

Some inequalities for distance-regular graphs

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1 Definitions

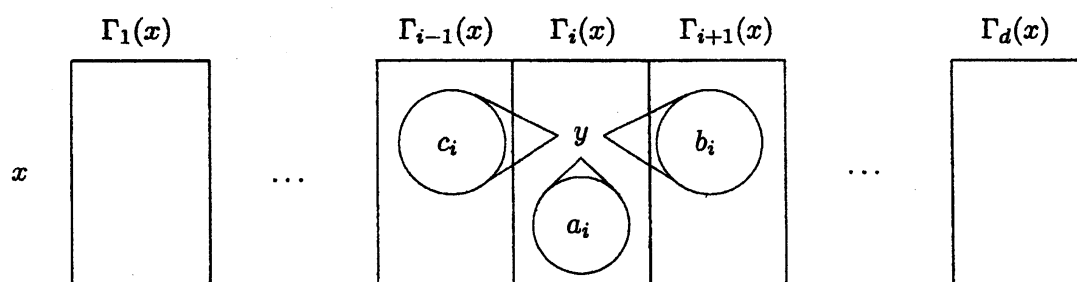
All graphs considered are undirected finite graphs without loops or multiple edges. Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph with usual shortest path distance ∂_Γ . The *diameter* of Γ , denoted by d , is the maximal distance of two vertices in Γ . Let $u \in \Gamma$. We denote by $\Gamma_j(u)$ the set of vertices which are at distance j from u . For $x, y \in V\Gamma$ with $\partial_\Gamma(x, y) = i$, let

$$\begin{aligned} C_i(x, y) &= \Gamma_{i-1}(x) \cap \Gamma_1(y), \\ A_i(x, y) &= \Gamma_i(x) \cap \Gamma_1(y), \\ B_i(x, y) &= \Gamma_{i+1}(x) \cap \Gamma_1(y). \end{aligned}$$

A connected graph Γ is said to be *distance-regular* if

$$c_i = |C_i(x, y)|, \quad a_i = |A_i(x, y)| \quad \text{and} \quad b_i = |B_i(x, y)|$$

depend only on $i = \partial_\Gamma(x, y)$ rather than individual vertices.



The numbers c_i , a_i and b_i are called the *intersection numbers* of Γ . In particular, $k := b_0$ is called the *valency* of Γ . The array

$$\iota(\Gamma) = \begin{pmatrix} * & c_1 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_i & \cdots & b_{d-1} & * \end{pmatrix}$$

is called the *intersection array* of Γ . Define

$$r = r(\Gamma) := \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}.$$

Let $\Gamma = (V\Gamma, E\Gamma)$ be a distance-regular graph of diameter $d \geq 2$ and valency $k \geq 3$. Define $R_i := \{(x, y) \mid \partial_\Gamma(x, y) = i\}$. Then $(V\Gamma, \{R_i\}_{0 \leq i \leq d})$ is an association scheme of class d and

$$p_{i,j}^t := \#\{z \mid \partial_\Gamma(x, z) = i, \partial_\Gamma(y, z) = j\},$$

where $\partial_\Gamma(x, y) = t$.

Conversely, for an *association scheme* $(X, \{R_i\}_{0 \leq i \leq d})$ satisfying *P-polynomial condition*, the graph (X, R_1) is a distance-regular graph. (See [1] and [3].)

2 The Odd graph and the doubled Odd graph

Let m be a positive integer and X a set of $2m + 1$ elements. For $0 \leq t \leq m$ define

$$X_t := \{Y \subseteq X \mid \#Y = t\}.$$

The *Odd graph* O_{m+1} is a graph with

$$V\Gamma = X_m, \quad E\Gamma = \{(Y, Y') \mid Y \cap Y' = \emptyset\}.$$

The *doubled Odd graph* $2O_{m+1}$ is a graph with

$$V\Gamma = X_m \cup X_{m+1} \quad E\Gamma = \{(Y, Z) \mid Y \in X_m, Z \in X_{m+1}, Y \subset Z\}.$$

Then Odd graph O_{m+1} is a distance-regular graph of diameter m with :

