

On Splitting Numbers

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Abstract. We shall discuss about splitting numbers of uncountable regular cardinals.

1. Question about splitting numbers

The author introduced a question about splitting numbers of uncountable regular cardinals at a talk in Aug. 4. 1992 at RIMS, Kyoto.

Let κ be an infinite cardinal. $\mathcal{S} \subseteq [\kappa]^\kappa (= \{X \subseteq \kappa : |X| = \kappa\})$ is called a splitting family on κ if for all $X \in [\kappa]^\kappa$ there exists an $A \in \mathcal{S}$ such that $|X \cap A| = |X \setminus A| = \kappa$. The splitting number of κ is the minimum cardinality of a splitting family on κ . We denote it by $s(\kappa)$.

In particular, $s(\omega)$ is the original splitting number, which is one of the so-called six cardinals and the following is well-known.

FACT. [2]

- (1) $s(\omega) \geq \omega_1$.
- (2) $\text{Con}(ZFC)$ implies $\text{Con}(ZFC + s(\omega) \geq \omega_2)$.

What about uncountable regular cardinals? In 1991, M. Motoyoshi studied splitting numbers of uncountable regular cardinals under the supervision of S. Kamo.

FACT. [5] Let κ be an uncountable regular cardinal. Then κ is strongly inaccessible $\iff s(\kappa) \geq \kappa$.

At that time I pointed out the following fact.

PROPOSITION. Let κ be an uncountable regular cardinal. Then κ is weakly compact $\iff s(\kappa) \geq \kappa^+$.

PROOF: ($\Rightarrow$) Suppose κ is weakly compact. Assume for a contradiction that $\mathcal{S} = \{S_\alpha : \alpha < \kappa\}$ is a splitting family on κ . For each α , let $S_\alpha^0 = S_\alpha$ and $S_\alpha^1 = \kappa \setminus S_\alpha$. Put $T = \{f \in {}^{<\kappa}2 : |\kappa \cap \bigcap \{S_\alpha^{f(\alpha)} : \alpha \in \text{dom } f\}| = \kappa\}$. Then $(T, \subseteq)$ makes a κ -tree. Hence there is a $g : \kappa \rightarrow 2$ such that for all $\alpha < \kappa$ $g \restriction \alpha \in T$. Consider the next two cases. (i) When the sequence $\langle \bigcap \{S_\alpha^{g(\alpha)} : \alpha < \xi\} : 0 < \xi < \kappa \rangle$ is eventually constant. (ii) Otherwise. In either cases, using the sequence above, we can get an $X \in [\kappa]^\kappa$ which witnesses that $\mathcal{S}$ is not a splitting family on κ , a contradiction.

($\Leftarrow$) Suppose κ is uncountable regular and $s(\kappa) \geq \kappa^+$. By Motoyoshi's result above, κ is strongly inaccessible. Assume for a contradiction that κ is not weakly compact. Let $(T, <)$ be a well-pruned κ -Aronszajn tree which witnesses this assumption. For each $t \in T$, let $S_t = \{u \in T : t \preceq u\}$ and put $\mathcal{S} = \{S_t : t \in T \text{ and } |S_t| = \kappa\}$. Fix an arbitrary $X \in [T]^\kappa$. Then it is easily verified that for some $\alpha < \kappa$, the α -th level of T has two distinct members, say u and v , so that both $X \cap S_u$ and $X \cap S_v$ has size κ ; if not, one can get a chain of length κ in T , contradicting that T is κ -Aronszajn. Since X was arbitrary, $\mathcal{S}$ makes a splitting family on T . However, the size of $\mathcal{S}$ is clearly at most κ . Therefore $s(\kappa) \leq \kappa$, a contradiction (Q.E.D.)

Question 1 Does $\text{Con}(ZFC + \exists \text{ a measurable cardinal})$ imply $\text{Con}(ZFC + \exists \kappa \text{ s.t. } \kappa \text{ is an uncountable regular cardinal and } s(\kappa) \geq \kappa^{++})$?

REMARK: $\text{Con}(s(\kappa) \geq \kappa^{++}, \kappa \text{ uncountable regular})$ implies $\text{Con}(\exists \text{ a measurable})$. Suppose κ is an uncountable regular cardinal and $s(\kappa) \geq \kappa^{++}$. Let K be the Dodd-Jensen core model. There exists an $X \in [\kappa]^\kappa$ such that no $Y \in P(\kappa) \cap K$ can split X . Then $U = \{Y \in P(\kappa) \cap K : |X \setminus Y| < \kappa\}$ makes a K - κ -complete ultrafilter. Hence by Dodd-Jensen's result [1], there is an inner model of a measurable cardinal.

2. Answer(s)

In August 1992, S.Kamo and T.Miyamoto independently showed a slight stronger result than the following.

THEOREM. [3], [4] $\text{Con}(ZFC + \exists \text{ a supercompact cardinal})$ implies

$Con(ZFC + \exists \kappa \text{ s.t. } \kappa \text{ is an uncountable regular cardinal and } s(\kappa) \geq \kappa^{++})$

After that, in October 1992, J. Zapletal in U. S. told us that he improved the lower bound.

Answer. [6] $Con(ZFC + \exists \kappa \text{ s.t. } \kappa \text{ is an uncountable regular cardinal and } s(\kappa) \geq \kappa^{++})$ implies $Con(ZFC + \exists \text{ a measurable cardinal } \kappa \text{ such that } o(\kappa) = \kappa^{++})$.

Thus we had negative answer for **Question 1**. However, I don't know the exact consistency strength yet.

Question 2 Find a large cardinal axiom which is equi-consistent to the existence of uncountable regular κ s. t. $s(\kappa) \geq \kappa^{++}$.

References

- [1] A.Dodd, B.Jensen, *The covering lemma for K*, Ann. Math. Logic (1982).
- [2] E.K.van Douwen, *the integers and topology*, (in Handbook of Set-theoretic Topology) (1984).
- [3] Shizuo Kamo, *Splitting numbers on uncountable regular cardinals*, preprint (Aug. 1992).
- [4] Tadatoshi Miyamoto, *Personal letters*, (Aug. 1992).
- [5] Minoru Motoyoshi, *On the cardinalities of splitting families of uncountable regular cardinals (written in Japanese)*, Master thesis (Univ. of Osaka Prefecture) (Mar. 1992).
- [6] Jindrich Zapletal, *Splitting number and the core model*, preprint (Oct. 1992).