

September 6, 1995

An explicit formula for zeta functions associated with quadratic forms¹

TOMOYOSHI IBUKIYAMA, OSAKA UNIVERSITY

Here we shall consider the following problem taking up the space of quadratic forms.

Problem. *Explicitly write down the zeta functions of prehomogeneous vector spaces.*

§1.

1.1. Let

$$V = V_n := \{x \in M_n \mid x = {}^t x\},$$

$$G := GL_n,$$

$$\rho(g)x := gx {}^t g \quad (g \in G, x \in V),$$

$$V_n(\mathbb{R}) \setminus \{\det x = 0\} = V_n^n \cup V_n^{n-1} \cup \cdots \cup V_n^0, \text{ where } x \in V_n^i \text{ if and only if } x$$

has i positive and $n - i$ negative eigenvalues,

$$L \subset V_n(\mathbb{R}) \text{ an } SL_n(\mathbb{Z})\text{-invariant lattice,}$$

$$L^{(i)} := L \cap V_n^i,$$

$$c_n := \frac{2 \prod_{k=1}^n \Gamma(\frac{k}{2})}{\pi^{n(n+1)/4}},$$

$$\mu(x) := \text{'size' of } \rho(SL_n(\mathbb{Z}))x \text{ for } x \in L \text{ with } \det x \neq 0 \text{ (see [Sa, (1.5)] and}$$

(1.2) below),

$$\zeta_i(s, L) := c_n \sum_{x \in L_n^{(i)} / SL_n(\mathbb{Z})} \frac{\mu(x)}{|\det x|^s}.$$

¹Notes by Akihiko Gyoja.

1.2. Example. For $x \in L^{(n)}$, we have

$$c_n \mu(x) = \varepsilon(x)^{-1},$$

where

$$\varepsilon(x) := |\{\gamma \in SL_n(\mathbb{Z}) \mid \gamma x {}^t \gamma = x\}|.$$

In particular,

$$\zeta_n(s, L) = \sum_{x \in L^{(n)}/SL_n(\mathbb{Z})} \frac{1}{|\det x|^s \varepsilon(x)}.$$

1.3. Remark. If $n \geq 3$, there are exactly 2 possibilities of the choice of L (up to constant multiple), i.e.,

$$\begin{aligned} L_n &:= \{(x_{ij}) \in V_n \mid x_{ij} \in \mathbb{Z}\} \\ &= \text{the integral lattice,} \end{aligned}$$

and

$$\begin{aligned} L_n^* &:= \{(x_{ij}) \in V_n \mid x_{ii} \in \mathbb{Z}, \quad x_{ij} \in \tfrac{1}{2}\mathbb{Z} \ (i \neq j)\} \\ &= \text{the half integral lattice.} \end{aligned}$$

Cf. [IS2]. Note that L_n^* is the dual lattice of L_n with respect to the bilinear form $\text{tr}(xy)$, and its elements can be identified with the integral quadratic forms.

If $n = 2$, there are 4 lattices, i.e., besides L_2 and L_2^* , we have the lattices

$$\begin{aligned} M &:= \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in L_2 \mid a + b + c \equiv 0 \pmod{2} \right\}, \text{ and} \\ N &:= \left\{ \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in L_2^* \mid a \equiv b \equiv c \pmod{2} \right\}. \end{aligned}$$

Cf. [IS3]. (Since $SL_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we can see that M and N are $SL_2(\mathbb{Z})$ -invariant simply by noting that they are invariant under these two matrices.)

§2. Review of quadratic forms (1).

2.1. Genera and classes. For $x_1, x_2 \in V_n(\mathbb{Q})$, we say that

(2.1.1) x_1 and x_2 belong to the *same genus* if $x_1 \sim_{GL_n(\mathbb{Z}_v)} x_2$ for all places $v \leq +\infty$, where $\mathbb{Z}_v = \mathbb{Z}_p$ if $v = p$ and $\mathbb{Z}_\infty = \mathbb{R}$, and

(2.1.2) x_1 and x_2 belong to the *same class* if $x_1 \sim_{SL_n(\mathbb{Z})} x_2$.

Then each genus consists of several classes, whose cardinality is known to be finite, and is called the *class number*. The class number measures the difference between the local theory and the global theory. It is important but rarely calculable.

2.1.3. Remark. If x_1 and x_2 belong to the same genus, then $\det x_1 = \det x_2$. (In fact, $c := \det x_1 / \det x_2 \in \mathbb{Q}^\times$ belongs to $\mathbb{Z}_v^{\times 2}$ for all $v \leq +\infty$. Hence $c = 1$.)

