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A MODULAR BRANCHING RULE FOR WREATH PRODUCTS

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ABSTRACT. We give a modular branching rule for wreath products as a generalization of Kleshchev's modular branching rule for the symmetric groups.

1. INTRODUCTION

In [Tsu], the author gave a modular branching rule for certain wreath products, the wreath products $\{G \wr \mathfrak{S}_n\}_{n\geq 0}$ where any irreducible FG-module is 1-dimensional on a splitting field F of G. In this paper we give a modular branching rule for any wreath products, after a survey of the representation theory of the symmetric groups with a special emphasis on the connection with Lie theory.

As our motivation, let us review previously known branching rules. We shall start off with the symmetric groups $\{\mathfrak{S}_n\}_{n\geq 0}$ with the default embeddings. In characteristic zero, we can summarize the classically known branching rule as follows (for the details, see Theorem 2.3).

- for any irreducible $\mathfrak{S}_{n+1}$-module V, $\mathsf{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V)$ is multiplicity-free.
- Young's lattice controls the structure of $\mathsf{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V)$ as an $\mathfrak{S}_n$-module.

An analogue in positive characteristics is discovered and proved by Kleshchev and it is now known as Kleshchev's modular branching rule [Kl1, Kl2, Kl3, Kl4]. The language of quantum groups and Kashiwara's crystal bases [HK, Kas] lets us state Kleshchev's modular branching rule in characteristic $p > 0$ succinctly and beautifully as follows (for the details, see Theorem 2.7).

- for any irreducible $\mathfrak{S}_{n+1}$-module V, $\mathsf{Soc}(\mathsf{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V))$ is multiplicity-free.
- the crystal basis $B(\Lambda_0)$ of the fundamental irreducible $U_q(\mathfrak{g}(A^{(1)}_{p-1}))$-module $L(\Lambda_0)$ controls the structure of $\mathsf{Soc}(\mathsf{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V))$ as an $\mathfrak{S}_n$-module.

Here for an A-module X, we denote by $\mathsf{Soc}(X)$ the largest completely reducible A-submodule of X.

Recently, two analogues for Hecke algebras are obtained. The first one is for the cyclotomic degenerate Hecke algebras, which we survey in detail in this paper. Let λ be a dominant integral weight $\lambda \in P^+$ associated with the Cartan datum of $A^{(1)}_{p-1}$ [HK, Definition 2.1.1] and consider the inductive system of the cyclotomic degenerate Hecke algebras $\{\mathcal{H}^{\lambda}_n\}_{n\geq 0}$ [Kl5, Chapter 7]. Then the following generalization holds [Kl5, Theorem 10.3.5]. Note that Kleshchev's modular branching rule is a special case of $\lambda = \Lambda_0$.

- for any irreducible $\mathcal{H}^{\lambda}_{n+1}$-module V, $\mathsf{Soc}(\mathsf{Res}^{\mathcal{H}^{\lambda}_{n+1}}_{\mathcal{H}^{\lambda}_n}(V))$ is multiplicity-free.

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- the crystal basis $B(\lambda)$ of $U_q(\mathfrak{g}(A^{(1)}_{p-1}))$-module $L(\lambda)$ controls the structure of $\mathrm{Soc}(\mathrm{Res}^{\mathcal{H}^\lambda_{n+1}}_{\mathcal{H}^\lambda_n}(V))$ as an $\mathcal{H}^\lambda_n$-module.

The second one is for the cyclotomic Hecke algebras. Let q be a primitive e-th root of unity where $e \geq 2$ and let $v_i = q^{\gamma_i}$ for $\gamma_i \in \mathbb{Z}/e\mathbb{Z}$ and consider the inductive system of the cyclotomic Hecke algebras $\{\mathcal{H}_n(v, q)\}_{n\geq 0}$ [Ari, Definition 2.1]. Then the following holds [Ari, Theorem 6.1]. See also [Bru, GV].

- for any irreducible $\mathcal{H}_{n+1}(v, q)$-module V, $\mathrm{Soc}(\mathrm{Res}^{\mathcal{H}_{n+1}(v,q)}_{\mathcal{H}_n(v,q)}(V))$ is multiplicity-free.
- the crystal basis $B(\Lambda_{\gamma_1} + \cdots + \Lambda_{\gamma_m})$ of $U_q(\mathfrak{g}(A^{(1)}_{e-1}))$-module $L(\Lambda_{\gamma_1} + \cdots + \Lambda_{\gamma_m})$ controls the structure of $\mathrm{Soc}(\mathrm{Res}^{\mathcal{H}_{n+1}(v,q)}_{\mathcal{H}_n(v,q)}(V))$ as an $\mathcal{H}_n(v, q)$-module.

In [Tsu], the author gave a generalization of Kleshchev's modular branching rule for the symmetric groups to certain wreath products. Let G be a finite group and F be its splitting field and further assume that any irreducible FG-module is 1-dimensional. We denote by α the number of inequivalent irreducible representations of FG. For example, $\alpha = 1$ if $\mathrm{char}\, F = p > 0$ and G is a p-group. Although the main result of [Tsu] gives a modular branching rule in any characteristics, we now repeat the succinct form of it only when $\mathrm{char}\, F = p > 0$ for simplicity (for the details, see [Tsu, Theorem 5.2]).

- for any irreducible $G \wr \mathfrak{S}_{n+1}$-module V, $\mathrm{Soc}(\mathrm{Res}^{G\wr\mathfrak{S}_{n+1}}_{G\wr\mathfrak{S}_n}(V))$ is multiplicity-free.
- the crystal basis of $U_q(\mathfrak{g}(A^{(1)}_{p-1}))^{\otimes\alpha} (\cong U_q(\mathfrak{g}((A^{(1)}_{p-1})^{\oplus\alpha})))$-module $L(\Lambda_0)^{\otimes\alpha}$ controls the structure of $\mathrm{Soc}(\mathrm{Res}^{G\wr\mathfrak{S}_{n+1}}_{G\wr\mathfrak{S}_n}(V))$ as a $G \wr \mathfrak{S}_n$-module.

Hence we can give a modular branching rule for the complex reflection groups $G(m, 1, n) = (\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$, which are often called the generalized symmetric groups, in splitting fields for $\mathbb{Z}/m\mathbb{Z}$. Especially for $m = 2$, which is the case of the Weyl groups of type B, we can give a modular branching rule in any field. It was the aim of [Tsu]. Note that if we choose the parameters $q = 1$ and $v_k = \exp(\frac{2\pi\sqrt{-1}k}{m})$, then $\mathcal{H}_n(v, q) \cong FG(m, 1, n)$. Thus, it can be said that the different crystals appear in modular branching rule for $FG(m, 1, n)$ and for its deformation, cyclotomic Hecke algebras.

The purpose of this paper is to give a modular branching rule for any sequence of wreath products $\{G \wr \mathfrak{S}_n\}_{n\geq 0}$ on splitting fields. We should remark that in characteristic zero it is already known (see Remark 2.4 and Remark 2.5). For any sequence of wreath products $\{G \wr \mathfrak{S}_n\}_{n\geq 0}$, socle of restriction is not necessarily multiplicity-free and socle multiplicity-freeness holds if and only if any irreducible FG-module is 1-dimensional (see Corollary 3.11. For the details, see Theorem 3.8). As in [Tsu], our proof for wreath products is elementary in that it is essentially a combination of Frobenius reciprocity, Nakayama relations, Mackey theorem, Clifford theory and Kleshchev's modular branching rule.

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Notations and conventions In the following, we assume for simplicity that any group is finite and any module is finite dimensional.

- For a finite dimensional algebra A, we denote by $\mathsf{Irr}(A)$ the set of isomorphism classes of irreducible A-modules.
- For an A-module V, we denote by $[V]_A$ the isomorphism class of A-modules isomorphic to V. If A is clear from the context, we often omit the suffix.
- By the word directed graph, we mean a directed graph with no loops but allow multiple-edges. We write directed graph as $G = (V, E)$ where V is the set of vertices and $E : V \times V \to \mathbb{Z}_{\geq 0}$ is its adjacent relation meaning there are $E(v_1, v_2)$ directed arrow from v_1 to v_2. Note that it is a different convention from that in [Tsu].
- For a given sequence of non-negative integers $\overrightarrow{n} = (n_1, \cdots, n_\alpha)$ such that $\sum_{\beta=1}^{\alpha} n_\beta = n$ (such $\overrightarrow{n}$ is called a composition of n), we denote by $\mathfrak{S}_{\overrightarrow{n}}$ the Young subgroup of $\mathfrak{S}_n$, that is
$$\begin{aligned}\mathfrak{S}_{\overrightarrow{n}} &\stackrel{\text{def}}{=} \mathfrak{S}_{\{1,\cdots,n_1\}} \times \mathfrak{S}_{\{n_1+1,\cdots,n_1+n_2\}} \times \cdots \times \mathfrak{S}_{\{n-n_\alpha+1,\cdots,n\}}.\\ &\cong \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_\alpha}.\end{aligned}$$
- Given a field F and an inductive system of groups $\mathfrak{I} = (\{G_n\}_{n\geq 0}, \{\varphi_n : G_n \to G_{n+1}\}_{n\geq 0})$, we define a directed graph $\mathbb{B}_F(\mathfrak{I}) = (V, E)$ as follows.
  - $V = \displaystyle\bigsqcup_{n\geq 0} \mathsf{Irr}(FG_n)$.
  - for $[W] \in \mathsf{Irr}(FG_n)$ and $[U] \in \mathsf{Irr}(FG_m)$,
$$E([W],[U]) = \begin{cases} \dim \mathrm{Hom}_{FG_n}(W, \mathrm{Res}^{FG_{n+1}}_{FG_n} U) & (m = n+1) \\ 0 & (\text{otherwise}). \end{cases}$$