Case : $m = 2t$,

$$\left\{ \begin{array}{cccccccc} * & 1 & 1 & 2 & 2 & \cdots & t & t \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & t+1 \\ 2t+1 & 2t & 2t & 2t-1 & 2t-1 & \cdots & t+1 & * \end{array} \right\}.$$

Case : $m = 2t - 1$,

$$\left\{ \begin{array}{cccccccc} * & 1 & 1 & 2 & 2 & \cdots & t-1 & t \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & t \\ 2t & 2t-1 & 2t-1 & 2t-2 & 2t-2 & \cdots & t+1 & * \end{array} \right\}.$$

The doubled Odd graph $2O_{m+1}$ is distance-regular of diameter $2m + 1$ with :

$$\left\{ \begin{array}{cccccccccc} * & 1 & 1 & 2 & 2 & \cdots & m-1 & m & m & m+1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ m+1 & m & m & m-1 & m-1 & \cdots & 2 & 1 & 1 & * \end{array} \right\}.$$

More information for these graphs can be found in [3, §9.1.D].

These graphs had been characterized by some of their intersection numbers. In particular, Ray-Chaudhuri and Sprague [8], Cuypers [4, Theorems 4.6-4.7] and Koolen [7, Theorem 16] proved the following result.

Theorem 1 *Let Γ be a distance-regular graph of diameter $d \geq 5$ such that $c_1 = c_2 = 1$, $c_3 = c_4 = 2$ and $a_1 = \cdots = a_4 = 0$. Then Γ is either the Odd graph or the doubled Odd graph.* ■

3 Some Inequalities for distance-regular graphs

The following are well known basic properties of the intersection numbers.

- (1) $1 = c_1 \leq c_2 \leq \dots \leq c_{d-1} \leq c_d \leq k$.
- (2) $k = b_0 \geq b_1 \geq \dots \geq b_{d-2} \geq b_{d-1} \geq 1$.
- (3) $c_i \leq b_j$ if $i + j \leq d$.

We recall the proof of these properties:

Let $(x_0, x_1, \dots, x_d)$ be a path of length d such that $\partial_\Gamma(x_0, x_d) = d$. Then we have

$$C_1(x_1, x_0) \subseteq C_2(x_2, x_0) \subseteq \dots \subseteq C_{d-1}(x_{d-1}, x_0) \subseteq C_d(x_d, x_0) \subseteq \Gamma_1(x_0),$$

$$B_0(x_0, x_0) \supseteq B_1(x_1, x_0) \supseteq \dots \supseteq B_{d-2}(x_{d-2}, x_0) \supseteq B_{d-1}(x_{d-1}, x_0) \supseteq \{x_1\}$$

and

$$C_i(x_0, x_i) \subseteq B_j(x_{i+j}, x_i).$$

■

The idea of this proof is : “ Let $X \subseteq Y$ be subsets of $V\Gamma$. Then $\#X \leq \#Y$. ”

We prove several inequalities for intersection numbers by applying this idea. For example:

Theorem 2 Suppose $1 \leq c_{t-1} < c_t$. Then $c_{2t-1} \neq c_t$.

Sketch of the Proof.

Suppose $c_{2t-1} = c_t$ and derive a contradiction. Let $u, x, y \in V\Gamma$ such that

$$\partial_\Gamma(u, y) = 2t - 1, \partial_\Gamma(u, x) = t - 1 \quad \text{and} \quad \partial_\Gamma(x, y) = t.$$

Then there exist $v \in C_{2t-1}(y, u) \setminus C_{t-1}(x, u)$ and $u' \in \Gamma_1(v) \cap \Gamma_{t-1}(x)$ such that

$$B_{t-1}(u', x) \subseteq B_{t-1}(u, x) \setminus C_{t-1}(y, x).$$

Then this implies $b_{t-1} \leq b_{t-1} - c_{t-1}$ which is a contradiction.

■

Theorem 3 Let Γ be a distance-regular graph with:

$$\left\{ \begin{array}{ccccccc} * & 1 & \cdots & 1 & 2 & \cdots & 2 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ k & k-1 & \cdots & k-1 & k-2 & \cdots & k-2 \end{array} \quad \cdots \right\}.$$

$\underbrace{\hspace{10em}}_r \quad \underbrace{\hspace{10em}}_s$

Then $s \leq \frac{r+4}{3}$.