2.2. Siegel Mass formula. For $x \in L_n^*$,

$$(2.2.1) \quad 2^{-1} \cdot p^{-\frac{n(n-1)\nu}{2}} \times (\#O(x \bmod p^\nu))$$

becomes stable as $\nu \rightarrow +\infty$, where $O(x \bmod p^\nu)$ is the orthogonal group contained in $GL_n(\mathbb{Z}/p^\nu\mathbb{Z})$. We put $\alpha_p(x) := \lim_{\nu \rightarrow +\infty} (2.2.1)$, and call it the *local density*. An explicit formula for $\alpha_p(x)$ will be given in (3.3).

Let $\mathcal{L}$ be a genus, and d the common value of $\det x$ ($x \in \mathcal{L}$). Cf. (2.1.3). The Siegel Mass formula says

$$(2.2.2) \quad \sum_{\substack{x \in \mathcal{L}/SL_n(\mathbb{Z}) \\ \det x = d}} \mu(x) = \frac{2|d|^{\frac{n+1}{2}}}{\prod_p \alpha_p(x)}.$$

§3. Review of quadratic forms (2).

3.1. Equivalence of quadratic forms over fields.

3.1.1. Discriminant. For $x \in V_n(\mathbb{Q}_v)$, if $x \sim_{GL_n(\mathbb{Q}_v)} \text{diag}(a_1, \dots, a_m, 0, \dots, 0)$ with $a_i \in \mathbb{Q}_v^\times$, put

$$\Delta_v(x) := \prod_{i=1}^m a_i.$$

Then $\Delta_v(x)$ is well-defined as an element of $\mathbb{Q}_v^\times / \mathbb{Q}_v^{\times 2}$ and called the *discriminant* of x .

3.1.2. Hasse invariant. For $0 \neq a, b \in \mathbb{Z}_v$ ($v \leq +\infty$), the Hilbert symbol $(a, b)_v = \pm 1$ is defined so that $= +1$ iff $ax^2 + by^2 = 1$ has a solution (x, y) in $\mathbb{Q}_v$.² The Hilbert symbol defines a symmetric bilinear form on $\mathbb{Q}_v^\times / \mathbb{Q}_v^{\times 2}$ (i.e., $(aa', b)_v = (a, b)_v(a', b)_v$ and $(a, b)_v = (b, a)_v$). Moreover it is non-degenerate and satisfies $(a, -a)_v = (a, 1 - a)_v = 1$. For any $x \in V_n(\mathbb{Q}_v)$ there exists $g \in GL_n(\mathbb{Q}_v)$ such that

$$gx^t g = \text{diag}(a_1, \dots, a_m, 0, \dots, 0) \quad (a_i \in \mathbb{Q}_v^\times)$$

(Cf. [Se, Chapter 4, Theorem 1].) Put

$$S_v(x) := \prod_{1 \leq i \leq j \leq m} (a_i, a_j)_v.$$

Then $S_v(x)$ depends only on x , and is independent of the choice of g .³ The invariant $S_v(x)$ is called the *Hasse invariant* of x .

²Note that $(a, b)_v = +1$ iff $ax^2 + by^2 = z^2$ has a solution $(x, y, z) \neq (0, 0, 0)$ in $\mathbb{Q}_v$. In fact, if it has a solution (x_0, y_0, z_0) with $z_0 \neq 0$ (resp. $z_0 = 0$), then $(x, y) = (x_0/z_0, y_0/z_0)$ (resp. $(x, y) = (\frac{1}{2}x_0((ax_0^2)^{-1} + 1), \frac{1}{2}y_0((ax_0^2)^{-1} - 1))$) is a solution of $ax^2 + by^2 = 1$. Then the above definition is equivalent to the one given in [Se, Chapter 3].

³In fact, the well-definedness of $\varepsilon_v(x) := \prod_{i < j} (a_i, a_j)_v$ is proved in [Se, Chapter 4, Theorem 5]. Since

$$\prod_i (a_i, a_i)_v = \prod_i (a_i, -a_i)_v (a_i, -1)_v = (\Delta_v(x), -1)_v,$$

we have $S_v(x) = \varepsilon_v(x)(\Delta_v(x), -1)_v$, and we get the well-definedness.

Remark. As we have seen above, our definition of the Hasse invariant is different from the invariant $\varepsilon(x)$ studied in [Se]. Our convention is the same as [O] and [Kit].