  If $\mathbb{B}_F(\mathfrak{I})$ has only single-edges, we say that $\mathfrak{I}$ is socle multiplicity-free over F. Note that it is the same as that in [Tsu, Definition 2.1]. We just rewrite it by adapting our directed graph notation.
- Let $\{G_i = (V_i, E_i)\}_{i=1}^k$ be k directed graphs and $\{d_i\}_{i=1}^k$ be a sequence of positive integers. We define a directed graph $\mathbb{I}(\{G_i\}_{i=1}^k, \{d_i\}_{i=1}^k) = (V, E)$ as follows.
  - $V = V_1 \times \cdots \times V_k$.
  - for $v = (v_1, \cdots, v_k)$ and $w = (w_1, \cdots, w_k) \in V$,
$$E(v, w) = \begin{cases} d_j E_j(v_j, w_j) & (\exists j \text{ s.t. } E_j(v_j, w_j) > 0 \text{ and } \forall j' \neq j, v_{j'} = w_{j'}) \\ 0 & (\text{otherwise}). \end{cases}$$

  Note that the operator $\mathbb{I}$ is a generalization of the operator $*$ in [Tsu, Definition 2.4]. If G_i is single-edged and $d_i = 1$ for all $1 \leq i \leq k$, then $\mathbb{I}(\{G_i\}_{i=1}^k, \{d_i\}_{i=1}^k) = G_1 * \cdots * G_k$.

2. Review of modular representations of the symmetric groups

We shall first recall the representation theory of the symmetric groups. It is well-known that for each partition λ of n, we can construct a $\mathbb{Z}$-free, $\mathbb{Z}$-finite rank, $\mathbb{Z}\mathfrak{S}_n$-module S^λ which is called the Specht module [Jam, Chapter 4]. Each S^λ has an $\mathfrak{S}_n$-invariant symmetric bilinear form. For a field F, we write $S^\lambda_F = F \otimes_{\mathbb{Z}} S^\lambda$ and denote by D^λ_F the quotient of S^λ_F by the radical of its invariant form. It is also well-known that $D^\lambda_F = S^\lambda_F$ if $\operatorname{char} F = 0$ [Jam, Chapter 4]. The following is the fundamental theorem of the representation theory of the symmetric groups.

Theorem 2.1 ([Jam, Theorem 11.5]). *Suppose that our ground field F has characteristic $p(\geq 0)$. As μ varies over p-regular partitions of n, D_F^μ varies over a complete set of inequivalent irreducible $F\mathfrak{S}_n$-modules. Each D_F^μ is self-dual and absolutely irreducible. Every field is a splitting field for $\mathfrak{S}_n$.*

Definition 2.2. *We denote by $\mathfrak{S}$ the inductive system of the symmetric groups*

$$\mathfrak{S} = (\mathfrak{S}_0 \subsetneq \mathfrak{S}_1 \subsetneq \mathfrak{S}_2 \subsetneq \cdots).$$

Here if $m < n$, the default embedding of $\mathfrak{S}_m$ into $\mathfrak{S}_n$ is with respect to the first m letters.

Theorem 2.3 ([Jam, Theorem 9.2]). *For any field F of characteristic zero, $\mathfrak{S}$ is socle multiplicity-free over F. Moreover, the map $\mathbb{B}_0 \to \mathbb{B}_F(\mathfrak{S}), \lambda \mapsto [S_F^\lambda]$ is an isomorphism as directed graphs where $\mathbb{B}_0 = (\mathcal{P}_0, E_0)$ is a single-edged directed graph obtained from the Young's lattice $\mathcal{P} \stackrel{def}{=} \{\lambda \vdash n \mid n \geq 0\} (= \mathcal{P}_0)$ in the trivial manner.*

Remark 2.4. *Usually we can reach the branching theorem as above only after constructing the irreducible representations. However another approach is known recently. In fact, Okounkov and Vershik's approach first establishes multiplicity-freeness and then goes on to the representation theory of the symmetric groups in characteristic zero* [OV]. *For any finite group G, Pushkarev apply a similar technique of Okounkov-Vershik and succeed in obtaining a branching rule for the sequence of wreath products $\{G \wr \mathfrak{S}_n\}_{n\geq 0}$* [Pus, Theorem 11].

Remark 2.5. *We can state much stronger result in a field F of characteristic zero. Although the following fact is well-known, we recall it to compare with Remark 2.11. In this case we have an isomorphism as graded $\mathbb{Z}$-algebras*

$$\bigoplus_{n\geq 0} K_0(F\mathfrak{S}_n\text{-}mod) \xrightarrow{\sim} \bigoplus_{n\geq 0} \varprojlim_m \mathbb{Z}[x_1, \cdots, x_m]_n^{\mathfrak{S}_m}$$

$$[V]_{F\mathfrak{S}_n} \longmapsto \sum_{\lambda\vdash n} \frac{1}{z_\lambda} \chi_V(C(\lambda)) p_\lambda$$

where we use the following notations.

- *$K_0(F\mathfrak{S}_n\text{-}mod)$ is the Grothendieck group of the abelian category $F\mathfrak{S}_n$-mod.*
- *the algebra structure of LHS is defined by induction, that is*
$$[V]_{F\mathfrak{S}_m} \cdot [W]_{F\mathfrak{S}_n} \stackrel{def}{=} \left[\mathrm{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}} V \otimes W\right]_{F\mathfrak{S}_{m+n}}.$$
- *for $\lambda = (\lambda_1, \cdots, \lambda_l) \vdash n$,*
$$p_\lambda \stackrel{def}{=} (\sum_{m=1}^{\infty} x_m^{\lambda_1}) \cdots (\sum_{m=1}^{\infty} x_m^{\lambda_l}) \in \varprojlim_m \mathbb{Z}[x_1, \cdots, x_m]_n^{\mathfrak{S}_m}.$$
- *for $\lambda \vdash n$, $C(\lambda)$ stands for the conjugacy class consists of the elements of $\mathfrak{S}_n$ whose cycle type is λ, and write $z_\lambda = n!/\#C(\lambda)$.*
- *$\chi_V(C(\lambda))$ stands for the character value of $\mathfrak{S}_n$-representation V on $C(\lambda)$ conjugacy class.*

Moreover, this isomorphism is an isometry with respect to the symmetric bilinear forms defined as follows.

- *on LHS, $\langle [S_F^\lambda], [S_F^\mu] \rangle = \delta_{\lambda\mu}$ for $\lambda, \mu \vdash n$.*

- *on RHS,* $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ *for* $\lambda, \mu \vdash n$*, where* s_λ *is the Schur function.*

From this result we can obtain many classical result such as Frobenius character formula, Murnaghan-Nakayama formula, Young's rule and Littlewood-Richardson rule by working in RHS, the ring of symmetric functions [Ful, Chapter 7], [FH, Chapter 4], [Mac, Chapter 1], [Sag, Chapter 3]. *As described in* [Mac, Appendix B], *we can determine the algebra structure of*

$$\bigoplus_{n\geq 0} K_0(\mathbb{C}[G \wr \mathfrak{S}_n]\text{-}mod)$$

by the similar argument for any finite group G.

Kleshchev successfully discovered and proved an analogue of Theorem 2.3 in positive characteristics. It is described in terms of the famous conormal node of a Young diagram which we recall briefly.

Definition 2.6. *Let* $l \geq 2$ *be a positive integer and* $i \in \mathbb{Z}/l\mathbb{Z}$*. Note that the following definitions depend on this given* l*. Let* λ *be a partition. We identify the partition* λ *with the Young diagram of shape* λ*, that is* $\{(r,s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} | s \leq \lambda_r\}$.

(1) *For a node* $A = (r,s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$*, we define its* ***residue*** *to be* $-r+s+l\mathbb{Z} \in \mathbb{Z}/l\mathbb{Z}$ *and denote it by* $\mathsf{res}(A)$.

