

A lower bound for the volume of Dirichlet fundamental polyhedrons

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1. Introduction. This paper is concerned with a lower bound for the volume of Dirichlet fundamental polyhedrons for Kleinian groups.

Let $H^3 = \{(x, y, t) \in \mathbb{R}^3; t > 0\}$ with metric $d(\cdot, \cdot)$ induced by the line element $ds^2 = (dx^2 + dy^2 + dt^2)/t^2$. Let f be an element of $PSL(2, \mathbb{C})$ and identifies with a Möbius transformation of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ onto itself, such as

$$f(z) = \frac{az+b}{cz+d}, \quad ad-bc=1.$$

Its action on the Riemann sphere $\hat{\mathbb{C}}$ can be naturally extended to H^3 .

Next for each f and g in $PSL(2, \mathbb{C})$, we let $[f, g]$ denote the commutator $fgf^{-1}g^{-1}$. We define the two complex numbers

$$(1) \quad \beta(f) = \text{tr}^2(f) - 4, \quad \gamma(f, g) = \text{tr}([f, g]) - 2,$$

for the two generator subgroup $\langle f, g \rangle$.

The following inequality [4] gives an important necessary condition for a two generator group $\langle f, g \rangle$ to be nonelementary and discrete.

Proposition 1([J1]). If $\langle f, g \rangle$ is nonelementary and discrete, then

$$(2) \quad |\gamma(f, g)| + |\beta(f)| \geq 1 \quad \text{and} \quad |\gamma(f, g)| + |\beta(g)| \geq 1.$$

The above inequality is called Jørgensen's inequality.

Proposition 2([R]). If $\langle f, g \rangle$ is a nonelementary discrete Fuchsian subgroup of $\text{PSL}(2, \mathbb{R})$, then

$$(3) \quad |\gamma(f, g)| \geq 2 - 2\cos(\pi/7).$$

On the other hand, if $\langle f, g \rangle$ is not Fuchsian, it is known that there is a nonelementary discrete group $\langle f, g \rangle$ with an arbitrary small number $|\gamma(f, g)|$ ([J2]). But if we have some restrictive conditions for elements of generators, one can obtain the lower bounds for $|\gamma(f, g)|$. If we have a restriction $\beta(f) = \beta(g)$ for two-generator group $\langle f, g \rangle$. Jørgensen proved in [J3] that

$$(4) \quad |\gamma(f, g)| \geq 1/8.$$

The first purpose of this paper is to show the existence of a collar from the consideration of a lower bound of $|\gamma(f, g)|$. This gives an improvement of a previous paper [2].

2 Preliminary. We collect some elementary results.

Lemma 1. If f and g are in $\text{PSL}(2, \mathbb{C})$ with $\gamma(f, g) = \gamma$ and $\beta(f) = \beta$, then

$$(5) \quad \gamma(f, gfg^{-1}) = \gamma(\gamma - \beta) \text{ and } \beta([f, g]) = \gamma(\gamma + 4).$$

Lemma 2. Let G be an elementary discrete subgroup of $\text{PSL}(2, \mathbb{C})$. If

$f, g \in G$ with $\gamma(f, g) \neq 0$, $\beta(f) = \beta(g) \neq -4$, then

$$(6) \quad |\gamma(f, g)| \geq 2 - 2\cos(\pi/5) (=0.3819\cdots).$$

It is easily seen that $\gamma(f,g) \neq 0$ if and only if $\text{fix}(f) \cap \text{fix}(g) = \emptyset$ where $\text{fix}(f)$ denotes the fixed point set in $\hat{C}$. Now we need to show the followings.

Lemma 3. If $\langle f,g \rangle$ is discrete with $\gamma(f,g) = \beta(f) \neq 0$, then either f is elliptic of order 2,3,4, or 6 or g is elliptic of order 2.

3. A lower bound for the commutator. We show here a lower bound for the commutator of two generator groups.

Next we show Theorem 1 which is applicable to the collar lemma.