Example. Assume that $p \neq 2$, $d_1, d_2 \in \mathbb{Z}_p^\times$ and $\sigma_1, \sigma_2 \in \mathbb{Z}$. Then

$$(p^{\sigma_1} d_1, p^{\sigma_2} d_2)_p = \left(\frac{-1}{p} \right)^{\sigma_1 \sigma_2} \left(\frac{d_1}{p} \right)^{\sigma_2} \left(\frac{d_2}{p} \right)^{\sigma_1},$$

where $\left(\frac{*}{p} \right)$ is the Legendre symbol. [Se, Chapter 3, Theorem 1].

Local Theory.

3.1.3. (1) A $GL_n(\mathbb{Q}_p)$ -isomorphism class of $x \in V_n(\mathbb{Q}_p)$ is determined by

- (a) rank x ,
- (b) $\Delta_p(x) \in \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2}$, and
- (c) $S_p(x) = \text{Hasse invariant}$.

Cf. [Se, Chapter 4, Theorem 7]. (Note that if $\det x \neq 0$, $\Delta_p(x) = (\det x \bmod \mathbb{Q}_p^{\times 2})$.)

(2) A $GL_n(\mathbb{R})$ -isomorphism class of $x \in V_n(\mathbb{R})$ is determined by the signature of x .

Global theory.

The determination of the equivalence relation of quadratic forms over $\mathbb{Q}$ consists of the following two steps.

3.1.4. Hasse principle. The diagonal mapping

$$\delta : V_n(\mathbb{Q})/GL_n(\mathbb{Q}) \rightarrow \prod_{v \leq +\infty} V_n(\mathbb{Q}_v)/GL_n(\mathbb{Q}_v)$$

is injective. Cf. [Se, Chapter 4, Theorem 9].

3.1.5. Image of δ . Assume that $(x_v)_v \in \prod_{v \leq +\infty} V_n(\mathbb{Q}_v)$ is given. In order that the equivalence class of $(x_v)_v$ belongs to the image of δ , it is necessary and sufficient that the following four conditions are satisfied.

- (0) For all $v \leq +\infty$, $\text{rad}(x_v) := \{a \in \mathbb{Q}_v^n \mid ax_v b = 0 \text{ for all } b \in \mathbb{Q}_v^n\}$ are defined over $\mathbb{Q}$, and $\text{rad}(x_v) \cap \mathbb{Q}^n$ are independent of v .

- (1) There exists $d \in \mathbb{Q}^\times$ such that $d \in \Delta_v(x_v) \cdot \mathbb{Q}_v^{\times 2}$ for all $v \leq +\infty$;
- (2) $S_v(x_v) = 1$ for almost all v ;
- (3) $\prod_{v \leq +\infty} S_v(x_v) = 1$.

Cf. [Se, Chapter 4, Prop. 7].

3.2. Equivalence of quadratic forms over rings. Here and below, we assume that the $\mathbb{Z}$ -structure of V_n is given by $V_n(\mathbb{Z}) = L_n^*$.

3.2.1. Local theory. First assume that $p \neq 2$. A complete list of representatives of $\{x \in V_n(\mathbb{Z}_p) \mid \det x \neq 0\}/GL_n(\mathbb{Z}_p)$ is given by

$$(1) \quad \text{diag}(p^{\sigma_1} \underbrace{\text{diag}(1, \dots, 1, d_1)}_{n_1}, \dots, p^{\sigma_s} \underbrace{\text{diag}(1, \dots, 1, d_s)}_{n_s})$$

with $d_i \in \mathbb{Z}_p^\times / \mathbb{Z}_p^{\times 2}$ ($1 \leq i \leq s$). The complete set of invariants is given by⁴

$$\left\{ \begin{array}{l} 0 \leq \sigma_1 < \sigma_2 < \dots < \sigma_s \\ n_i \in \mathbb{Z}_{>0}, \quad \left(\frac{d_i}{p}\right) = \pm 1, \quad (1 \leq i \leq s) \end{array} \right\},$$

i.e.,

- (a) two elements x and y are $GL_n(\mathbb{Z}_p)$ -equivalent iff these invariants for x and y are the same and
- (b) every such $\{\sigma_i, n_i, \pm 1\}$ are given by some x . ((b) is trivial.)