(2) *A node* A *inside* λ *is called* ***i-removable*** *if* $\mathsf{res}(A) = i$ *and* $\lambda \setminus \{A\}$ *is still a Young diagram.*

(3) *A node* A *outside* λ *is called* ***i-addable*** *if* $\mathsf{res}(A) = i$ *and* $\lambda \cup \{A\}$ *is again a Young diagram.*

(4) *Label all* i*-addable nodes of* λ *by* $+$ *and all* i*-removable nodes of* λ *by* $-$*. The* ***i-signature*** *of* λ *is the sequence of pluses and minuses obtained by going along the rim of the Young diagram from bottom left to top right and reading off all the signs.*

(5) *The* ***reduced i-signature*** *of* λ *is the sequence of pluses and minuses obtained from the* i*-signature of* λ *by successively erasing all neighboring pairs of the form* $-+$*. Note that the reduced* i*-signature of* λ *always looks like a sequence that starts with* $+$*s followed by* $-$*s.*

(6) *Nodes that correspond to* $+$*s of the reduced* i*-signature of* λ *are called* ***i-conormal***.

(7) *The node that corresponds to the rightmost* $+$ *of the reduced* i*-signature of* λ *is called* ***i-cogood***. *It is the rightmost* i*-conormal node.*

(8) *We set* $\varphi_i(\lambda) = \#\{i\text{-conormal nodes of } \lambda\}$.

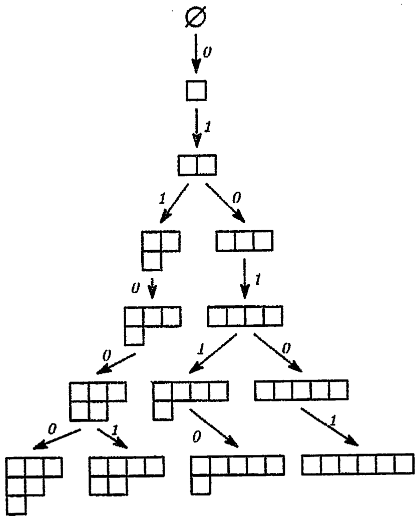
(9) *If* $\varphi_i(\lambda) > 0$*, we set* $\tilde{f}_i(\lambda) = \lambda \cup \{A\}$ *where* A *is the (unique)* i*-cogood node.*

(10) *Let* p *be a prime number and put* $l = p$ *in the above definitions. We define a single-edged directed graph* $\mathbb{B}_p = (\mathcal{P}_p, E_p)$ *by* $\mathcal{P}_p = \{\lambda \in \mathcal{P} \mid \lambda \text{ is } p\text{-regular}\}$ *and*

$$E_p(\lambda,\mu) = \begin{cases} 1 & (\exists i \in \mathbb{Z}/p\mathbb{Z}, \varphi_i(\lambda) > 0 \text{ and } \tilde{f}_i(\lambda) = \mu) \\ 0 & (\text{otherwise}). \end{cases}$$

Theorem 2.7 ([Kl2, Kl5]). *For any field* F *of characteristic* $p > 0$*,* $\mathfrak{S}$ *is socle multiplicity-free over* F*. Moreover, the map* $\mathbb{B}_p \to \mathbb{B}_F(\mathfrak{S}), \lambda \mapsto [D^\lambda_F]$ *is an isomorphism as directed graphs.*

Example 2.8. *The following directed graph is a part of* $\mathbb{B}_2$*. For more of it, see* [Kl5, Example 11.1.2].



Remark 2.9. *Let l, i, λ be as in Definition 2.6. We redefine*

$$(9^{\text{new}})\quad \tilde{f}_i(\lambda) = \begin{cases} \lambda \cup \{A\} & (\textit{if } \varphi_i(\lambda) > 0 \textit{ and } A \textit{ is the (unique) } i\textit{-cogood node of } \lambda) \\ 0 & (\textit{if } \varphi_i(\lambda) = 0), \end{cases}$$

where 0 *is a special symbol. Next we define the dual concepts.*

(6') *Nodes that correspond to $-$s of the reduced i-signature of λ are called **i-normal**.*

(7') *The node that corresponds to the leftmost $-$ of the reduced i-signature of λ is called **i-good**. It is the leftmost i-normal node.*

(8') *We set $\varepsilon_i(\lambda) = \#\{i\text{-normal nodes of } \lambda\}$.*

$$(9^{\text{new}}{}')\quad \tilde{e}_i(\lambda) = \begin{cases} \lambda \setminus \{A\} & (\textit{if } \varepsilon_i(\lambda) > 0 \textit{ and } A \textit{ is the (unique) } i\textit{-good node of } \lambda) \\ 0 & (\textit{if } \varepsilon_i(\lambda) = 0). \end{cases}$$

*Put $l = p$ and consider the affine generalized Cartan matrix $A^{(1)}_{p-1}$. We use the Lie theoretic notations associated with $A^{(1)}_{p-1}$. For example, the **weight** of λ*

$$\mathrm{wt}(\lambda) \stackrel{def}{=} \Lambda_0 - \sum_{i \in \mathbb{Z}/p\mathbb{Z}} \#\{A \in \lambda \mid \mathsf{res}(A) = i\} \cdot \alpha_i$$

is defined as the element of the weight lattice, which is a constituent of the Cartan datum associated with $A^{(1)}_{p-1}$ [HK, Definition 2.1.1]. *Then it is known that the 6-tuple $(\mathcal{P}_p, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt})$ is a crystal in the sense of Kashiwara and it is isomorphic to the crystal basis $B(\Lambda_0)$ of $U_q(\mathfrak{g}(A^{(1)}_{p-1}))$-module $L(\Lambda_0)$ as crystals (known as Misra-Miwa realization* [MM]*). It suggests us that there is a connection between the modular representations of the symmetric groups in characteristic $p > 0$ and the Lie theory associated with $A^{(1)}_{p-1}$.*

Remark 2.10. *In this Remark, we assume F is an algebraically closed field of characteristic $p > 0$ for simplicity. To explain the previously mentioned connection, we need variants of affine Hecke algebra of type A. For a non-negative integer n, we define the **degenerate affine Hecke algebra** $\mathcal{H}_n$ by generators*

$x_1, \cdots, x_n, s_1, \cdots, s_{n-1}$ *and the following relations* [Dri], [Lus].

$$\begin{cases} x_ix_j = x_jx_i & (1 \le i,j \le n) \\ s_i^2 = 1, s_is_j = s_js_i, s_is_{i+1}s_i = s_{i+1}s_is_{i+1} & (1 \le i,j < n, |i-j| > 1) \\ s_ix_j = x_js_i & (1 \le i < n, 1 \le j \le n, j \ne i, i+1) \\ s_ix_i = x_{i+1}s_i - 1, s_ix_{i+1} = x_is_i + 1 & (1 \le i < n). \end{cases}$$

Note there is a natural isomorphism just as F-vector spaces [Kl5, Theorem 3.2.2]

$$F[x_1, \cdots, x_n] \otimes F\mathfrak{S}_n \xrightarrow{\sim} \mathcal{H}_n, \quad f(x_1, \cdots, x_n) \otimes \sigma \longmapsto f(x_1, \cdots, x_n)\sigma,$$

and the center of $\mathcal{H}_n$ is $Z(\mathcal{H}_n) = F[x_1, \cdots, x_n]^{\mathfrak{S}_n}(\subseteq \mathcal{H}_n)$ [Kl5, Theorem 3.3.1]. *Hence we can decompose any $\mathcal{H}_n$-module M as*

$$M = \bigoplus_{\gamma \in F^n/\mathfrak{S}_n} M[\gamma],$$

where we denote by $M[\gamma]$ the generalized eigenspace

$$M[\gamma] \stackrel{def}{=} \{m \in M \mid \forall f \in F[x_1, \cdots, x_n]^{\mathfrak{S}_n}, \exists d_f \ge 1, (f(x_1, \cdots, x_n) - f(\gamma))^{d_f} m = 0\}.$$

In particular, for any irreducible $\mathcal{H}_n$-module there exists an unique $\gamma \in F^n/\mathfrak{S}_n$ such that $M = M[\gamma]$. For an irreducible $\mathcal{H}_n$-module M and for each $a \in F$, we define

$$\begin{cases} e_a(M) = \mathsf{Res}^{\mathcal{H}_{n-1} \otimes F[x_n]}_{\mathcal{H}_{n-1}}(M_a) \\ \tilde{e}_a(M) = \mathsf{Soc}(e_a(M)) \\ \tilde{f}_a(M) = \mathsf{Top}(\mathsf{Ind}^{\mathcal{H}_{n+1}}_{\mathcal{H}_n \otimes F[x_{n+1}]} M \otimes (F[x_{n+1}]/\langle x_{n+1} - a \rangle)), \end{cases}$$

where Top *is the dual concept of* Soc *and we define M_a by*

$$M_a = \{m \in M \mid \exists d \ge 1, (x_n - a)^d m = 0\}.$$

Since x_n and $\mathcal{H}_{n-1}$ commute, M_a is an $\mathcal{H}_{n-1} \otimes F[x_n](\subseteq \mathcal{H}_n)$-module. It is known [Kl5, Lemma 5.1.5, Corollary 5.1.8], [GV] *that*

