b) \right. \end{array} \right]$$

74) Wu, Tai Te and Wu Tai Tsum. Iterative soluts. of W-H.  
 int. eqs.  
 Quart. Appl. Math. 20(1962/63) 341-352 (MR25,5357)

$$\left[ \text{eq. } f(\lambda) - \int_0^{\infty} K(\lambda - t) f(t) dt = g(\lambda), \quad (\lambda > 0) \right]$$

75) Kranzer, H.C. and Radlow, J. Asymptotic factorization  
 for perburbed Wiener-Hopf problems.  
 J. Math. Anal. Appl. 4(1962) 240-256 (MR27,1795)

76) Ivanov, V.V. On the W-H. eqs. of the first kind.  
 Sov. Math. 4(1963) 1021-1024

## 7. Monte Carlo method

77) "Symposium on Monte Carlo methods." Univ. of Florida,  
 1954, John Wiley, N.Y., Chapman and Hall, London,  
 1956.

78) Cutkosky, R.E. A Monte Carlo method for solving a class  
 of integ. eqs.  
 J. Research Nat. Bur. Standards 47,113-115(1951)  
 (MR13,p590)

79) Nygaard, K. On the solut. of integ. eqs. by M.C. method.  
 Norwegian Defence Research Establishment, Rep.no.F-R94  
 (1952)

80) Curtis, J.H. "Monte Carlo" methods for the iteration of  
 linear operator.  
 J. Math. Phys. 32, (1954) 209-232 (MR15,p560).

81) Page, E.S. The Monte Carlo solution of some int. eqs.  
 Proc. Cambridge Philos. Soc. 50(1954)414-425 (MR16,p291)

- (82) Pavis, P. and Rabinowitz, P. Some M.C. experiments in computing multiple integrals.  
Math. Tables Aids Comput. 10(1956)1-8(MR17,p901)
- (83) Sobol, I.M. Solut. of the Peierl integ. eq. by the M.C. method.  
Tetr. Verojatnost i Primenen 5(1960)360-366  
(MR26,5381)
- (84) Sugiyama, Hiroshi and Joh, Kenzo A numerical procedure of conformal mapping in case of simply, doubly and multiply connected domains from the view point of M.C. approach, I.  
Tech. Rep. Osaka Univ. 12(1962) 1-9

8. Nonlinear eqs.

- (85) Busbridge, I.W. On soluts. of Chandrasekhar's integ. eq.  
Trans. Amer. Math. Soc. 105(1962) 112-117  
eq.  $\left[ \frac{1}{f(\lambda)} = 1 - \lambda \int_0^1 \psi(t)(\lambda+t)^{-1} f(t) dt \right]$   
(MR25,5358)
- (86) Bückner, H.F. An iterative method for solving non-linear integ. eqs.  
Symposium of the numerical treatment of ordinary diff. eqs., integral and integro-differential eqs. (1960)  
(MR23 ).
- (87) Schmeidler, W. Algebraisch Integralgleichungen, I.II.  
Math. Nachr. 8(1952) 31-40 (MR14,p382)  
ibid. 10(1953)247-255 (MR15,p434)
- (88) -----, und Morgenstern, D. Zur Übertragung des Alternative-Satzes der Fredholmschen Theorie auf algebraischer Integ.gleich.  
Arch. Math. 5(1954) 452-457
- (89) -----, ReeLe Lösungen algebraischen Integ-gleich.  
Univ. Nac. Jacuman Rev. Ser A 13(1960)136-157  
(MR27,556)