Theorem 1. Let $\langle f,g \rangle$ be a discrete subgroup of $\text{PSL}(2,C)$ with $\gamma(f,g) \neq 0$,

$\beta(g) \neq -4$ and

(7) $0 < |\beta(f)| < 2\{2\cos(2\pi/7) - 2\cos(\pi/7) + 1\}$, then

$$|\gamma(f,g)| > 2 - 2\cos(\pi/7).$$

Proof. Put $\gamma = \gamma(f,g)$ and $\beta = \beta(f)$. Suppose $|\gamma(f,g)| \leq 2 - 2\cos(\pi/7)$.

If $\gamma = \beta$, then Lemma 3 yields $|\gamma| = |\beta| \geq 1$ or $\beta(g) = -4$ which contradict the assumption of Theorem 1. Therefore $\gamma \neq \beta$ and we have that $\langle [f,g], f[f,g]f^{-1} \rangle$

is nonelementary discrete subgroup of $\langle f,g \rangle$ by Lemma 2 with

$$0 < |\gamma([f,g], f[f,g]f^{-1})| = |\gamma^2(\gamma - \beta)(\beta + 4)| < 0.3 \text{ and } |\beta([f,g])| = |\gamma(\gamma + 4)| < 1.$$

We have the following result from Jørgensen's inequality (2) that

$$(8) \quad 1 \leq |\gamma^2(\gamma - \beta)(\beta + 4)| + |\gamma(\gamma + 4)|.$$

Specially $|\beta| > |\gamma|$, if not we have

$$1 \leq |\gamma^2(\gamma - \beta)(\beta + 4)| + |\gamma(\gamma + 4)| \leq (2|\gamma^3| + |\gamma|)(|\gamma| + 4) < 1.$$

Set $S = \{z; |z| \leq 2 - 2\cos(\pi/7)\}$. Let R be the union of the convex hulls of $S \cup \{-4\}$ and $S \cup \{\beta\}$. The function $u(z) = |z+4| + |z-\beta|$ is subharmonic in

$D = \text{int}(R)$ and hence there exists a point ξ in ∂D such that

$$|\gamma + 4| + |\gamma - \beta| \leq u(\xi). \text{ Put } 0 \leq \theta = |\arg \beta| \leq \pi, \text{ then we have follows by}$$

estimating $u(\xi)$ from the above by the length of the component of $\partial D \setminus \{-4, \beta\}$

which contains ξ , that is

$$(9) \quad |\gamma + 4| + |\gamma - \beta| \leq r_1 + r_2 + |\gamma|\theta,$$

where $r_1 = (4^2 - |\gamma|^2)^{1/2} + |\gamma| \sin^{-1}(|\gamma|/4)$ and

$r_2 = (|\beta|^2 - |\gamma|^2)^{1/2} + |\gamma| \sin^{-1}(|\gamma|/|\beta|)$. Set $r = r_1 + r_2$ where

$4.91 < r < 4.918$. Then (12) is reformed by

$$(10) \quad 1 \leq |\beta + 4|(\theta - 1)|\gamma|^3 + \{|\beta + 4|(r - 4) + 1\}|\gamma|^2 + 4|\gamma|.$$

Our goal is to estimate that the right hand side of above inequality is less than 1 and this contradicts the assumption $|\gamma| \leq 2 - 2\cos(\pi/7) = d$.

Now $|\beta + 4|^2 = |\beta|^2 + 4^2 + 8|\beta|\cos\theta = a + b\cos\theta$ where $16.7 \leq a \leq 16.8$ and

$7.1 \leq b \leq 7.2$. Set $F(x) = (a + b\cos\theta)^{1/2}(\theta - 1)x^3 + \{(a + b\cos\theta)^{1/2}(r - 4) + 1\}x^2 + 4x$ where

$0 \leq x \leq d$. The derivative $F'(x)$ is a increasing function with respect to x ,

then $F(x) \leq F(d)$ where $d = 2 - 2\cos(\pi/7)$. Let $f(\theta) = F(d)$ and therefore we have

$f'(\theta) = (a + b\cos\theta)^{-1/2}(d^2/2)g(\theta)$ where

$$g(\theta) = 2(a + b \cos \theta)d - b\{(\theta - 1)d + (r - 4)\} \sin \theta.$$

We divide θ into 13 cases;

(I) If $0 \leq \theta \leq 4\pi/9$, then $g(\theta) > 0$ and $f(\theta) \leq f(4\pi/9) \leq 0.997789 < 1$.