- $\tilde{e}_a$ *maps each irreducible $\mathcal{H}_n$-module to an irreducible $\mathcal{H}_{n-1}$-module or zero.*
- $\tilde{f}_a$ *maps each irreducible $\mathcal{H}_n$-module to an irreducible $\mathcal{H}_{n+1}$-module.*

Write $I = \mathbb{Z} \cdot 1_F = \{0, \cdots, p-1\} \subseteq F$ and recall that we are using Lie theoretic notations associated with $A^{(1)}_{p-1}$ as in the previous Remark. For a dominant integral weight $\lambda \in P^+$ associated with the Cartan datum $(A^{(1)}_{p-1}, \Pi, \Pi^\vee = \{h_i\}_{i \in I}, \mathcal{P}, \mathcal{P}^\vee)$ [HK, Definition 2.1.1], *we define the* **degenerate cyclotomic Hecke algebra** *$\mathcal{H}^\lambda_n \stackrel{def}{=} \mathcal{H}_n/\langle f_\lambda \rangle$ where $\langle f_\lambda \rangle$ is the two-sided ideal generated by $f_\lambda \stackrel{def}{=} \prod_{i \in I}(x_1 - i)^{\lambda(h_i)} \in \mathcal{H}_n$. Note that $\mathcal{H}^\lambda_n$ is finite dimensional* [Kl5, Theorem 7.5.6].

We define two functors by

$$\begin{cases} \mathsf{pr}^\lambda : \mathcal{H}_n\text{-mod} \longrightarrow \mathcal{H}^\lambda_n\text{-mod}, & M \longmapsto \mathcal{H}^\lambda_n \otimes_{\mathcal{H}_n} M (= M/\langle f_\lambda \rangle M) \\ \mathsf{infl}^\lambda : \mathcal{H}^\lambda_n\text{-mod} \longrightarrow \mathcal{H}_n\text{-mod}, & M \longmapsto {}_{\mathcal{H}_n}\mathcal{H}^\lambda_n \otimes_{\mathcal{H}^\lambda_n} M (= M), \end{cases}$$

where by ${}_{\mathcal{H}_n}\mathcal{H}_n^\lambda$ we regard $\mathcal{H}_n^\lambda$ as an $\mathcal{H}_n$-module obtained from the inflation along the canonical surjection $\mathcal{H}_n \twoheadrightarrow \mathcal{H}_n^\lambda$. Further, define for an $\mathcal{H}_n^\lambda$-module M and $i \in I$,

$$\begin{cases} e_i^\lambda(M) = \mathsf{pr}^\lambda(e_i(\mathsf{infl}^\lambda M)) \\ \tilde{e}_i^\lambda(M) = \mathsf{pr}^\lambda(\tilde{e}_i(\mathsf{infl}^\lambda M)) \\ \tilde{f}_i^\lambda(M) = \mathsf{pr}^\lambda(\tilde{f}_i(\mathsf{infl}^\lambda M)) \\ f_i^\lambda(M) = \varprojlim_m \mathsf{pr}^\lambda(\mathsf{Ind}_{\mathcal{H}_n\otimes F[x_{n+1}]}^{\mathcal{H}_{n+1}}(\mathsf{infl}^\lambda M) \otimes (F[x_{n+1}]/\langle (x_{n+1}-i)^m\rangle)). \end{cases}$$

It is known [Kl5, Theorem 8.2.5], [GV] *that*

- *$\tilde{e}_i^\lambda$ maps each irreducible $\mathcal{H}_n^\lambda$-module to an irreducible $\mathcal{H}_{n-1}^\lambda$-module or zero.*
- *$\tilde{f}_i^\lambda$ maps each irreducible $\mathcal{H}_n^\lambda$-module to an irreducible $\mathcal{H}_{n+1}^\lambda$-module or zero.*

Finally, for an irreducible $\mathcal{H}_n^\lambda$-module $M = M[\gamma]$ we define

$$\begin{cases} \mathrm{wt}^\lambda(M) = \lambda - \sum_{i\in I} k_i\alpha_i \\ \varepsilon_i^\lambda(M) = \max\{m \geq 0 | (\tilde{e}_i^\lambda)^m M \neq 0\} \\ \varphi_i^\lambda(M) = \max\{m \geq 0 | (\tilde{f}_i^\lambda)^m M \neq 0\}, \end{cases}$$

where $k_i = \#\{1 \leq j \leq n \mid \gamma_j + p\mathbb{Z} = i\}$ for $\gamma = (\gamma_1, \cdots, \gamma_n) \pmod{\mathfrak{S}_n}$.

It is known [Kl5, Theorem 9.5.1], [Gro, GV] *that there is an isomorphism*

$$\mathbb{C} \otimes_{\mathbb{Z}} (\bigoplus_{n\geq 0} K_0(\mathcal{H}_n^\lambda\text{-}mod)) \cong L(\lambda)$$

as $\mathfrak{g}(A_{p-1}^{(1)})$-modules where the Chevalley generators e_i, f_i act on LHS by e_i^λ, f_i^λ respectively and $h \in \mathfrak{h}^*$ acts by $h([M]) = (wt^\lambda([M])(h))[M]$ for an irreducible $\mathcal{H}_n^\lambda$-module M. Moreover the 6-tuple

$$(\bigsqcup_{n\geq 0} \mathsf{Irr}(\mathcal{H}_n^\lambda), \varepsilon_i^\lambda, \varphi_i^\lambda, \tilde{e}_i^\lambda, \tilde{f}_i^\lambda, \mathrm{wt}^\lambda)$$

is a crystal in the sense of Kashiwara and it is isomorphic to $B(\lambda)$ as a crystal [Kl5, Theorem 10.3.5], [Gro, GV].

Remark 2.11. *In this Remark, let F be an arbitrary field of characteristic $p > 0$. Put $\lambda = \Lambda_0$ in Remark 2.10. Then there is a natural F-algebra isomorphism* [Kl5, Theorem 7.5.6]

$$F\mathfrak{S}_n \xrightarrow{\sim} \mathcal{H}_n^{\Lambda_0}, \quad w \longmapsto w \pmod{\langle x_1\rangle}.$$

Hence we can state much stronger result than Theorem 2.7.

- *We have $\mathbb{C} \otimes_{\mathbb{Z}} (\bigoplus_{n\geq 0} K_0(F\mathfrak{S}_n\text{-}mod)) \cong L(\Lambda_0)$ as $\mathfrak{g}(A_{p-1}^{(1)})$-modules.*
- *The map*

$$\mathcal{P}_p \longrightarrow \bigsqcup_{n\geq 0} \mathsf{Irr}(F\mathfrak{S}_n), \quad \lambda \longmapsto [D_F^\lambda]$$

induces an isomorphism as $U_q(\mathfrak{g}(A_{p-1}^{(1)}))$-crystals

$$B(\Lambda_0) \cong (\mathcal{P}_p, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt}) \xrightarrow{\sim} (\bigsqcup_{n\geq 0} \mathsf{Irr}(F\mathfrak{S}_n), \varepsilon_i^{\Lambda_0}, \varphi_i^{\Lambda_0}, \tilde{e}_i^{\Lambda_0}, \tilde{f}_i^{\Lambda_0}, \mathrm{wt}^{\Lambda_0}).$$

3. MAIN RESULT

Recall that the wreath product $G \wr \mathfrak{S}_n$ is a semi-direct product $G^n \rtimes_\theta \mathfrak{S}_n$ where

$$\theta : \mathfrak{S}_n \longrightarrow \mathrm{Aut}(G^n), \quad \sigma \mapsto \theta(\sigma)((g_1, \cdots, g_n)) = (g_{\sigma^{-1}(1)}, \cdots, g_{\sigma^{-1}(n)}).$$

Hence any element $x \in G \wr \mathfrak{S}_n$ is written $x = (f; \sigma)$ for uniquely determined $f = (g_1, \cdots, g_n) \in G^n$ and $\sigma \in \mathfrak{S}_n$. The multiplication rule is given by

$$(f_1; \sigma_1) \cdot (f_2; \sigma_2) = (f_1 \cdot (f_2)_{\sigma_1}; \sigma_1 \cdot \sigma_2) \tag{1}$$

where $(f_2)_{\sigma_1} \stackrel{\text{def}}{=} \theta(\sigma_1)(f_2)$. The normal subgroup $(G^n \cong)\{(f; 1_{\mathfrak{S}_n}) \mid f \in G^n\} \subseteq G \wr \mathfrak{S}_n$ is often called the base group of $G \wr \mathfrak{S}_n$ and denoted by G^*.