3.2.2. Global theory. Let us consider when

- (1) a collection $(x_v)_{v \leq +\infty} \in \prod_{v \leq +\infty} V_n(\mathbb{Z}_v)/GL_n(\mathbb{Z}_v)$ comes from an element of $V_n(\mathbb{Z})$,

⁴See [O] for the detail. It is enough to read pp. 81–89 in order to understand basic concepts. Then read pp.227–233 and pp.243–247 in order to understand the above result. In the terminology of [O], the decomposition into the direct sum of $p^{\sigma_i} \text{diag}(1, \dots, 1, d_i)$ is the Jordan splitting. Each block gives a p^{σ_i} -modular lattice. Concerning the isomorphism class of each block, see [O, 92:1], which, in fact, can be proved directly. Note that we are assuming $p \neq 2$. Glance over pp. 250–279 and see how complicated the similar result in the case $p = 2$.

assuming that $d \in \mathbb{Q}^\times$ is given, and that $x_v \in V_n(\mathbb{Z}_v)$ and $\det(x_v) = d$ for all $v \leq +\infty$.

As we have seen in (3.1.5), it is necessary that (3.1.5, (1)–(3)) are satisfied. By our assumption, (3.1.5, (1)) is satisfied. Note that $d \in \mathbb{Z}_p^\times$ for almost all p . If $x_v \underset{GL_n(\mathbb{Z}_v)}{\sim} \text{diag}(1, \dots, 1, \varepsilon)$, then $\varepsilon \in d \cdot \mathbb{Z}_v^{\times 2}$ and hence $S_v(x_v) = (\varepsilon, \varepsilon)_v = (d, d)_v = 1$ for almost all $v \leq +\infty$ by [Se, Chapter 3, Theorem 3], i.e., (3.1.5, (2)) is satisfied. Then only (3.1.5, (3)) is essential. In fact, we can show the following.

For $d \in \mathbb{Q}^\times$, the diagonal mapping

$$\begin{aligned} \delta : \{x \in V_n(\mathbb{Z}) \mid \det(x) = d\} / \approx \\ \rightarrow \{(x_v)_v \in \prod_{v \leq +\infty} V_n(\mathbb{Z}_v) \mid \det(x_v) = d\mathbb{Z}_v^{\times 2} \text{ for all } v, \\ \text{and } \prod_{v \leq +\infty} S_v(x_v) = 1\} / \prod_{v \leq +\infty} GL_n(\mathbb{Z}_v) \end{aligned}$$

*is a bijection, where $x_1 \approx x_2$ means that x_1 and x_2 belong to the same genus.*⁵

Proof. The injectivity is trivial. Let us prove that δ is surjective. By (3.1.5), there exists $y \in V_n(\mathbb{Q})$ such that $y \underset{GL_n(\mathbb{Q}_v)}{\sim} x_v$ for all $v \leq +\infty$. Note that $y \in V_n(\mathbb{Z}_p)$ and $\det(y) \in \mathbb{Z}_p^\times$ for almost all $p < \infty$. For such $p \neq 2$, $y \underset{GL_n(\mathbb{Z}_p)}{\sim} x_p$ by (3.2.1), i.e., there exists $g \in GL_n(\mathbb{Q}_A)$ such that $y = g(x_v)_v {}^t g$. Decompose $g = \gamma^{-1} g'$ according to the decomposition $GL_n(\mathbb{Q}_A) = GL_n(\mathbb{Q}) \times \prod_{v \leq +\infty} GL_n(\mathbb{Z}_v)$.⁶ Put $x := \gamma y {}^t \gamma$. Then $x = g'(x_v)_v {}^t g'$ with $g' \in \prod_{v \leq +\infty} GL_n(\mathbb{Z}_v)$, and hence $x = \gamma y {}^t \gamma \in V_n(\mathbb{Q}) \cap \prod_v V_n(\mathbb{Z}_v) = V_n(\mathbb{Z})$. ■

We record a modification of (3.2.2).

⁵By the same argument as above, we can see that, if $\det(x_v) \in d\mathbb{Z}_v^{\times 2}$ for all v , then $S_v(x_v) = 1$ for almost all v , and hence $\prod_v S_v(x_v)$ is a finite product.

⁶In fact, $|GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{Q}_A) / \prod_{v \leq +\infty} GL_n(\mathbb{Z}_v)| = (\text{class number of } \mathbb{Q}) = 1$.

3.2.3. In the same notation as (3.2.2), for $0 < d \in 2^{-n}\mathbb{Z}$, the diagonal mapping

$$(1) \quad \begin{aligned} & \delta : \{x \in L_n^{*(n)} \mid \det(x) = d\} / \approx \\ & \rightarrow \{(x_p)_p \in \prod_{p < \infty} V_n(\mathbb{Z}_p) \mid \det(x_p) \in d\mathbb{Z}_p^{\times 2} \text{ for all } p < +\infty, \text{ and} \\ & \quad \prod_{p < +\infty} S_p(x_p) = 1\} / \prod_{p < \infty} GL_n(\mathbb{Z}_p). \end{aligned}$$

is a bijection.

