

Necessary Conditions for the Gevrey–Well–Posedness of Schrödinger Type Equations

Michael Dreher, University of Tsukuba *

1 Introduction

We study necessary conditions under which the following Cauchy problem of Schrödinger type,

$$Lu = \left(i\partial_t + \Delta + \sum_{j=1}^n b_j(x)\partial_{x_j} + c(x) \right) u = f(t, x), \quad u(0, x) = \varphi(x), \quad (1.1)$$

is well-posed in Gevrey spaces G^s , $1 < s < \infty$. Here $G^s = \varinjlim_{\varrho > 0} G_{\varrho}^s$, and G_{ϱ}^s is the Hilbert space $G_{\varrho}^s = \{v \in L^2(\mathbb{R}^n) : \|v\|_{s,\varrho} = \left\| \exp(\varrho \langle \xi \rangle^{1/s}) \hat{v}(\xi) \right\|_{L^2} < \infty\}$, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ and $\hat{v}$ is the usual Fourier transform of v .

Definition 1.1. We say that the Cauchy problem for the operator L is *forward G^s well-posed* if for every $T > 0$ and every $\varrho_0 > 0$ there are constants $C = C(T, \varrho_0)$ and $\varrho > 0$ such that for every $\varphi \in G_{\varrho_0}^s$, $f \in C([0, T], G_{\varrho_0}^s)$ there is a unique solution $u \in C([0, T], G_{\varrho}^s)$ to (1.1) with

$$\|u(t, \cdot)\|_{s,\varrho} \leq C \|\varphi\|_{s,\varrho_0} + C \int_0^t \|f(\tau, \cdot)\|_{s,\varrho_0} d\tau, \quad 0 \leq t \leq T.$$

If the coefficients b_j are purely imaginary valued, then *a priori* estimates of a solution u to (1.1) in the spaces L^2 , H^∞ , and G_{ϱ}^s can be easily derived, and the well-posedness of this Cauchy problem follows by standard arguments. The situation is more delicate when $\Re b_j \not\equiv 0$. For example, the Cauchy problem for the operator

*Faculty of Mathematics and Computer Science, Technical University Bergakademie Freiberg, Agricolastrasse 1, 09596 Freiberg, Germany

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$L = i\partial_t + \partial_x^2 + \partial_x$ is neither well-posed in L^2 nor in G^s , $1 < s < \infty$, as can be shown by an explicit representation of the solution, see also [12]. Generally, well-posedness requires a certain decay of $\Re b_j(x)$ at infinity.

Therefore, we propose the following condition:

Condition 1. There is a constant $M = M(d_0)$ such that

$$\sup_{x \in \mathbb{R}^n, \omega \in S^{n-1}} \left| \int_0^\sigma \sum_{j=1}^n \Re b_j(x + 2\theta\omega) \omega_j d\theta \right| \leq M(1 + |\sigma|)^{d_0}, \quad \forall \sigma \in \mathbb{R}.$$

We assume that the coefficients b_j and c belong to Gevrey spaces $G_{L^\infty}^{s_b}$, $G_{L^\infty}^s$:

$$\begin{aligned} \|\partial_x^\alpha b_j(\cdot)\|_{L^\infty} &\leq C^{1+|\alpha|} \alpha!^{s_b}, \quad \forall \alpha, \\ \|\partial_x^\alpha c(\cdot)\|_{L^\infty} &\leq C^{1+|\alpha|} \alpha!^s, \quad \forall \alpha. \end{aligned} \tag{1.2}$$

Theorem 1. *Let (1.2) be satisfied, and let d_0 be a number with $d_0 > 3/(s+1)$ and $d_0 > 2/(s+1-s_b)$. If Condition 1 is violated, then the Cauchy problem for the operator L is not G^s well-posed.*

Sufficient conditions for the G^s well-posedness of the Cauchy problem for the operator $L = i\partial_t + \Delta + \sum_{j=1}^n b_j(t, x) \partial_{x_j} + c(t, x)$ were given in [2], namely $\Re b_j(t, x) = o(\langle x \rangle^{1/s-1})$. In case of the model operator $L = i\partial_t + \Delta + \langle x \rangle^{d-1} \partial_x$ with $x \in \mathbb{R}^1$, and $0 < d < 1$, the Cauchy problem is therefore well-posed if $d < 1/s$. On the other hand, Theorem 1 implies ill-posedness for $d > 3/(s+1)$.

This gap can be closed if we suppose that the coefficients b_j decay not too rapidly:

Condition 2. There are $x_0 \in \mathbb{R}^n$, $\omega_0 \in S^{n-1}$, and $\varepsilon_0 > 0$, $c_0 > 0$ such that

$$-\sum_{j=1}^n \Re b_j(x + \tau\omega') \omega_j \geq 2c_0 \langle \tau \rangle^{d_0-1},$$

for all $\tau \geq 0$, $|x - x_0| < \varepsilon_0$, and all $\omega, \omega' \in S^{n-1}$ with $|\omega - \omega_0| < \varepsilon_0$, $|\omega' - \omega_0| < \varepsilon_0$.