Definition 3.1. *Denote by $G \wr \mathfrak{S}$ the inductive system $G \wr \mathfrak{S}_0 \subseteq G \wr \mathfrak{S}_1 \subseteq G \wr \mathfrak{S}_2 \subseteq \cdots$ of the wreath product groups of G.*

We need a representation theory of wreath products, which is a typical application of Clifford theory. An excellent presentation is given in [JK, Chapter 4] and we briefly quote it. Definitions in this chapter depend on a labeling of $\mathrm{Irr}(FG) = \{[V_1], \cdots, [V_\alpha]\}$.

Definition 3.2. *For a given composition $\vec{n} = (n_1, \cdots, n_\alpha)$ of n, we define an irreducible FG^*-module $E(\vec{n}) = V_1^{\otimes n_1} \otimes \cdots \otimes V_\alpha^{\otimes n_\alpha}$.*

Theorem 3.3 ([JK, 4.3.27]). *Let $\vec{n}$ be as above. The inertia group for $E(\vec{n})$ is given by $\{(f; \sigma) \mid f \in G^n, \sigma \in \mathfrak{S}_{\vec{n}}\} (\subseteq G \wr \mathfrak{S}_n)$. We denote it by $G^*\mathfrak{S}_{\vec{n}}$.*

Definition 3.4 ([JK, p.154]). *Let $\vec{n}$ be as above. To the underlying vector space $E(\vec{n})$, we can define an $F[G^*\mathfrak{S}_{\vec{n}}]$-module structure by*

$$(f; \sigma)(v_1 \otimes \cdots \otimes v_n) \stackrel{def}{=} g_1 v_{\sigma^{-1}(1)} \otimes \cdots \otimes g_n v_{\sigma^{-1}(n)}$$

where $f = (g_1, \cdots, g_n) \in G^n$ and $\sigma \in \mathfrak{S}_{\vec{n}}$. We denote this $F[G^*\mathfrak{S}_{\vec{n}}]$-module by $\widetilde{E}(\vec{n})$.

Definition 3.5. *For a given sequence of p-regular partitions $\vec{\lambda} = (\lambda_1, \cdots, \lambda_\alpha)$ such that $\sum_{\beta=1}^{\alpha} |\lambda_\beta| = n$, we write $\vec{n} = (|\lambda_1|, \cdots, |\lambda_\alpha|)$ and define an irreducible $F\mathfrak{S}_{\vec{n}}$-module $D(\vec{\lambda}) = D_F^{\lambda_1} \otimes \cdots \otimes D_F^{\lambda_\alpha}$.*

Definition 3.6 ([JK, 4.3.31]). *Let $\vec{\lambda}$ and $\vec{n}$ be as above. To the underlying vector space $D(\vec{\lambda})$, we define an $F[G^*\mathfrak{S}_{\vec{n}}]$-module structure by inflating along $G^*\mathfrak{S}_{\vec{n}} \twoheadrightarrow G^*\mathfrak{S}_{\vec{n}}/G^* \cong \mathfrak{S}_{\vec{n}}$. We denote this $F[G^*\mathfrak{S}_{\vec{n}}]$-module by $\widetilde{D}(\vec{\lambda})$.*

Definition and Theorem 3.7 ([JK, 4.4.3]). *Let $\vec{\lambda}$, $\vec{n}$ be as above. We define an $F[G \wr \mathfrak{S}_n]$-module $C(\vec{\lambda})$ by*

$$C(\vec{\lambda}) = \mathrm{Ind}_{F[G^*\mathfrak{S}_{\vec{n}}]}^{F[G\wr\mathfrak{S}_n]} \widetilde{E}(\vec{n}) \otimes \widetilde{D}(\vec{\lambda}).$$

Then the following map is bijective:

$$\{\vec{\lambda} = (\lambda_1, \cdots, \lambda_\alpha) \in \mathcal{P}_p^\alpha \mid \sum_{\beta=1}^{\alpha} |\lambda_\beta| = n\} \xrightarrow{\sim} \mathrm{Irr}(F[G \wr \mathfrak{S}_n]), \quad \vec{\lambda} \longmapsto C(\vec{\lambda}).$$

Under these preparations, we state the main result of this paper, a modular branching rule for wreath products.

Theorem 3.8. *Let G be a group and F be a splitting filed of characteristic $p(\geq 0)$. We denote by α the number of p-regular conjugacy classes of G and fix a labeling of $\mathsf{Irr}(FG) = \{[V_1], \cdots, [V_\alpha]\}$. Then, the map*

$$\mathbb{I}(\{\mathbb{B}_p\}_{i=1}^{\alpha}, \{\dim V_i\}_{i=1}^{\alpha}) \longrightarrow \mathbb{B}_F(G \wr \mathfrak{S}), \quad \overrightarrow{\lambda} = (\lambda_1, \cdots, \lambda_\alpha) \longmapsto [C(\overrightarrow{\lambda})]$$

is an isomorphism as directed graphs.

Proof. The first half of the proof is similar to that in [Tsu], but for readers' convenience we insert and begin with it.

We show the equivalent statement that for any sequence of p-regular partitions $\overrightarrow{\lambda} = (\lambda_1, \cdots, \lambda_\alpha)$ and $\overrightarrow{\mu} = (\mu_1, \cdots, \mu_\alpha)$ such that $\sum_{i=1}^{\alpha} |\lambda_i| = n$ and $\sum_{i=1}^{\alpha} |\mu_i| = n+1$, we have

$$\dim \mathrm{Hom}_{G \wr \mathfrak{S}_n}(C(\overrightarrow{\lambda}), \mathsf{Res}_{G \wr \mathfrak{S}_n}^{G \wr \mathfrak{S}_{n+1}} C(\overrightarrow{\mu})) = \begin{cases} \dim V_\gamma & (\exists \gamma \text{ s.t. the condition } \diamond_\gamma \text{ is satisfied}) \\ 0 & (\text{otherwise}), \end{cases} \tag{2}$$

where $\diamond_\gamma$ stands for the following condition (see Definition 2.6(10)):

$$\diamond_\gamma : E_p(\lambda_\gamma, \mu_\gamma) > 0 \text{ and } \lambda_{\gamma'} = \mu_{\gamma'} \text{ for any } \gamma' \neq \gamma. \tag{3}$$

Let $\overrightarrow{a} = (|\lambda_1|, \cdots, |\lambda_\alpha|)$, $\overrightarrow{b} = (|\mu_1|, \cdots, |\mu_\alpha|)$, $\overrightarrow{c} = (|\lambda_1|, \cdots, |\lambda_\alpha|, 1)$ and let $X = G \wr \mathfrak{S}_{n+1}, Y = G^{n+1}\mathfrak{S}_{\overrightarrow{b}} (\subseteq X), Z = G^n \mathfrak{S}_{\overrightarrow{c}} (= G^n \mathfrak{S}_{\overrightarrow{a}} \subseteq X), W = G \wr \mathfrak{S}_n (\subseteq X)$. By Frobenius reciprocity,

$$\begin{aligned} &\dim \mathrm{Hom}_W(C(\overrightarrow{\lambda}), \mathsf{Res}_W^X C(\overrightarrow{\mu})) \\ &= \dim \mathrm{Hom}_W(\mathsf{Ind}_Z^W \widetilde{E}(\overrightarrow{a}) \otimes \widetilde{D}(\overrightarrow{\lambda}), \mathsf{Res}_W^X C(\overrightarrow{\mu})) \\ &= \dim \mathrm{Hom}_Z(\widetilde{E}(\overrightarrow{a}) \otimes \widetilde{D}(\overrightarrow{\lambda}), \mathsf{Res}_Z^X C(\overrightarrow{\mu})). \end{aligned}$$

Let D be a (Z, Y)-double coset representatives in X. By Mackey theorem,

$$\begin{aligned} \mathsf{Res}_Z^X C(\overrightarrow{\mu}) &= \mathsf{Res}_Z^X \mathsf{Ind}_Y^X (\widetilde{E}(\overrightarrow{b}) \otimes \widetilde{D}(\overrightarrow{\mu})) \\ &\cong \bigoplus_{d \in D} \mathsf{Ind}_{dYd^{-1} \cap Z}^Z {}^d(\mathsf{Res}_{Y \cap d^{-1}Zd}^Y \widetilde{E}(\overrightarrow{b}) \otimes \widetilde{D}(\overrightarrow{\mu})) \end{aligned}$$

as FZ-modules where ${}^dM$ for an $F[Y \cap d^{-1}Zd]$-module M stands for an $F[dYd^{-1} \cap Z]$-module which is obtained by the pullback through the group isomorphism

$$dYd^{-1} \cap Z \xrightarrow{\sim} Y \cap d^{-1}Zd, \quad x \mapsto d^{-1}xd.$$

Since $dYd^{-1} \cap Z$ and $Y \cap d^{-1}Zd$ appear many times, we abbreviate $dYd^{-1} \cap Z$ and $Y \cap d^{-1}Zd$ to $V_{1,d}$ and $V_{2,d}$ respectively.