Sketch of the Proof.

Suppose $s > \frac{r+4}{3}$ and derive a contradiction. Let $u, v, x \in V\Gamma$ such that

$$\partial_\Gamma(u, v) = r+1, \quad \partial_\Gamma(u, x) = 1, \quad \partial_\Gamma(x, v) = r.$$

Then we can take a path $(u = y_0, y_1, y_2, \dots, y_{r+1} = v)$ of length $r+1$ such that

$$C_{r+1}(v, y) \subseteq C_{r+1}(u, v) \setminus C_r(x, v).$$

This is a contradiction. ■

We have the following result by Theorem 1 and 3.

Corollary 4 Let Γ be a distance-regular graph as in theorem. If $s = r \geq 2$, then $r = 2$ and Γ is either the Odd graph or the doubled Odd graph. ■

Theorem 5 Let Γ be a distance-regular graph with

$$\left\{ \begin{array}{ccccccc} * & 1 & \cdots & 1 & c & \cdots & c \\ 0 & a_1 & \cdots & a_1 & a & \cdots & a \\ k & b_1 & \cdots & b_1 & b & \cdots & b \end{array} \quad \cdots \right\}.$$

$\underbrace{\hspace{10em}}_r \quad \underbrace{\hspace{10em}}_s$

Suppose $s = r \geq 2$. Then $a = a_1 = 0$ and $r = 2$. In particular, Γ is either the Odd graph, the doubled Odd graph or the doubled Grassmann graph. ■

Proposition 6 Let Γ be a distance-regular graph of diameter $d \geq 2$ and $k \geq 3$. Let q, h be positive integers with $q + h \leq d$.

- (1) If $c_q < c_{q+1}$ and $a_q = 0$, then $c_h \leq c_{q+h} - c_m$ and $b_{q+h} \leq b_h - c_q$.
 (2) If $b_{q+h} < b_{q+h-1}$ and $a_{q+h} = 0$, then $c_h \leq b_q - b_{q+h}$.

Sketch of the Proof of (1).

Let $u, x, y \in V\Gamma$ such that

$$\partial_\Gamma(u, x) = h, \quad \partial_\Gamma(x, y) = q \quad \text{and} \quad \partial_\Gamma(u, y) = q + h.$$

Define

$$W := \bigcup_{z \in C_h(u, x)} \{C_{q+1}(z, y) \setminus C_q(x, y)\}.$$

Then

$$\#C_h(u, x) \leq \#W \quad \text{and} \quad W \subseteq C_{q+h}(u, y) \setminus C_q(x, y).$$

Hence $c_h \leq c_{q+h} - c_q$. ■

4 A characterization of O_k and $2O_k$

Theorem 7 Let Γ be a distance-regular graph of diameter $d \geq 5$ valency $k \geq 3$ and $r = \max\{i \mid (c_i, b_i) = (c_1, b_1)\} \geq 2$. Suppose one of the following conditions holds. Then Γ is either the Odd graph O_k , or the doubled Odd graph $2O_k$.

- (i) $a_{m+r} = 0$ and $1 + c_m = c_{m+r} \leq k - 2$ hold for some m with $r \leq m \leq d - r - 1$.
 (ii) $a_m = 0$ and $2 \leq b_{m+r} = b_m - 1$ hold for some m with $r \leq m \leq d - r - 1$.

Sketch of the Proof of (i).

We have $a_{m+r} = \dots = a_1 = 0$ and $c_{r+1} > c_r$. Since $c_m = c_{m+r} - c_r$, the equality holds in Proposition 6. Then we obtain that $c_{m-1} = c_{m+r-1} - c_r$ holds.

Inductively, we have $c_j = c_{j+r} - c_r$ for all $j < m$. This implies that

$$c_{r+1} = \dots = c_{2r} = 2.$$

The desired result is proved by Corollary 3. ■

The reader is referred to [2, 5, 6] for more detailed proofs of the results.

References

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