(II) If $4\pi/9 \leq \theta \leq 5\pi/11$, then we have

$$f(\theta) \leq \{16.8 + 7.2 \cos(4\pi/9)\}^{1/2} (5\pi/11 - 1)d^3 + \\ [\{16.8 + 7.2 \cos(4\pi/9)\}^{1/2} (r - 4) + 1]d^2 + 4d < 0.99884 < 1.$$

(III) If $5\pi/11 \leq \theta \leq 8\pi/17$, similarly we have $f(\theta) \leq 0.999444 < 1$.

(IV) If $8\pi/17 \leq \theta \leq 9\pi/19$, similarly we hold $f(\theta) \leq 0.998055 < 1$.

(V) If $9\pi/19 \leq \theta \leq \pi/2$, similarly $f(\theta) \leq 0.99785 < 1$.

(VI) If $10\pi/21 \leq \theta \leq 11\pi/23$, then $f(\theta) \leq 0.99785 < 1$

(VII) If $11\pi/23 \leq \theta \leq \pi/2$, also we have $f(\theta) \leq 0.999901$

(VIII) If $\pi/2 \leq \theta \leq 5\pi/9$, then $g(\theta) < -0.64$ and $f(\theta) \leq f(\pi/2) < 0.9974 < 1$.

(IX) If $5\pi/9 \leq \theta \leq 2\pi/3$, also we have $g(\theta) < -0.47$ and $f(\theta) < 1$

(X) If $2\pi/3 \leq \theta \leq 11\pi/15$, we have $f(\theta) \leq 0.99936 < 1$.

(XI) If $11\pi/15 \leq \theta \leq 7\pi/9$, then we have $f(\theta) \leq 0.995193 < 1$.

(XII) If $7\pi/9 \leq \theta \leq 8\pi/9$, also we have $f(\theta) \leq 0.9995 < 1$.

(XIII) If $8\pi/9 \leq \theta \leq \pi$, also $f(\theta) \leq 0.9986 < 1$.

This completes the proof.

Theorem 2([G&M2]). Let $\langle f, g \rangle$ be a discrete subgroup of $\text{PSL}(2, \mathbb{C})$ with

$\gamma(f, g) \neq 0$, $\beta(f) = \beta(g) \neq -4$ then

$$(11) \quad |\gamma(f, g)| \geq 0.193.$$

The above constant is not sharp. The following theorem is sharp.

Theorem 3([G&M2]). Let $\langle f, g \rangle$ be a discrete subgroup of $\text{PSL}(2, \mathbb{C})$ with

$$\gamma(f, g) \neq 0, \quad \beta(f) = \beta(g) \neq -4 \text{ and}$$

$$(12) \quad \min\{|\beta(f)|, |\beta(fg)|, |\beta(fg^{-1})|\} \geq 2\{\cos(2\pi/7) + 2\cos(\pi/7) - 1\}, \text{ then}$$

$$|\gamma(f, g)| \geq 2 - 2\cos(\pi/7).$$

The following two theorems are also proved in [F2].

Theorem 4([F2]). Let $\langle f, g \rangle$ be a nonelementary discrete group with

$$\beta(g) \neq -4 \text{ and } 0 < |\beta(f)| < 2\{\cos(2\pi/7) + 2\cos(\pi/7) - 1\}, \text{ then}$$

$$(13) \quad |\gamma(f, g)| \geq 2 - 2\cos(\pi/7) \text{ or } |\gamma(f, g) - \beta(f)| > 1.$$