Now we recall a necessary fact about the double coset representatives of the symmetric groups. Let $\overrightarrow{\nu_1}, \overrightarrow{\nu_2}$ be compositions of n. Denote by $D_{\overrightarrow{\nu_2}}$ the set of minimal length left $\mathfrak{S}_{\overrightarrow{\nu_2}}$-coset representatives in $\mathfrak{S}_n$. Then $D_{\overrightarrow{\nu_1}\overrightarrow{\nu_2}} \stackrel{\mathrm{def}}{=} D_{\overrightarrow{\nu_1}}^{-1} \cap D_{\overrightarrow{\nu_2}}$ is the set of minimal length $(\mathfrak{S}_{\overrightarrow{\nu_1}}, \mathfrak{S}_{\overrightarrow{\nu_2}})$-double coset representatives in $\mathfrak{S}_n$ [DJ]. In the following discussion, it is important that we know $D_{\overrightarrow{\nu_2}}$ explicitly, that is for $\overrightarrow{\nu_2} = (\eta_1, \cdots, \eta_\kappa)$ we have

$$D_{\overrightarrow{\nu_2}} = \left\{ \sigma \in \mathfrak{S}_n \,\middle|\, \begin{matrix} \sigma(1) < \sigma(2) < \cdots < \sigma(\eta_1) \\ \vdots \\ \sigma(n-\eta_\kappa+1) < \cdots < \sigma(n) \end{matrix} \right\}.$$

By the multiplication rule (1), it is clear that $D_{\vec{c}\vec{b}}(\subseteq \mathfrak{S}_{n+1} \subseteq X)$ is a (Z,Y)-double coset representatives in X. Thus, we need to compute

$$(4)\quad \begin{aligned} L(d) &\stackrel{\text{def}}{=} \dim \operatorname{Hom}_Z\left(\widetilde{E}(\vec{a}) \otimes \widetilde{D}(\vec{\lambda}), \mathsf{Ind}_{V_{1,d}}^{Z}{}^{d}(\mathsf{Res}_{V_{2,d}}^{Y} \widetilde{E}(\vec{b}) \otimes \widetilde{D}(\vec{\mu}))\right) \\ &= \dim \operatorname{Hom}_{V_{1,d}}\left(\mathsf{Res}_{V_{1,d}}^{Z} \widetilde{E}(\vec{a}) \otimes \widetilde{D}(\vec{\lambda}), {}^{d}(\mathsf{Res}_{V_{2,d}}^{Y} \widetilde{E}(\vec{b}) \otimes \widetilde{D}(\vec{\mu}))\right) \end{aligned}$$

for each $d \in D_{\vec{c}\vec{b}}$ by Nakayama relations. By restricting to the subgroup $G^n \subseteq V_{1,d}(= G^n(d\mathfrak{S}_{\vec{b}}d^{-1} \cap \mathfrak{S}_{\vec{c}}) \subseteq X)$, we have

$$\begin{aligned} L(d) &\leq \dim \operatorname{Hom}_{G^n}\left(\mathsf{Res}_{G^n}^{Z} \widetilde{E}(\vec{a}) \otimes \widetilde{D}(\vec{\lambda}), \mathsf{Res}_{G^n}^{V_{1,d}}{}^{d}(\mathsf{Res}_{V_{2,d}}^{Y} \widetilde{E}(\vec{b}) \otimes \widetilde{D}(\vec{\mu}))\right) \\ &= \dim \operatorname{Hom}_{G^n}\left(E(\vec{a})^{\oplus \dim D(\vec{\lambda})}, (V_{\xi_1} \otimes \cdots \otimes V_{\xi_n})^{\oplus \dim D(\vec{\mu}) \dim V_{\xi_{n+1}}}\right) \end{aligned}$$

where we denote by $\xi(\chi)$ for $1 \leq \chi \leq n+1$ the unique $1 \leq \xi \leq \alpha$ such that

$$|\mu_1| + \cdots + |\mu_{\xi-1}| < \chi \leq |\mu_1| + \cdots + |\mu_\xi|$$

and we abbreviate $\xi(d^{-1}(i))$ to ξ_i for $1 \leq i \leq n+1$.

Lemma 3.9. *If $L(d) > 0$, then there exists an unique $1 \leq j \leq \alpha$ such that the followings conditions are satisfied.*

(A) $|\lambda_{j'}| = |\mu_{j'}|$ *for any* $j' \neq j$.
(B) $|\lambda_j| + 1 = |\mu_j|$.
(C) $d(|\lambda_1| + \cdots + |\lambda_j| + 1) = n + 1$.
(D) $d^{-1}(1) < \cdots < d^{-1}(n)$.

Moreover, we have

$$(5)\quad \begin{aligned} U_{1,d} &\stackrel{def}{=} d\mathfrak{S}_{\vec{b}}d^{-1} \cap \mathfrak{S}_{\vec{c}} = \mathfrak{S}_{(|\mu_1|,\cdots,|\mu_{j-1}|,|\mu_j|-1,|\mu_{j+1}|,\cdots,|\mu_\alpha|,1)}, \\ U_{2,d} &\stackrel{def}{=} \mathfrak{S}_{\vec{b}} \cap d^{-1}\mathfrak{S}_{\vec{c}}d = \mathfrak{S}_{(|\mu_1|,\cdots,|\mu_{j-1}|,|\mu_j|-1,1,|\mu_{j+1}|,\cdots,|\mu_\alpha|)}. \end{aligned}$$

Proof. We show that $j = \xi_{n+1}$. Note that

$$\begin{aligned} &\dim \operatorname{Hom}_{G^n}(E(\vec{a})^{\oplus \dim D(\vec{\lambda})}, (V_{\xi_1} \otimes \cdots \otimes V_{\xi_n})^{\oplus \dim D(\vec{\mu}) \dim V_{\xi_{n+1}}}) \\ &= \dim D(\vec{\lambda}) \dim D(\vec{\mu}) \dim V_{\xi_{n+1}} \dim \operatorname{Hom}_{G^n}(E(\vec{a}), V_{\xi_1} \otimes \cdots \otimes V_{\xi_n}) \end{aligned}$$

and the isomorphism between F-vector spaces

$$\operatorname{Hom}_{G^n}\left(E(\vec{a}), V_{\xi_1} \otimes \cdots \otimes V_{\xi_n}\right) \cong \operatorname{Hom}_G(V_{\zeta(1)}, V_{\xi_1}) \otimes \cdots \otimes \operatorname{Hom}_G(V_{\zeta(n)}, V_{\xi_n}),$$

where we denote by $\zeta(\chi)$ for $1 \leq \chi \leq n$ the unique $1 \leq \zeta \leq \alpha$ such that

$$|\lambda_1| + \cdots + |\lambda_{\zeta-1}| < \chi \leq |\lambda_1| + \cdots + |\lambda_\zeta|.$$

Hence, if $L(d) > 0$, we have (by recalling $d \in D_{\vec{c}}^{-1} \cap D_{\vec{b}} \subseteq D_{\vec{c}}^{-1}$)

$$\begin{cases} 1 \leq d^{-1}(1) < \cdots < d^{-1}(|\lambda_1|) \leq |\mu_1| \\ |\mu_1| + 1 \leq d^{-1}(|\lambda_1| + 1) < \cdots < d^{-1}(|\lambda_1| + |\lambda_2|) \leq |\mu_1| + |\mu_2| \\ \qquad \vdots \\ (n+1) - |\mu_\alpha| + 1 \leq d^{-1}(n - |\lambda_\alpha| + 1) < \cdots < d^{-1}(n) \leq n+1. \end{cases}$$

This implies that we have (A) and (B). Hence it is enough to show that (C) holds. Suppose to the contrary, we have

$$|\mu_1| + \cdots + |\mu_{j-1}| + 1 \leq d^{-1}(n+1) \leq |\mu_1| + \cdots + |\mu_j| - 1.$$

Because $d \in D_{\overrightarrow{c}}^{-1} \cap D_{\overrightarrow{b}} \subseteq D_{\overrightarrow{b}}$, we get the following contradiction:

$$d(d^{-1}(n+1)) = n+1 < d(|\mu_1| + \cdots + |\mu_j|).$$

Thus, we have (A),(B),(C),(D). Now (5) follows by the routine calculation. $\square$