3.3. Local density $\alpha_p(x)$ for x as (3.2.1, (1)) is given by

$\alpha_p(x) = 2^{s-1} \cdot p^w \cdot P \cdot E$, where

$$w := \sum_{i=1}^s \frac{n_i(n_i+1)}{2} \sigma_i + \sum_{j < i} n_i n_j \sigma_j,$$

$$P := \prod_{i=1}^s P\left(\left[\frac{n_i}{2}\right]\right) \text{ with } P(m) := \prod_{i=1}^m \left(1 - \frac{1}{p^{2^i}}\right),$$

$$E := \prod_{n_i : \text{even}} \left(1 + \frac{1}{p^{n_i/2}} \left(\frac{(-1)^{n_i/2} d_i}{p}\right)\right)^{-1}.$$

See [Ko] and [Kit].

§4. Calculation in some simple cases.

4.1. Now we calculate

$$(4.1.1) \quad \zeta_n(s, L_n^*) = c_n \sum_{x \in L_n^{*(n)}/SL_n(\mathbb{Z})} \frac{\mu(x)}{|\det x|^x}.$$

(For the sake of simplicity, we restrict ourselves to $\zeta_i(s, L)$ with $i = n$ and $L = L_n^*$.)

For general i with $L = L_n$ or $L = L_n^*$, see [IS1]. For $L = M$ or $L = N$, see [IS3].)

By (3.2.3), we have

$$(4.1.2) \quad \zeta_n(s, L_n^*) = c_n \sum_{d \in 2^{-n}\mathbb{Z}_{>0}} d^{-s} \left\{ \sum_{(x_p)_p} \left(\sum_{\substack{x \in L_n^{*(n)}/SL_n(\mathbb{Z}) \\ x \sim_{GL_n(\mathbb{Z}_p)} x_p, \text{ for all } p}} \mu(x) \right) \right\},$$

where in the second summation on the right hand side, $(x_p)_p$ runs over the right hand side of (3.2.3, (1)). By Siegel's Mass formula (2.2.2), the inside of $\{ \}$ of (4.1.2) is equal to

$$\begin{aligned}
 & 2d^{\frac{n+1}{2}} \sum_{\substack{(x_p)_p \\ \det x_p \in d\mathbb{Z}_p^{\times 2} \\ \prod_p S_p(x_p)=1}} \frac{1}{\prod_p \alpha_p(x)} \\
 (4.1.3) \quad & = d^{\frac{n+1}{2}} \left\{ \sum_{\substack{(x_p)_p \\ \det x_p \in d\mathbb{Z}_p^{\times 2}}} \frac{1}{\prod_p \alpha_p(x_p)} + \sum_{\substack{(x_p)_p \\ \det x_p \in d\mathbb{Z}_p^{\times 2}}} \frac{1}{\prod_p \alpha_p(x_p) S_p(x_p)} \right\}
 \end{aligned}$$

By (4.1.2) and (4.1.3), we get

$$\begin{aligned}
 & c_n^{-1} \zeta_n(s, L_n^*) \\
 (4.1.4) \quad & = \sum_{d \in 2^{-n}\mathbb{Z}_{>0}} d^{-s+\frac{n+1}{2}} \prod_p \left(\sum_{\substack{x_p \in V(\mathbb{Z}_p)/GL_n(\mathbb{Z}_p) \\ \det x_p \in d\mathbb{Z}_p^{\times 2}}} \alpha_p(x_p)^{-1} \right) \\
 & + \sum_{d \in 2^{-n}\mathbb{Z}_{>0}} d^{-s+\frac{n+1}{2}} \prod_p \left(\sum_{\substack{x_p \in V(\mathbb{Z}_p)/GL_n(\mathbb{Z}_p) \\ \det x_p \in d\mathbb{Z}_p^{\times 2}}} (\alpha_p(x_p) S_p(x_p))^{-1} \right).
 \end{aligned}$$

4.2. We proceed to the remaining calculation taking up as an example the case $n = 3$.

Then the complete list of representatives x as in (3.2.1, (1)) with $\det(x) = d \in 2^{-n}\mathbb{Z}_{>0}$

($d = p^t d_0$, $(d_0, p) = 1$) is given by

$$(1, 1, 1) : x = \text{diag}(p^{\sigma_1} d_1, p^{\sigma_2} d_2, p^{\sigma_3} d_3), t = \sigma_1 + \sigma_2 + \sigma_3, d_0 = d_1 d_2 d_3,$$

$$(1, 2) : x = \text{diag}(p^{\sigma_1} d_1, p^{\sigma_2}, p^{\sigma_2} d_2), t = \sigma_1 + 2\sigma_2, d_0 = d_1 d_2,$$

$$(2, 1) : x = \text{diag}(p^{\sigma_1}, p^{\sigma_1} d_1, p^{\sigma_2} d_2), t = 2\sigma_1 + \sigma_2, d_0 = d_1 d_2,$$

$$(3) : x = \text{diag}(p^{\sigma_1}, p^{\sigma_1}, p^{\sigma_1} d_1), t = 3\sigma_1, d_0 = d_1,$$

where $(1, 1, 1)$ etc. indicates $(n_1, n_2, \dots)$ in (3.2.1, (1)).