- 90) ----- Math. Z. 72(1959/60) 279-293
- 91) ----- J. Reine Angew. Math. 210(1962) 105-112  
(MR27,557)
- 92) Cameron, R.H. and Shapiro, J.M. Nonlinear integ. eqs.  
Ann. of Math. (2) 62(1955) 472-497 (MR17,p632)
- 93) Miller, G.F. A note on the numerical solut. of certain  
nonlinear integ. eqs.  
Proc. Roy. Soc. London Ser A 236(1956) 529-534
- 94) Salehov, G.S. Dokl. Akad. Nauk SSSR (N.S.) 82(1952)  
525-528 (MR14,p91)
- 95) Mertvecoa, M.A. ibid. 88(1953) 611-614
- 96) Lumiste, Yu. The method of steepest descent for nonlin.  
eqs. Tartu. Gos. Univ. Trudy Estest. Math. Fak. 37  
(1955) 106-113 (MR17,p900)
- 97) Ul'm, S. On convergence of certain iterat. processes  
in Banach space.  
Uc. Zap. Tartu. Gos. Univ. 42(1956) 135-142 (MR19,p687)
- 98) Kaazik, Yu Ya. On approx. solution of nonlinear operator  
eqs. by iterative method.  
Uspehi Mat. Nauk (N.S.) 12(1957) no1(73)195-199  
(MR19,p687)
- 99) Kivistik, L. The method of steepest descent for nonlin.  
eqs. Eesti NSV Tead. Akad. Toimentised Fuus-Mat. Tehn.  
Tead. Ser 9(1960) 145-159 (MR24B, 523)
- 100) Babaev, A.A. Perturbation method for a class of nonlin.  
singul. integ. eqs. with Cauchy kernel.  
Akad. Nauk Azerbaidzan SSR Trudy Inst. Mat. Mech. 1(9)  
(1961) 158-163 (MR27,555)
- 101) Pohozaev, S.I. Analog of Schmidt's method for nonlin. eqs.  
Sov. Math. 2(1961) 103-105
- 102) Simeonov, S.V. A process of successive approximations and  
its use in solving functional eqs. with nonlin. opera-  
tor of monotonic type.  
Sov. Math. 2(1961) 790-792
- 103) Fridman, V.M. An iterative process with minimum errors for  
nonlinear operator equation.  
Sov. Math. 2(1961) 1058-1062

- 104) Sokolov, J.D. On a method of approximate solut. of systems of nonlin. integ. eqs. with constant limits.  
Ukrain. Mat. Z. 15(1963) 58-70 (MR26,5385)

etc.

- 105) Erdelyi, A. An example in singular perturbation.  
Monatsch. Math. 66(1962) 123-128
- 106) ----- A result on non-lin. Volterra integ. eqs.  
"Studies in mathemat. analysis and related topics."  
pp104-109 Stanford Univ. Press, 1962
- 107) Sato, T. Mem. Fac. Sci. Kyushu Univ. A5(1950) 65-67  
(MR12,p519)
- 108) Sato, T. and Iwasaki, A.  
Proc. Japan Acad. 31(1955) 395-398 (MR17,p751)
- 109) Iwasaki, A. On non-lin. integ. eqs. of Volterra-type.  
Funktial Ekvac. 3(1960/61) 105-114 (MR26,2838)
- 110) Nohel, J.A. Some problems in nonlin. Volterra integ. eqs. Bull. Amer. Math. Soc. 68(1962) 323-329  
(MR26,2838)

---

9 その他

- 111) Crout, P.D. J. Math. Phys. Mass. Inst. Tech. 19(1940) 34-92  
(MR1,p127)
- 112) Hildebrand, F.B. and Crout, P.D.  
ibid. 20(1941) 310-335 (MR3,p154)
- 113) Bergman, S. Some methods for solutions of boundary-value problems of linear partial diff. eqs.  
Proceedings of Symposia in Applied Math., Vol. VI  
Numerical Analysis, p11-29 MacGraw-Hill, 1956
- 114) Kupradse, W.D. "Randwertaufgaben der Schwingungstheorie und Integralgleichungen." Hochschulbuecher fuer Math. Band 21. VEB Deutscher Verlag der DE Wissenschaften 1956 (transl. from Russian)
- 115) Copson, E.T.  
Proc. Edinburgh Math. Soc. (2) 8(1947) 14-19  
(MR9,318, 10,855)

- 116) Collins, W.D.  
Proc. Cambridge Philos. Soc. 57(1961) 367-384  
(MR23B,204)
- 117) Williams, W.E. The reduction of boundary value  
problems to Fredholm integral eqs. of the second  
kind.  
Z. Angew. Math. Phys. 13(1962) 133-152. (MR25,414)
- 118) ----- Note on the reduction of dual and  
triple series eqs. to dual and triple integ. eqs.  
Proc. Cambridge Philos. Soc. 59(1963)731-734  
(MR27,4036)
- 119) Todd, T. The condition of certain integ. eqs.  
Symposium of numerical treatment of ordin. diff.  
eqs. and integ. and integro-diff. eqs. (Rohm), 1960
- 120) Jaswon, M.A. Integ. eq. methods in potential theory, I.  
Proc. Roy. Soc. Ser A 275(1963) 23-32
- 121) Loseth, A.T. Approximate soluts. of Fredholm-type  
integ. eqs.  
Bull. Amer. Math. Soc. 60, 415-430 (MR16,p291)  
(This paper mentions results of about 25 authors.)