Theorem 5([F2]). Let $\langle f, g \rangle$ be a nonelementary discrete group with

$$\beta(g) \neq -4 \text{ and } 0 < |\beta(f)| < 2\{\cos(2\pi/7) + 2\cos(\pi/7) - 1\}, \text{ then}$$

$$(14) \quad \max\{|\gamma(f, g)|, |\gamma(f, gfg^{-1})|\} \geq 2 - 2\cos(\pi/7).$$

4. The collar lemma. Let G be a discrete subgroup of $\text{PSL}(2, \mathbb{C})$ acting on the

upper half space H^3 . If $f \in G \setminus \{\text{id.}\}$ is not a parabolic element, then we denote A_f the geodesic in H^3 joining the fixed points of f on $\hat{C}$ the boundary of H^3 in

$\mathbb{R}^3$. For a positive number k , we define a tubular neighborhood about A_f as

$$N_k(f) = \{x \in H^3; d(x, A_f) \leq k\},$$

where d is the hyperbolic metric. Let G_f be the subgroup of G which leaves A_f invariant. We call $N_k(f)$ a collar for f in G , if $g(N_k(f)) \cap N_k(f) = \emptyset$ for all $g \in G \setminus G_f$ and $g(N_k(f)) = N_k(f)$ for all $g \in G_f$. The number k is called the width of the collar $N_k(f)$.

Following [F1], we introduce the notion of complex distance between two geodesics in H^3 and also state the cosine rule. Denote a directed geodesic L by the ordered pair of its endpoints; so $L=(a,b)$ for its endpoints $a, b \in \mathbb{C}$, $a \neq b$. The complex distance $t = \delta(L_1, L_2) \in \mathbb{C}$ between two directed geodesics $L_1=(a_1, b_1)$ and $L_2=(a_2, b_2)$ is defined as follows: $|\operatorname{Re}(t)| \geq 0$ is the hyperbolic distance between the geodesics and $\operatorname{Im}(t)$ is the angle made by the geodesics along their common perpendicular and is determined modulo 2π unless $\operatorname{Re}(t) \neq 0$, in which case $\pm \operatorname{Im}(t)$ is determined modulo 2π . We can compute the complex distance by the formula ([F1]),

$$(15) \quad \cosh^2(t/2) = (a_1, a_2, b_2, b_1).$$

The right hand side of this equality denotes the cross ratio of these four points. Therefore, for any $f \in \operatorname{PSL}(2, \mathbb{C})$, we see $\delta(L_1, L_2) = \delta(f(L_1), f(L_2))$.

Let $f \in \operatorname{PSL}(2, \mathbb{C})$ be non-parabolic and let A_f be directed geodesics in the hyperbolic space joining the fixed points of f . If L is a perpendicular to A_f then the complex distance t between L and $f(L)$ is called the complex

translation length of f . In this case, we have

$$(16) \quad \text{tr}^2(f) = 4\cosh^2(t/2),$$

which makes sense even if f is not loxodromic.

For the geodesics L_0, L_1, L_2 , put $\omega = \delta(L_1, L_2)$, $t_1 = \delta(L_0, L_1)$, $t_2 = \delta(L_0, L_2)$ and denote by α the complex distance from the perpendicular between L_0 and L_1 to the perpendicular between L_0, L_2 . Then we have the so-called cosine rule:

$$(17) \quad \cosh(\omega) = \cosh(t_1)\cosh(t_2) - \cosh(\alpha)\sinh(t_1)\sinh(t_2).$$

Let ω be the complex distance between A_f and $A_{gfg^{-1}}$. Then we can normalize f and gfg^{-1} as follows:

$$f = \begin{bmatrix} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{bmatrix}, gfg^{-1} = \begin{bmatrix} \cosh(t/2) & \exp(\omega)\sinh(t/2) \\ \exp(-\omega)\sinh(t/2) & \cosh(t/2) \end{bmatrix}$$

We have $\gamma = \text{tr}(fgf^{-1}g^{-1}) - 2 = -(1 - \cosh(t))(1 - \cosh(\omega))$. Recall the cosine rule

(17) and take $L_2 = A_f$, $L_0 = A_g$ and $L_1 = A_{gfg^{-1}} = g(A_f)$. It is easy to show that

$$\mu = \delta(A_g, A_f) = \delta(A_g, A_{gfg^{-1}}). \text{ Thus we have } \cosh(\omega) = \cosh^2(\mu) - \cosh(t')\sinh^2(\mu)$$

where t' is a complex translation length of g . Therefore we have the following

lemma([F1], [K]).