Let $p \neq 2$. Fix an element $\varepsilon \in \mathbb{Z}_p^\times \setminus \mathbb{Z}_p^{\times 2}$. For $d = p^t d_0$ with $(d_0, p) = 1$, we get using (3.3) that

$$\begin{aligned}
(4.2.1) \quad & \sum_{\substack{x_p \in V(\mathbb{Z}_p)/GL_n(\mathbb{Z}_p) \\ \det x_p \in d\mathbb{Z}_p^{\times 2}}} \alpha_p(x_p)^{-1} \\
&= \sum_{\substack{0 \leq \sigma_1 < \sigma_2 < \sigma_3 \\ d_1, d_2, d_3 \in \{1, \varepsilon\} \\ \sigma_1 + \sigma_2 + \sigma_3 = t \\ d_1 d_2 d_3 \in d_0 \mathbb{Z}_p^{\times 2}}} (2^2 \cdot p^{3\sigma_1 + 2\sigma_2 + \sigma_3})^{-1} \quad \text{contribution of } (1, 1, 1) \\
&+ \sum_{\substack{0 \leq \sigma_1 < \sigma_2 \\ d_1, d_2 \in \{1, \varepsilon\} \\ \sigma_1 + 2\sigma_2 = t \\ d_1 d_2 \in d_0 \mathbb{Z}_p^{\times 2}}} \left(2^1 \cdot p^{3\sigma_1 + 3\sigma_2} \left(1 - \frac{1}{p^2} \right) \right)^{-1} \left(1 + \frac{1}{p} \left(\frac{-d_2}{p} \right) \right) \quad \text{contribution of } (1, 2) \\
&+ \sum_{\substack{0 \leq \sigma_1 < \sigma_2 \\ d_1, d_2 \in \{1, \varepsilon\} \\ 2\sigma_1 + \sigma_2 = t \\ d_1 d_2 \in d_0 \mathbb{Z}_p^{\times 2}}} \left(2^1 \cdot p^{5\sigma_1 + \sigma_2} \left(1 - \frac{1}{p^2} \right) \right)^{-1} \left(1 + \frac{1}{p} \left(\frac{-d_1}{p} \right) \right) \quad \text{contribution of } (2, 1) \\
&+ \sum_{\substack{0 \leq \sigma_1 \\ d_1 \in \{1, \varepsilon\} \\ 3\sigma_1 = t \\ d_1 \in d_0 \mathbb{Z}_p^{\times 2}}} \left(2^0 \cdot p^{6\sigma_1} \left(1 - \frac{1}{p^2} \right) \right)^{-1} \quad \text{contribution of } (3) \\
(4.2.2) \quad &= \sum_{\substack{0 \leq \sigma_1 < \sigma_2 < \sigma_3 \\ \sigma_1 + \sigma_2 + \sigma_3 = t}} (p^{3\sigma_1 + 2\sigma_2 + \sigma_3})^{-1} \\
&+ \sum_{\substack{0 \leq \sigma_1 < \sigma_2 \\ \sigma_1 + 2\sigma_2 = t}} \left(p^{3\sigma_1 + 3\sigma_2} \left(1 - \frac{1}{p^2} \right) \right)^{-1} \\
&+ \sum_{\substack{0 \leq \sigma_1 < \sigma_2 \\ 2\sigma_1 + \sigma_2 = t}} \left(p^{5\sigma_1 + \sigma_2} \left(1 - \frac{1}{p^2} \right) \right)^{-1} \\
&+ \sum_{\substack{0 \leq \sigma_1 \\ 3\sigma_1 = t}} \left(p^{6\sigma_1} \left(1 - \frac{1}{p^2} \right) \right)^{-1}
\end{aligned}$$