Theorem 2. *Suppose (1.2) with $s_b < s$ and Condition 2. Then $d_0 \leq 1/s$ is necessary for the G^s well-posedness.*

A necessary condition for H^∞ well-posedness was given in [7]:

$$\sup_{x \in \mathbb{R}^n, \omega \in S^{n-1}} \left| \int_0^\sigma \sum_{j=1}^n \Re b_j(x + 2\theta\omega) \omega_j d\theta \right| \leq M \log(1 + |\sigma|) + N, \quad \forall \sigma \in \mathbb{R}.$$

This condition is sufficient in the case of one space dimension; and it is sufficient in the cases of two or more space dimensions if one supposes certain relations on derivatives of the coefficients b_j , see [8].

The investigation of an operator with variable coefficients in the principal part, $L = i\partial_t + \sum_{j,k} a_{jk}(x)\partial_{x_j}\partial_{x_k} + \sum_j b_j(x)\partial_{x_j} + c(x)$, where $a(x, \xi) = \sum_{j,k} a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2$, $c_0 > 0$, requires the introduction of the bicharacteristic strip $(X, P) = (X, P)(t, x, p)$, which is the solution to the Hamilton–Jacobi equations,

$$\partial_t X_j = \partial_{P_j} a(X, P), \quad \partial_t P_j = -\partial_{X_j} a(X, P), \quad (X, P)(0, x, p) = (x, p).$$

Then a necessary condition for the H^∞ well-posedness is

$$\sup_{x, \omega} \left| \int_0^\sigma \sum_{j=1}^n \Re b_j(X(\theta, x, \omega)) P_j(\theta, x, \omega) d\theta \right| \leq M \log(1 + |\sigma|) + N, \quad \forall \sigma \in \mathbb{R},$$

under some additional condition. For details, see [6].

Sufficient and necessary conditions for H^s well-posedness were discussed in [3], [4] and [13]. These conditions are similar to the conditions for H^∞ well-posedness if a loss of regularity is allowed, otherwise similar to the conditions of L^2 well-posedness.

In [9] and [11], the following necessary condition for L^2 well-posedness was shown:

$$\sup_{x \in \mathbb{R}^n, \omega \in S^{n-1}} \left| \int_0^\sigma \sum_{j=1}^n \Re b_j(X(\theta, x, \omega)) P_j(\theta, x, \omega) d\theta \right| \leq M, \quad \forall \sigma \in \mathbb{R}.$$

This condition is also sufficient, see [10].

Schrödinger type equations with a lower order term of order strictly less than 1 were investigated in [1]; and sufficient conditions for G^s well-posedness were proved.

Theorem 1 and Theorem 2 will be discussed simultaneously; and the both cases will be called Case I and Case II, respectively. The following lemma, which gives us an integrated estimate of $\Re b_j$ from below, is quite essential.

Lemma 1.1. *Assume that $0 < d_0 < 1$ and that Condition 1 is violated. Then, for each $k \in \mathbb{N}$, there are $x_k \in \mathbb{R}^n$, $\sigma_k \in \mathbb{R}_+$, $\omega_k \in S^{n-1}$ with the property that*

$$\begin{aligned} - \int_0^{\sigma_k} \sum_{j=1}^n \Re b_j(x_k + 2\theta\omega_k) \omega_{k,j} d\theta &= k(1 + \sigma_k)^{d_0}, \\ - \int_0^\sigma \sum_{j=1}^n \Re b_j(x_k + 2\theta\omega_k) \omega_{k,j} d\theta &\geq k d_0 \sigma (1 + \sigma_k)^{d_0-1}, \quad 0 \leq \sigma \leq \sigma_k, \end{aligned}$$

where σ_k tends to infinity for $k \rightarrow \infty$.

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This lemma gives us a sequence $\{\sigma_k\}_k$ tending to infinity in Case I. In Case II, we choose this sequence arbitrarily, but still tending to infinity. Now we fix special initial data, $\varphi_k(x) = \varphi(x - x_k)$ (in Case I), and $\varphi_k(x) = \varphi(x - x_0)$ (in Case II), where $\varphi \in G_{\varrho_0}^s$ is given by $\hat{\varphi}(\xi) = \langle \xi \rangle^{-(n+1)/2} \exp(-\varrho_0 \langle \xi \rangle^{1/s})$. Assuming that (1.1) is G^s well-posed, there is a unique solution $u_k \in C^1([0, T], G_{\varrho}^s)$ of

$$Lu_k = 0, \quad u_k(0, x) = \varphi_k(x). \quad (1.3)$$

Next we define a seminorm $E_k(t)$ for the function $u_k(t, \cdot)$.