Now we assume $L(d) > 0$ and d be the form in Lemma 3.9 for uniquely determined j. For $1 \le k \le \alpha$, define $\overrightarrow{\lambda}[k] = (\lambda'_1, \cdots, \lambda'_\alpha)$ to be a sequence of p-regular partitions determined by

$$\lambda'_i = \begin{cases} \lambda_k & (i = k) \\ () & (i \neq k) \end{cases}$$

where () stands for the empty partition and we define $\overrightarrow{\mu}[k]$ similarly. Under the identification $V_{1,d} \cong (G \wr \mathfrak{S}_{|\lambda_1|}) \times \cdots \times (G \wr \mathfrak{S}_{|\lambda_\alpha|})$, $\mathsf{Res}^Z_{V_{1,d}} \widetilde{E}(\overrightarrow{a}) \otimes \widetilde{D}(\overrightarrow{\lambda})$ and ${}^d(\mathsf{Res}^Y_{V_{2,d}} \widetilde{E}(\overrightarrow{b}) \otimes \widetilde{D}(\overrightarrow{\mu}))$ are identified with

$$\begin{cases} C(\overrightarrow{\lambda}[1]) \otimes \cdots \otimes C(\overrightarrow{\lambda}[\alpha]) \\ C(\overrightarrow{\mu}[1]) \otimes \cdots \otimes \mathsf{Res}^{G\wr\mathfrak{S}_{|\mu_j|}}_{G\wr\mathfrak{S}_{|\lambda_j|}} C(\overrightarrow{\mu}[j]) \otimes \cdots \otimes C(\overrightarrow{\mu}[\alpha]) \end{cases}$$

respectively. Hence, we have

$$\begin{aligned}(6)\qquad L(d) &\overset{(4)}{=} \dim \mathrm{Hom}_{V_{1,d}} \left(\mathsf{Res}^Z_{V_{1,d}} \widetilde{E}(\overrightarrow{a}) \otimes \widetilde{D}(\overrightarrow{\lambda}), {}^d(\mathsf{Res}^Y_{V_{2,d}} \widetilde{E}(\overrightarrow{b}) \otimes \widetilde{D}(\overrightarrow{\mu}))\right) \\ &= \dim \mathrm{Hom}_{G\wr\mathfrak{S}_{|\lambda_1|}}(C(\overrightarrow{\lambda}[1]), C(\overrightarrow{\mu}[1])) \times \cdots \\ &\qquad \times \cdots \times \dim \mathrm{Hom}_{G\wr\mathfrak{S}_{|\lambda_j|}}(C(\overrightarrow{\lambda}[j]), \mathsf{Res}^{G\wr\mathfrak{S}_{|\mu_j|}}_{G\wr\mathfrak{S}_{|\lambda_j|}} C(\overrightarrow{\mu}[j])) \\ &\qquad \times \cdots \times \dim \mathrm{Hom}_{G\wr\mathfrak{S}_{|\lambda_\alpha|}}(C(\overrightarrow{\lambda}[\alpha]), C(\overrightarrow{\mu}[\alpha])).\end{aligned}$$

For any $j' \neq j$, by (A) in Lemma 3.9 and by Definition and Theorem 3.7, we have

$$\begin{cases} \overrightarrow{\lambda}[j'] = \overrightarrow{\mu}[j'], \text{ i.e., } \lambda_{j'} = \mu_{j'} \\ \dim \mathrm{Hom}_{G\wr\mathfrak{S}_{|\lambda_{j'}|}}(C(\overrightarrow{\mu}[j']), C(\overrightarrow{\lambda}[j'])) = 1. \end{cases}$$

Next, consider the middle term of (6).

$$\begin{aligned}&\dim \mathrm{Hom}_{G\wr\mathfrak{S}_{|\lambda_j|}}(C(\overrightarrow{\lambda}[j]), \mathsf{Res}^{G\wr\mathfrak{S}_{|\mu_j|}}_{G\wr\mathfrak{S}_{|\lambda_j|}} C(\overrightarrow{\mu}[j])) \\ &= \dim V_j \dim \mathrm{Hom}_{G\wr\mathfrak{S}_{|\lambda_j|}}(\widetilde{E}(\overrightarrow{N}) \otimes \widetilde{D}(\overrightarrow{\lambda}[j]), \widetilde{E}(\overrightarrow{N}) \otimes \mathsf{Res}^{G\wr\mathfrak{S}_{|\lambda_j|+1}}_{G\wr\mathfrak{S}_{|\lambda_j|}} \widetilde{D}(\overrightarrow{\mu}[j])),\end{aligned}$$

where $\overrightarrow{N} = (0, \cdots, 0, \underbrace{|\lambda_j|}_{j\text{th place}}, 0, \cdots, 0)$. Thus to show (2), it is enough to show

$$\begin{aligned}(7)\qquad &\dim \mathrm{Hom}_{G\wr\mathfrak{S}_{|\lambda_j|}}(\widetilde{E}(\overrightarrow{N}) \otimes \widetilde{D}(\overrightarrow{\lambda}[j]), \widetilde{E}(\overrightarrow{N}) \otimes \mathsf{Res}^{G\wr\mathfrak{S}_{|\lambda_j|+1}}_{G\wr\mathfrak{S}_{|\lambda_j|}} \widetilde{D}(\overrightarrow{\mu}[j])) \\ &= E_p(\lambda_j, \mu_j).\end{aligned}$$

Lemma 3.10. *Let $\mathcal{G}, \mathcal{N}, \mathcal{H}$ be groups such that $\mathcal{G} = \mathcal{N} \rtimes \mathcal{H}$. For an $\mathcal{H}$-module $\mathcal{W}$, we can define a $\mathcal{G}$-module structure by inflating along $\mathcal{G} \twoheadrightarrow \mathcal{G}/\mathcal{N} \cong \mathcal{H}$ and denote this $\mathcal{G}$-module by $\widehat{\mathcal{W}}$. For an irreducible $\mathcal{G}$-module $\mathcal{U}$ and an $\mathcal{H}$-module $\mathcal{W}$, we have*

$$\mathsf{Soc}(\mathcal{U} \otimes \widehat{\mathcal{W}}) = \mathcal{U} \otimes \widehat{\mathsf{Soc}(\mathcal{W})} (= \mathcal{U} \otimes \mathsf{Soc}(\widehat{\mathcal{W}})),$$

under the following assumptions:

(i) $\mathrm{Res}^{\mathcal{G}}_{\mathcal{N}}(\mathcal{U})$ *is irreducible.*
(ii) *For any irreducible $\mathcal{H}$-module $\mathcal{V}$ which appears in the composition factors of $\mathcal{W}$, $\mathcal{U} \otimes \widehat{\mathcal{V}}$ is an irreducible $\mathcal{G}$-module.*

Proof. It is enough to show that for any irreducible $\mathcal{G}$-module $\mathcal{X}$ and for any $f \in \mathrm{Hom}_{\mathcal{G}}(\mathcal{X}, \mathcal{U} \otimes \widehat{\mathcal{W}})$ we have $\mathrm{Im}(f) \subseteq \mathcal{U} \otimes \widehat{\mathrm{Soc}(\mathcal{W})}$ since RHS is completely reducible by the assumption (ii).

Let us assume $\mathrm{Im}(f) \neq 0$ and take a filtration of $\mathcal{W}$ as an $\mathcal{H}$-module

$$\{0\} = \mathcal{W}_0 \subsetneq \mathcal{W}_1 \subsetneq \cdots \subsetneq \mathcal{W}_s = \mathcal{W}$$

whose successive quotients are all irreducible. By the assumption (ii),

$$\{0\} = (\mathcal{U} \otimes \widehat{\mathcal{W}_0}) \subsetneq (\mathcal{U} \otimes \widehat{\mathcal{W}_1}) \subsetneq \cdots \subsetneq (\mathcal{U} \otimes \widehat{\mathcal{W}_s}) = \mathcal{U} \otimes \widehat{\mathcal{W}}$$

is a filtration of $\mathcal{U} \otimes \widehat{\mathcal{W}}$ as a $\mathcal{G}$-module whose successive quotient

$$(\mathcal{U} \otimes \widehat{\mathcal{W}_i})/(\mathcal{U} \otimes \widehat{\mathcal{W}_{i-1}}) \cong \mathcal{U} \otimes (\widehat{\mathcal{W}_i/\mathcal{W}_{i-1}})$$

is irreducible. Take an unique $i \geq 1$ such that $\mathrm{Im}(f) \subseteq \mathcal{U} \otimes \widehat{\mathcal{W}_i}$ and $\mathrm{Im}(f) \not\subseteq \mathcal{U} \otimes \widehat{\mathcal{W}_{i-1}}$ whose existence is guaranteed by $\mathrm{Im}(f) \neq 0$. Then it is easy to see that $\mathrm{Im}(f) \oplus (\mathcal{U} \otimes \widehat{\mathcal{W}_{i-1}}) = \mathcal{U} \otimes \widehat{\mathcal{W}_i}$. Hence we have $\mathcal{X} \cong \mathrm{Im}(f) \cong \mathcal{U} \otimes (\widehat{\mathcal{W}_i/\mathcal{W}_{i-1}})$ and