What is important concerning the last expression is that it does not contain d any more. Similar expression can be obtained also for $p = 2$. (This case is much more

difficult. See [IS2, pp.17–24].) Hence we can express the first summation of (4.1.4) as a product, for all prime numbers p , of

$$\begin{aligned}
 & \sum_{t \geq 0} p^{t(-s+2)} \left\{ \sum_{\substack{0 \leq \sigma_1 < \sigma_2 < \sigma_3 \\ \sigma_1 + \sigma_2 + \sigma_3 = t}} p^{-3\sigma_1 - 2\sigma_2 - \sigma_3} \right. \\
 & + \left(1 - \frac{1}{p^2}\right)^{-1} \sum_{\substack{0 \leq \sigma_1 < \sigma_2 \\ \sigma_1 + 2\sigma_2 = t}} p^{-3\sigma_1 - 3\sigma_2} \\
 & + \left(1 - \frac{1}{p^2}\right)^{-1} \sum_{\substack{0 \leq \sigma_1 < \sigma_2 \\ 2\sigma_1 + \sigma_2 = t}} p^{-5\sigma_1 - \sigma_2} \\
 & \left. + \left(1 - \frac{1}{p^2}\right)^{-1} \sum_{\substack{0 \leq \sigma_1 \\ 3\sigma_1 = t}} p^{-6\sigma_1} \right\} \\
 & = \sum_{0 \leq \sigma_1 < \sigma_2 < \sigma_3} u^{\sigma_1 + \sigma_2 + \sigma_3} p^{-\sigma_1 + \sigma_3} + \dots \\
 (4.2.3) \quad & = \frac{p^2 u^3}{(1 - u^3)(1 - pu^2)(1 - pu)} \\
 & + \left(1 - \frac{1}{p^2}\right)^{-1} \frac{pu^2}{(1 - u^3)(1 - pu^2)} \\
 & + \left(1 - \frac{1}{p^2}\right)^{-1} \frac{pu}{(1 - u^3)(1 - pu)} \\
 & + \left(1 - \frac{1}{p^2}\right)^{-1} \frac{1}{1 - u^3} \\
 (4.2.4) \quad & = \left(1 - \frac{1}{p^2}\right)^{-1} \frac{1}{(1 - pu)(1 - pu^2)},
 \end{aligned}$$

where $u = p^{-s}$. In the case $p = 2$, we have in place of (4.2.4)

$$(4.2.5) \quad \frac{1}{3} \frac{u^{-2}}{(1 - 2u)(1 - 2u^2)} = \left(1 - \frac{1}{p^2}\right)^{-1} \frac{1}{(1 - pu)(1 - pu^2)} \cdot \frac{2^{2s}}{4}.$$

Next we calculate the second summand of (4.1.4). For x of type $(1, 1, 1)$, $(1, 2)$, $(2, 1)$ and (3) , the respective Hasse invariants are given as follows.

$$\begin{aligned}
(1, 1, 1) : S_p(x) &= \left(\frac{-1}{p}\right)^{\sum_{i < j} \sigma_i \sigma_j} \left(\frac{d_0}{p}\right)^{\sigma_1 + \sigma_2} \left(\frac{d_1}{p}\right)^{\sigma_1 + \sigma_3} \left(\frac{d_2}{p}\right)^{\sigma_2 + \sigma_3} \\
(1, 2) : S_p(x) &= \left(\frac{-d_2}{p}\right)^{\sigma_1 + \sigma_2} \\
(2, 1) : S_p(x) &= \left(\frac{-d_1}{p}\right)^{\sigma_1 + \sigma_2} \\
(3) : S_p(x) &= 1
\end{aligned}$$

Hence replacing $\alpha_p(x_p)^{-1}$ in (4.2.1) with $\alpha_p(x_p)^{-1} \cdot S_p(x_p)$, we get in place of (4.2.2)