Let $h = h(x) \in G^{s_0}$ (with $s_0 > 1$ very close to 1) be a function with

$$h(x) = \begin{cases} 0 & : |x| \geq 1, \\ 1 & : |x| \leq 1/2, \end{cases} \quad 0 \leq h(x) \leq 1.$$

We choose the symbols

$$w_k(t, x, \xi) = h\left(\frac{x - x_k - 2t\sigma_k^{\delta_3}\omega_k}{\sigma_k^{-\delta_1}}\right) h\left(\frac{\xi - \sigma_k^{\delta_3}\omega_k}{\sigma_k^{\delta_2}}\right), \quad (\text{Case I}),$$

$$w_k(t, x, \xi) = h\left(\frac{x - x_0 - 2t\sigma_k\omega_0}{\varepsilon \langle 2t\sigma_k \rangle}\right) h\left(\frac{\xi - \sigma_k\omega_0}{\sigma_k^{\delta_2}}\right), \quad (\text{Case II}),$$

where $0 < \varepsilon \ll \varepsilon_0$, $\delta_1 = 1 - d_0$, and δ_2, δ_3 are certain positive constants. For multiindices $\alpha, \beta \in \mathbb{N}^n$, we specify

$$w_k^{(\alpha\beta)}(t, x, \xi) = \partial_y^\alpha h(y) \partial_\eta^\beta(\eta) \Big|_{y=\sigma_k^{\delta_1}(x-x_k-2t\sigma_k^{\delta_3}\omega_k), \eta=\sigma_k^{-\delta_2}(\xi-\sigma_k^{\delta_3}\omega_k)},$$

$$w_k^{(\alpha\beta)}(t, x, \xi) = \partial_y^\alpha h(\varepsilon^{-1}y) \partial_\eta^\beta(\eta) \Big|_{y=\langle 2t\sigma_k \rangle^{-1}(x-x_0-2t\sigma_k\omega_0), \eta=\sigma_k^{-\delta_2}(\xi-\sigma_k\omega_0)},$$

in Case I, Case II, respectively. These cut-off symbols are supported near the bicharacteristic strip. With some positive constant κ , we set $N \ni N_0 = \lfloor \sigma_k^{\kappa/s_1} \rfloor$, choose $s_1 > s_0$, and define the seminorm

$$E_k(t) = \sum_{|\alpha| \leq N_0, |\beta| \leq N_0-2} (\alpha! \beta!)^{-s_1} \left\| W_k^{(\alpha\beta)}(t, x, D_x) u_k(t, x) \right\|_{L^2(\mathbb{R}_x^n)}.$$

The ill-posedness of the Cauchy problem can be proved by estimates of $E_k(t)$ from above and below which contradict for large σ_k if we choose $\delta_1, \delta_2, \delta_3, \kappa, \varepsilon$ suitably. For reasons of space, we omit the tedious calculations, which can be found in [5], and only sketch the proof.

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It is easy to estimate E_k from above: the symbols $w_k^{(\alpha\beta)}$ belong to the Hörmander class $S_{0,0}^0$, then the Calderon–Vaillancourt theorem and the presumed well-posedness of the Cauchy problem give

$$E_k(t) \leq C \sigma_k^C \|\varphi\|_{s,\ell_0}.$$

To get an estimate from below, we write

$$\begin{aligned} v_k^{(\alpha\beta)}(t, x) &= W_k^{(\alpha\beta)}(t, x, D_x) u_k(t, x), \\ B(x, D_x) &= - \sum_{j=1}^n \Re b_j(x) D_{x_j}, \end{aligned}$$

and can deduce that

$$\begin{aligned} & \left\| v_k^{(\alpha\beta)} \right\|_{L^2} \partial_t \left\| v_k^{(\alpha\beta)} \right\|_{L^2} = \Re \left(\partial_t v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \right) \\ &= \Re \left(-i[L, W_k^{(\alpha\beta)}] u_k, v_k^{(\alpha\beta)} \right) + \Re \left(i \triangle v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \right) \\ & \quad + \sum_{j=1}^n \Re \left(i b_j \partial_{x_j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \right) + \Re \left(i c v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \right) \\ & \geq - \left\| [L, W_k^{(\alpha\beta)}] u_k \right\|_{L^2} \left\| v_k^{(\alpha\beta)} \right\|_{L^2} + \Re \left(B(x, D_x) v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \right) - C \left\| v_k^{(\alpha\beta)} \right\|_{L^2}^2. \end{aligned}$$

Now we need an estimate of $\left\| [L, W_k^{(\alpha\beta)}] u_k \right\|_{L^2}$ from above, and an estimate of $\Re \left(B(x, D_x) v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \right)$ from below.

The symbol of $[L, W_k^{(\alpha\beta)}]$ can be written as an asymptotic expansion, up to some remainder, and $\left\| [L, W_k^{(\alpha\beta)}] u_k \right\|_{L^2}$ can be estimated by certain norms $\left\| v_k^{(\alpha+\gamma, \beta+\delta)} \right\|_{L^2}$ plus some remainder which becomes negligible for $\sigma_k \rightarrow \infty$.

The term $\Re \left(B(x, D_x) v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \right)$ can be estimated using Condition 2 and Garding's inequality, or Lemma 1.1 and Gronwall's Lemma.

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