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{G}}(\mathcal{X}, \mathcal{U} \otimes \widehat{\mathcal{W}}) &\cong \mathrm{Hom}_{\mathcal{G}}(\mathcal{U} \otimes (\widehat{\mathcal{W}_i/\mathcal{W}_{i-1}}), \mathcal{U} \otimes \widehat{\mathcal{W}}) \\
&\cong \mathrm{Hom}_{\mathcal{G}}(\widehat{\mathcal{W}_i/\mathcal{W}_{i-1}}, \mathrm{End}_F(\mathcal{U}) \otimes \widehat{\mathcal{W}}) \\
&\cong \mathrm{Hom}_{\mathcal{H}}(\mathcal{W}_i/\mathcal{W}_{i-1}, \mathrm{End}_{\mathcal{N}}(\mathcal{U}) \otimes \mathcal{W}) \\
&\cong \mathrm{Hom}_{\mathcal{H}}(\mathcal{W}_i/\mathcal{W}_{i-1}, \mathcal{W}).
\end{aligned}$$

By tracing the isomorphisms, we know that there exists an non-zero homomorphism $g \in \mathrm{Hom}_{\mathcal{H}}(\mathcal{W}_i/\mathcal{W}_{i-1}, \mathcal{W})$ such that $\mathrm{Im}(f) = \mathcal{U} \otimes \widehat{\mathrm{Im}(g)} \subseteq \mathcal{U} \otimes \widehat{\mathrm{Soc}(\mathcal{W})}$. □

Finally, by applying Lemma 3.10 under

$$\mathcal{G} = G \wr \mathfrak{S}_{|\lambda_j|}, \mathcal{N} = G^{|\lambda_j|}, \mathcal{H} = \mathfrak{S}_{|\lambda_j|}, \mathcal{U} = \widetilde{E}(\overrightarrow{N}), \mathcal{W} = \mathrm{Res}^{\mathfrak{S}_{|\lambda_j|+1}}_{\mathfrak{S}_{|\lambda_j|}}(D^{\mu_j}_F),$$

we have

$$\begin{aligned}
&\dim \mathrm{Hom}_{G\wr\mathfrak{S}_{|\lambda_j|}}(\widetilde{E}(\overrightarrow{N}) \otimes \widetilde{D}(\overrightarrow{\lambda}[j]), \widetilde{E}(\overrightarrow{N}) \otimes \mathrm{Res}^{G\wr\mathfrak{S}_{|\mu_j|}}_{G\wr\mathfrak{S}_{|\lambda_j|}} \widetilde{D}(\overrightarrow{\mu}[j])) \\
&= \dim \mathrm{Hom}_{\mathcal{G}}(\mathcal{U} \otimes \widehat{D^{\lambda_j}_F}, \mathrm{Soc}(\mathcal{U} \otimes \widehat{\mathcal{W}})) \\
&= \dim \mathrm{Hom}_{\mathcal{G}}(\mathcal{U} \otimes \widehat{D^{\lambda_j}_F}, \mathcal{U} \otimes \widehat{\mathrm{Soc}(\mathcal{W})}) \\
&= \sum_{E_p(\nu,\mu_j)>0} \dim \mathrm{Hom}_{\mathcal{G}}(\mathcal{U} \otimes \widehat{D^{\lambda_j}_F}, \mathcal{U} \otimes \widehat{D^{\nu}_F}) \\
&= E_p(\lambda_j, \mu_j).
\end{aligned}$$

Hence (7) is proved. □

Corollary 3.11. *$G \wr \mathfrak{S}$ is socle-multiplicity over F if and only if any irreducible FG-module is 1-dimensional.*

Corollary 3.12. *For any sequence of p-regular partitions* $\overrightarrow{\lambda} = (\lambda_1, \cdots, \lambda_\alpha)$ *and* $\overrightarrow{\mu} = (\mu_1, \cdots, \mu_\alpha)$ *such that* $\sum_{i=1}^{\alpha} |\lambda_i| = n$ *and* $\sum_{i=1}^{\alpha} |\mu_i| = n+1$, *we have*

$$\dim \operatorname{Hom}_{F[(G\wr\mathfrak{S}_n)\times G]}(C(\overrightarrow{\lambda}) \otimes V_\gamma, \operatorname{Res}^{F[G\wr\mathfrak{S}_{n+1}]}_{F[(G\wr\mathfrak{S}_n)\times G]} C(\overrightarrow{\mu})) \leq 1$$

and the equality holds if and only if the condition $\diamond_\gamma$ *is satisfied (see (3)). Especially, for any irreducible* $F[G\wr\mathfrak{S}_{n+1}]$*-module* V, $\operatorname{Soc}(\operatorname{Res}^{F[G\wr\mathfrak{S}_{n+1}]}_{F[(G\wr\mathfrak{S}_n)\times G]} V)$ *is multiplicity-free.*

Proof. We use the notations introduced in the proof of Theorem 3.8. Further, we put $Z' = G^n\mathfrak{S}_{\overrightarrow{a}} \times G (= G^{n+1}\mathfrak{S}_{\overrightarrow{c}} \subseteq X)$. It is enough to show that we have

$$\dim \operatorname{Hom}_{W\times G}(C(\overrightarrow{\lambda}) \otimes V_\gamma, \operatorname{Res}^X_{W\times G} C(\overrightarrow{\mu})) \geq 1 \tag{8}$$

if the condition $\diamond_\gamma$ is satisfied.

Since D is also a (Z', Y)-double coset representatives in X, we have

$$\begin{aligned}
&\dim \operatorname{Hom}_{W\times G}(C(\overrightarrow{\lambda}) \otimes V_\gamma, \operatorname{Res}^X_{W\times G} C(\overrightarrow{\mu}))\\
&= \dim \operatorname{Hom}_{W\times G}((\operatorname{Ind}^W_Z \widetilde{E}(\overrightarrow{a}) \otimes \widetilde{D}(\overrightarrow{\lambda})) \otimes V_\gamma, \operatorname{Res}^X_{W\times G} C(\overrightarrow{\mu}))\\
&= \dim \operatorname{Hom}_{W\times G}(\operatorname{Ind}^{W\times G}_{Z'} \widetilde{E}(\overrightarrow{a}) \otimes \widetilde{D}(\overrightarrow{\lambda}) \otimes V_\gamma, \operatorname{Res}^X_{W\times G} C(\overrightarrow{\mu}))\\
&= \dim \operatorname{Hom}_{Z'}(\widetilde{E}(\overrightarrow{a}) \otimes \widetilde{D}(\overrightarrow{\lambda}) \otimes V_\gamma, \operatorname{Res}^X_{Z'} C(\overrightarrow{\mu}))\\
&= \sum_{d\in D} \dim \operatorname{Hom}_{Z'}(\widetilde{E}(\overrightarrow{a}) \otimes \widetilde{D}(\overrightarrow{\lambda}) \otimes V_\gamma, \operatorname{Ind}^{Z'}_{dYd^{-1}\cap Z'} {}^d(\operatorname{Res}^Y_{Y\cap d^{-1}Z'd} \widetilde{E}(\overrightarrow{b}) \otimes \widetilde{D}(\overrightarrow{\mu})))
\end{aligned}$$

by Frobenius reciprocity, Nakayama relations and Mackey theorem.

Assume the condition $\diamond_\gamma$ is satisfied for a given $(\overrightarrow{\lambda}, \overrightarrow{\mu}, \gamma)$, and we can take an unique $d \in D$ characterized as in Lemma 3.9 for $j = \gamma$. Then, it is easily checked that $dYd^{-1} \cap Z' = G^{n+1}U_{1,d}, Y \cap d^{-1}Z'd = G^{n+1}U_{2,d}$ and that the map

$$\operatorname{Res}^{Z'}_{G^{n+1}U_{1,d}} \widetilde{E}(\overrightarrow{a}) \otimes \widetilde{D}(\overrightarrow{\lambda}) \otimes V_\gamma \longrightarrow {}^d(\operatorname{Res}^Y_{G^{n+1}U_{2,d}} \widetilde{E}(\overrightarrow{b}) \otimes \widetilde{D}(\overrightarrow{\mu}))$$

given by

$$\begin{aligned}
&v_1 \otimes \cdots v_n \otimes w_1 \otimes \cdots w_\alpha \otimes v\\
&\longmapsto v_1 \otimes \cdots v_x \otimes v \otimes v_{x+1} \cdots v_n \otimes w_1 \otimes \cdots w_{\gamma-1} \otimes f(w_\gamma) \otimes w_{\gamma+1} \otimes \cdots \otimes w_\alpha
\end{aligned}$$

is an injective $F[G^{n+1}U_{1,d}]$-module homomorphism where $x = |\lambda_1| + \cdots + |\lambda_\gamma|$ and

$$f : D^{\lambda_\gamma}_F \hookrightarrow \operatorname{Res}^{\mathfrak{S}_{|\mu_\gamma|}}_{\mathfrak{S}_{|\lambda_\gamma|}} D^{\mu_\gamma}_F$$

is an injective $F[\mathfrak{S}_{|\lambda_\gamma|}]$-module homomorphism whose existence is guaranteed by Kleshchev's modular branching rule up to non-zero scalar. Thus, (8) is proved. □

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