

On certain conditions for starlikeness

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Abstract. The object of the present paper is to consider a sufficient condition for analytic functions in the open unit disk to be strongly starlike of order α .

1 Introduction.

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in A is said to be starlike in U if it satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U).$$

We denote by S^* the subclass of A consisting of all starlike functions $f(z)$ in U . Further a function $f(z)$ belonging to A is said to be strongly starlike of order α in U if it satisfies

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2} \alpha \quad (z \in U)$$

for some α ($0 < \alpha \leq 1$). We denote by $SS^*(\alpha)$ the subclass of A consisting of all strongly starlike functions of order α in U .

From the definition for strongly starlike functions of order α , we note that $f(z) \in SS^*(\alpha)$ is univalent and starlike in U . Recently, Tuneski [2] obtained the following theorem.

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Theorem A. Let a function $f(z) \in A$ satisfy

$$\frac{f(z)f''(z)}{f'(z)^2} \prec 2 - \frac{2}{(1-z)^2} \quad (z \in U),$$

where the symbol " $\prec$ " means the subordination. Then $f(z) \in S^*$.

To derive our main theorem, we need the following lemma due to Nunokawa [1].

Lemma. Let $p(z)$ be analytic in U with $p(0) = 1$ and $p(z) \neq 0$ ($z \in U$). If there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| \leq \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\alpha \quad (\alpha > 0),$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where $k \geq 1$ when $\arg(p(z_0)) = (\pi/2)\alpha$ and $k \leq -1$ when $\arg(p(z_0)) = -(\pi/2)\alpha$.

2 Strongly starlikeness of order α

Now we derive

Theorem. Let $f(z)$ in A satisfy the following inequalities

$$\pi - \frac{\pi}{2}\alpha - \tan^{-1}\alpha < \arg\left(\frac{f(z)f''(z)}{f'(z)^2} - 1\right) < \pi + \frac{\alpha}{2} + \tan^{-1}\alpha \quad (z \in U)$$

for some α ($0 < \alpha \leq 1$). Then $f(z)$ belongs to the class $SS^*(\alpha)$ in U .

Proof. From the assumption in the theorem, we see that $f'(z) \neq 0$ in U . Let us define the function $p(z)$ by $p(z) = zf'(z)/f(z)$. Then $p(z)$ satisfies

$$\frac{f(z)f''(z)}{f'(z)^2} = 1 + \frac{zp'(z)}{p(z)^2} - \frac{1}{p(z)}$$

and so

$$\frac{f(z)f''(z)}{f'(z)^2} - 1 = \frac{1}{p(z)} \left(-1 + \frac{zp'(z)}{p(z)} \right).$$

If there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\alpha,$$

then Lemma gives us that

(i) for the case $\arg(p(z_0)) = (\pi/2)\alpha$,

$$\begin{aligned} \arg \left(\frac{f(z_0)f''(z_0)}{f'(z_0)^2} - 1 \right) &= \arg \left\{ \frac{1}{p(z_0)} \left(\frac{z_0 p'(z_0)}{p(z_0)} - 1 \right) \right\} \\ &= -\frac{\pi}{2}\alpha + \arg \left(-1 + \frac{z_0 p'(z_0)}{p(z_0)} \right) \\ &= -\frac{\pi}{2}\alpha + \arg(-1 + ik\alpha) \\ &\leq \pi - \frac{\pi}{2}\alpha - \tan^{-1}\alpha. \end{aligned}$$

This contradicts our condition in the theorem.

(ii) for the case $\arg(p(z_0)) = -(\pi/2)\alpha$, the application of the same method as in (i) shows that

$$\arg \left(\frac{f(z_0)f''(z_0)}{f'(z_0)^2} - 1 \right) \geq \pi + \frac{\pi}{2}\alpha + \tan^{-1}\alpha.$$

This also contradicts the assumption of the theorem. Thus we complete the proof of our main theorem.

Putting $\alpha = 1$ in Theorem, we have the following corollary.

Corollary. *If $f(z) \in A$ satisfies*

$$\frac{\pi}{4} < \arg \left(\frac{f(z)f''(z)}{f'(z)^2} - 1 \right) < \frac{7\pi}{4} \quad (z \in U),$$

then $f(z) \in S^$.*

References

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