$$\begin{aligned}
& \sum_{\substack{0 \leq \sigma_1 < \sigma_2 < \sigma_3 \\ \sigma_1 + \sigma_2 + \sigma_3 = t \\ \sigma_1 \equiv \sigma_2 \equiv \sigma_3}} \left(p^{3\sigma_1 + 2\sigma_2 + \sigma_3}\right)^{-1} \\
& + \sum_{\substack{0 \leq \sigma_1 < \sigma_2 \\ \sigma_1 + 2\sigma_2 = t \\ \sigma_1 \equiv \sigma_2}} \left(p^{3\sigma_1 + 3\sigma_2} \left(1 - \frac{1}{p^2}\right)\right)^{-1} + \sum_{\substack{0 \leq \sigma_1 < \sigma_2 \\ \sigma_1 + 2\sigma_2 = t \\ \sigma_1 \equiv \sigma_2 + 1}} \left(p^{3\sigma_1 + 3\sigma_2} \left(1 - \frac{1}{p^2}\right)\right)^{-1} \frac{1}{p} \\
& + \sum_{\substack{0 \leq \sigma_1 < \sigma_2 \\ 2\sigma_1 + \sigma_2 = t \\ \sigma_1 \equiv \sigma_2}} \left(p^{5\sigma_1 + \sigma_2} \left(1 - \frac{1}{p^2}\right)\right)^{-1} + \sum_{\substack{0 \leq \sigma_1 < \sigma_2 \\ 2\sigma_1 + \sigma_2 = t \\ \sigma_1 \equiv \sigma_2 + 1}} \left(p^{5\sigma_1 + \sigma_2} \left(1 - \frac{1}{p^2}\right)\right)^{-1} \frac{1}{p} \\
& + \sum_{\substack{0 \leq \sigma_1 \\ 3\sigma_1 = t}} \left(p^{6\sigma_1} \left(1 - \frac{1}{p^2}\right)\right)^{-1},
\end{aligned}$$

where the congruence relation is considered modulo 2. Therefore we get in place of (4.2.3)

$$\begin{aligned}
& \frac{p^4 u^6}{(1 - u^3)(1 - p^2 u^4)(1 - p^2 u^2)} \\
& + \left(1 - \frac{1}{p^2}\right)^{-1} \frac{p^2 u^4 + u^2}{(1 - u^3)(1 - p^2 u^4)} \\
& + \left(1 - \frac{1}{p^2}\right)^{-1} \frac{p^2 u^2 + u}{(1 - u^3)(1 - p^2 u^2)} \\
& + \left(1 - \frac{1}{p^2}\right)^{-1} \frac{1}{1 - u^3} \\
(4.2.6) \quad & = \left(1 - \frac{1}{p^2}\right)^{-1} \frac{1}{(1 - u)(1 - p^2 u^2)}.
\end{aligned}$$

In the case $p = 2$, we have in place of (4.2.6)

$$(4.2.7) \quad -\frac{1}{3} \frac{u^{-2}}{(1-u)(1-4u^2)} = -\left(1 - \frac{1}{p^2}\right)^{-1} \frac{1}{(1-u)(1-p^2u^2)} \cdot \frac{2^{2s}}{4}.$$

Summing up (4.2.4)–(4.2.7), we get⁷

$$\zeta_3(s, L_3^*) = \frac{2^{2s}}{24} (\zeta(s-1)\zeta(2s-1) - \zeta(s)\zeta(2s-2)).$$

References

- [IS1] T.Ibukiyama, H.Saito, *On zeta functions associated to symmetric matrices and an explicit conjecture on dimensions of Siegel modular forms of general degree*, Int. Math. Research Notices, No.8, Duke Math. J. **67** (1992), 161–169.
- [IS2] T.Ibukiyama, H.Saito, *On zeta functions associated to symmetric matrices I, An explicit form of zeta functions*, preprint.
- [IS3] T.Ibukiyama, H.Saito, *On L-functions of ternary zero forms and exponential sums of Lee-Weintraub*, J. Number Theory **48** (1994), 252–257.
- [Kim] T.Kimura, *Introduction to the theory of prehomogeneous vector spaces*, in the same volume.
- [Kit] Y.Kitaoka, “Arithmetic of quadratic forms,” Cambridge Univ. Press.
- [Ko] O.Körner, *Die Masse der Geschlechter quadratischer Formen vom Range ≤ 3 in quadratischen Zahlkörper*, Math. Ann. **193** (1971), 279–314.

⁷If n is odd, $\zeta_n(s, L_n^*)$ can be expressed in terms of the Riemann zeta function $\zeta(s)$ in a similar way. If n is even, then d does not disappear in (4.2.2), and in the final expression of $\zeta_n(s, L_n^*)$ appears an infinite sum of Dirichlet L -functions for the quadratic Dirichlet characters associated with the quadratic fields. Cf. [IS1] and [IS2]. In the case of the space of quadratic forms, this infinite sum incidentally coincides with the Mellin transform of some Eisenstein series of one variable of half-integral weight $(n+1)/2$ belonging to $\Gamma_0(4)$. It is not clear whether this is of general feature, when we consider more general prehomogeneous vector spaces.

- [O] O.T.O'Meara, "Introduction to quadratic forms," Springer Verlag, 1973.
- [Sa] F.Sato, *Introduction to the theory of zeta functions of prehomogeneous vector spaces*, in the same volume.
- [Se] J.P.Serre, "Cours d'arithmétique," Presses Universitaires de France, Paris, 1970.