Lemma 4. Let f and g be non-parabolic elements in $\text{PSL}(2, \mathbb{C})$ and let μ be the complex distance between A_f and A_g . If $\gamma = \text{tr}(fgf^{-1}g^{-1}) - 2 \neq 0$, then

$$(18) \quad 4\gamma = \beta(f)\beta(g)\sinh^2(\mu).$$

Making use of Lemma 4 and Theorem 1, we will show the following so-called collar lemma.

Theorem 6. Let G be a nonelementary discrete subgroup of $\text{PSL}(2, \mathbb{C})$. Let f be an element of $G \setminus \{\text{id.}\}$ with $0 < |\beta(f)| = 2s < 2\{2\cos(2\pi/7) - 2\cos(\pi/7) + 1\} = 2c$, then there exist a collar $N_{k(s)}(f)$ with the width

$$(19) \quad \sinh^2 k(s) = (c/s - 1)/2.$$

And further let f and g be in G and suppose that f and g generate a nonelementary discrete group. If $0 < |\beta(f)| = 2s < 2c$ and $0 < |\beta(g)| = 2s' < 2c$, then the collars $N_{k(s)}(f)$ for f and $N_{k(s')}(g)$ for g are disjoint, where k is the function defined by (19).

Proof. Let f be an element of $G \setminus \{\text{id.}\}$ with $0 < |\beta(f)| < 2c$ and $g \in G \setminus G_f$.

Suppose f is elliptic. The condition $|\beta(f)| < 1$ implies that the order of f is not less than 7. Then $\langle f, gfg^{-1} \rangle$ is not elementary discrete subgroup of G . If f is not elliptic, then we see that $\mu \neq 0$ where μ is the complex distance between A_f and $A_{gfg^{-1}}$ for $g \in G \setminus G_f$. Thus we conclude that $\langle f, gfg^{-1} \rangle$ is non-elementary discrete group and we have from (2), $|\beta(f)| + |\gamma(f, gfg^{-1})| \geq 1$.

Therefore we have $\gamma(f, gfg^{-1}) \neq 0$ by the assumption of Theorem 7. Thus

$\langle f, gfg^{-1} \rangle$ is discrete with $\gamma(f, gfg^{-1}) \neq 0$, $\beta(f) = \beta(gfg^{-1}) \neq -4$, then we have

$|\gamma(f, gfg^{-1})| \geq 2 - 2\cos(\pi/7) = c^2$ for any $g \in G \setminus G_f$ by Theorem 1. It is already

known that $4\gamma(f, gfg^{-1}) = \beta^2(f) \sinh^2(\mu)$ from (18). By the simple calculation

we have $c = |\beta(f)| |\sinh(\mu)| \leq |\beta(f)| \{2\sinh^2(\operatorname{Re} \mu / 2) + 1\}$. This completes the

first part of theorem.

Next we prove the last part of theorem. Let μ be the complex distance between A_f and A_g . Then (18) and (19) imply

$$\begin{aligned} |\sinh^2(\mu)| &= 4|\gamma(f, g)| / (|\beta(f)| |\beta(g)|) \\ &\geq c^2 / (|\beta(f)| |\beta(g)|) \\ &= (2\sinh^2 k(s) + 1)(2\sinh^2 k(s') + 1) \\ &= (\cosh^2 k(s) + \sinh^2 k(s))(\cosh^2 k(s') + \sinh^2 k(s')) \\ &\geq \{\cosh k(s) \cosh k(s') + \sinh k(s) \sinh k(s')\}^2 \\ &= \cosh^2(k(s) + k(s')), \end{aligned}$$

where k is the function defined by (19). From $\cosh^2 \operatorname{Re} \mu \geq |\sinh^2 \mu|$, we have

$\operatorname{Re} \mu \geq k(s) + k(s')$, which proves the last part of the Theorem. We complete

the proof.

Remark. The function $k(s)$ defined on the above is decreasing function with respect to s and $k(s) \rightarrow \infty$ as $s \rightarrow \infty$.

Let f be a loxodromic element in $\text{PSL}(2, \mathbb{C})$ with the multiplier $\exp(2\alpha(f) + 2i\theta)$. The translation length $2\alpha(f)$ of f is also defined by $\inf\{d(\xi, f(\xi)); \xi \in \mathbb{H}^3\}$. Next lemma is given by Zagier ([Me2]).

Lemma 5. Let $x_1, x_2 \in \mathbb{R}$ and $0 < x_1 < \pi\sqrt{3}$, then there exist a positive integer n such that

$$(20) \quad \cosh nx_1 - \cos nx_2 \leq \cosh(\sqrt{4\pi x_1/\sqrt{3}}) - 1.$$

If the multiplier of f is given by $\exp(2\alpha(f) + 2i\theta)$, then

$$(21) \quad |\beta(f)| = 2(\cosh 2\alpha(f) - \cos 2\theta) \\ = 4(\sinh^2 \alpha(f) + \sin^2 \theta).$$

If f is a loxodromic, then the axes of f and $f^n (n \neq 0)$ are same. By a simple

computation and lemma 5 have $|\beta(f^n)| = 2(\cosh 2n\alpha(f) - \cos 2n\theta) \leq$

$2\{\cosh(\sqrt{8\pi\alpha(f)/\sqrt{3}}) - 1\}$ for some positive integer n . We restate Theorem 6,

setting $2\alpha(f) = L$, $\cosh(\sqrt{4\pi L/\sqrt{3}}) - 1 = s < c_1 = 2\cos(2\pi/7) - 2\cos(\pi/7) + 1 (< 0.445)$

and $\frac{\sqrt{3}}{4\pi} [\log(1 + c_1 + \sqrt{c_1^2 + 2c_1})]^2 = c_2 (= 0.114519)$, then

Theorem 7. Let g be a non trivial closed geodesic with the length

$L(g) < c_2$ in any complete hyperbolic 3-manifold M , then there exists a tubular

neighbourhood $N(g)$ around g in M . Let r be the the hyperbolic width of $N(g)$.

Then, the hyperbolic volume of $N(g)$ is $\pi \cdot L(g) \cdot \sinh^2 r$ which is a decreasing

function of $L(g)$.

Remark. If $L(g) \leq 0.10857$, $r \geq 0.17198$ and $\pi \cdot L(g) \cdot \sinh^2 r \geq 0.01018$.

5. A lower bound of the volume of $V(H^3/\Gamma)$. Let $q(z_1, z_2)$ be a chordal distance between $z_1, z_2 \in \hat{\mathbb{C}}$, that is $q(z_1, z_2) = 2|z_1 - z_2|(1 + |z_1|^2)^{-1/2}(1 + |z_2|^2)^{-1/2}$.

We introduce two different norms which measure the distance from f in mobius transformation groups to id . The first of the two norms for f is given in terms of the matrix,

$$(22) \quad m(f) = ||f - f^{-1}||$$

where for any matrix A in $SL(2, \mathbb{C})$ we let $||A||$ denote its euclidean norm

$||A||^2 = \text{tr}(AA^*)$ and A^* its Hermitian transpose. The second is defined by

$$(23) \quad \rho(f) = d(f(j), j)$$

where j is the point $(0, 0, 1)$ in H^3 .

If f is in $\hat{M}(\mathbb{C})$ with $\text{fix}(f) = \{z_1, z_2\}$, then $|\beta(f)| = \frac{1}{2} \frac{q(z_1, z_2)^2}{8 - q(z_1, z_2)^2} m(f)^2$

, $2\cosh(\rho(f)) = ||f||^2$ and simple computation leads $4||f||^2 = m(f)^2 + 2|\text{tr}^2(f)|$.

If f is in $\hat{M}(\mathbb{C}) \setminus \{\text{id.}\}$ with $\text{fix}(f) = \{z_1, z_2\}$ and multiplier $\exp(2\alpha(f) + 2i\theta)$,

then,

$$(24) \quad \sinh^2(\rho(f)/2) = \frac{4 - q^2}{8 - q^2} \frac{m(f)^2}{8} + \sinh^2 \alpha(f) ,$$

$$(25) \quad \sinh^2(\rho(f)/2) = \frac{4}{8 - q^2} \frac{m(f)^2}{8} - \sin^2 \theta ,$$

where $q = q(z_1, z_2)$.

Let N denote the set of positive integers and for each ρ in $[0, \infty)$ set

$$(26) \quad s(\alpha) = \sup_{\theta} \left(\inf_N (\sinh^2 k \alpha + \sin^2 k \theta) \right).$$

Then $s(x)$ is nonnegative, nondecreasing and continuous in $[0, \infty)$ with $s(0)=0$.

Moreover from a lemma due to Zagier [Me2], it follows that

$$(27) \quad s(t) \leq \sinh^2 \sqrt{a} t, \quad a = 2\pi / \sqrt{3},$$

for $0 \leq t < \sqrt{3}\pi/2$. Then we have the following lemma.

Lemma 7 ([G&M2]). Suppose that a and c are positive constants, and f is in $M(C) \setminus \{\text{id.}\}$ with two distinct fixed points and multiplier

$\exp(2\alpha(f) + 2i\theta)$. If $m(f^k)^2 \geq c$ for any k in N and

$$(28) \quad s(a) \leq c/8 + \sinh^2 \alpha(f) - \sinh^2 a,$$

then $\rho(f) \geq 2a$.

The following is an immediate result from Jørgensen's inequality.

Lemma 8. Let $\langle f, g \rangle$ be a non-elementary discrete group, then

$$(29) \quad m(f)m(g) \geq 4(\sqrt{2}-1).$$

If $m(f)m(g) < 4(\sqrt{2}-1)$ for a discrete group $\langle f, g \rangle$, then $\langle f, g \rangle$ is a elementary

group. Next lemma gives more simple result for generators if each generative

elements have the following restrictive conditions.

Lemma 9. Let $\langle f, g \rangle$ be a discrete group with

$m(f)^2 < 4(\sqrt{2}-1)$ and $m(g)^2 \leq 4(\sqrt{2}-1)$, then $\text{fix}(f) = \text{fix}(g)$.

The following proposition is an important role in this paper which is due to P. Waterman[W].

Proposition 4. Let G be a discrete group. Then there is a mobius transformation h for G satisfying

$$(30) \quad m(f) \geq 4(\sqrt{2}-1)$$

for any element f in $hGh^{-1} \setminus \{id.\}$.

Theorem 8. Let M be a complete hyperbolic 3-manifold, then

$$V(M) > 0.001.$$

Proof. It is known that M is represented by H^3/Γ for a torsion free Kleinian group Γ . If Γ contains parabolic transformations, then $V(M) \geq \sqrt{3}/4$ ([Me1]).

From now on, we consider that Γ is a purely loxodromic transformations group.

we set $2\alpha' = \inf[2\alpha(f); f \in \Gamma \setminus \{id.\}]$ where $\exp(2\alpha(f) + 2i\theta)$ is a multiplier

of f . If $2\alpha' \leq 0.10857$, then Theorem 7 and Remark shows that M has a collar

with the volume not less than 0.001. Otherwise, we consider a conjugate group

for Γ stated in Proposition 4 and reset this conjugate group Γ , then we have

$m(f)^2 \geq 4(\sqrt{2}-1)$ for f in $\Gamma \setminus \{id.\}$. Now we seek η satisfying the equation

$s(\eta) = (\sqrt{2}-1)/2 + \sinh^2 \alpha' - \sinh^2 \eta$. we have $2\eta \geq 0.11576$. Thus

$\rho(f) \geq 0.11576$, thus the Dirichlet fundamental polyhedron centered at j for Γ

contains a ball B with hyperbolic radius 0.11576. Then we have the result.

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