

On a density of the set of primes dividing the generalized Fibonacci numbers

By

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ABSTRACT J. C. Lagarias showed the set of prime numbers which divide some Lucas number L_n has positive density using Hasse's method [H]. In his paper he found several results for certain other special second-order linear recurrences [L], [W]. So we will research similar phenomena for slightly generalized second-order linear recurrences.

1 Introduction

In this note we will try to generalize a result of Lagarias on some second-order linear recurrences. Our method will be controlled by Hasse's one. Then we have to check whether these recurrences satisfy Hasse's conditions or not.

Now, any irreducible second-order recurrence $\{U_n\}$ whose terms U_n are rational numbers can be expressed in the form

$$U_n = \alpha\theta^n + \bar{\alpha}\bar{\theta}^n,$$

where α and θ are in the quadratic field K generated by the roots of the characteristic polynomial of $\{U_n\}$, and $\bar{\alpha}, \bar{\theta}$ are the algebraic conjugates of α, θ in K .

Hasse's conditions are as follows:

- (1) $\theta/\bar{\theta} = \pm\phi^k$, where $k = 1$ or 2 for some ϕ in K ,
- (2) $\bar{\alpha}/\alpha = \zeta\phi^j$, where ζ is a root of unity in K and j is an integer.

We put $S_U = \{p : p \text{ is a prime and } p|U_n \text{ for some } n\}$. These particular recurrences $\{U_n\}$, which satisfy the above conditions (1) and (2), have a special property.

Definition 1 A set Σ of primes is a Chebotarev set if and only if there is some finite normal extension L of the rationals $\mathbf{Q}$ such that a prime p is in Σ iff the Artin symbol $\left[\frac{L/\mathbf{Q}}{(p)}\right]$ is in specified conjugacy classes of the Galois group $\text{Gal}(L/\mathbf{Q})$.

Definition 2 Density $d(S_U)$ is defined

$$\lim_{X \rightarrow \infty} \frac{\#S_{U,X}}{\#\mathbf{P}_X} = d(S_U),$$

where $\#S_{U,X} = \#\{p; p \in S_U, p < X\}$ and $\#\mathbf{P}_X = \#\{p; p \text{ is a prime, } p < X\} \sim \frac{X}{\log X}$.

Property 1 Both the set S of primes and its complement

$$\bar{S} = \{p : p \text{ is a prime and } p \notin S\}$$

have a decomposition into disjoint countable unions of Chebotarev sets of primes. That is

$$S = \bigcup_{j=1}^{\infty} S^{(j)}, \quad \bar{S} = \bigcup_{j=1}^{\infty} \bar{S}^{(j)},$$

where $S^{(j)}$ and $\bar{S}^{(j)}$ are Chebotarev sets. Then the densities of the sets satisfy

$$\sum_{j=1}^{\infty} d(S^{(j)}) + \sum_{j=1}^{\infty} d(\bar{S}^{(j)}) = 1.$$

If S is any set of primes having Property 1, then S has a natural density $d(S)$ given by

$$d(S) = \sum_{j=1}^{\infty} d(S^{(j)}).$$

2 Known results

Hasse and Lagarias obtained the following prime densities for several types of sequences:

Theorem 1 (H. Hasse [H]) For the sequence $\{V_n\} = \{2^n + 1\}$, the set of primes

$$\begin{aligned} S_V &= \{p : p \text{ is a prime and } p \text{ divides } 2^n + 1 \text{ for some } n \geq 0\} \\ &= \{p \in \mathbf{P} ; p|V_n \text{ for some } n\}. \end{aligned}$$

has density $d(S_V) = \frac{17}{24}$.

Hasse's result actually covers all the sequences

$$\{A_n\} = \{a^n + 1 \mid n \geq 0\},$$

where a is an integer ≥ 3 , and the density of the associated set $S_A = \{p \in \mathbf{P} : p|A_n \text{ for some } n\}$ is

$$d(S_A) = \frac{2}{3}.$$

Theorem 2 (J. C. Lagarias [L]) For the sequence $\{L_n\}$ ($L_{n+1} = L_n + L_{n-1}$, $L_1 = 2$, $L_2 = 1$), the set of primes

$$S_L = \{p \in \mathbf{P} ; p|L_n \text{ for some } n\}$$

has density $d(S_L) = \frac{2}{3}$.

Theorem 3 (J. C. Lagarias [L2]) *For the sequence $\{W_n\}$ ($W_n = 5W_{n-1} - 7W_{n-2}$, $W_0 = 1$, $W_1 = 2$), the set of primes*

$$S_W = \{p \in \mathbf{P} ; p|W_n \text{ for some } n\}$$

has density $d(S_W) = \frac{3}{4}$.

Lagarias considered

$$\{A_n(m)\}, \quad \{B_n(m)\} \quad (m : \text{fixed})$$

where both series admit the condotion:

$$U_n = mU_{n-1} - U_{n-2}$$

with $A_0(m) = B_0(m) = 1$, $A_1(m) = m + 1$, $B_1(m) = m - 1$, to which Hasse's method is applicable. In the cases of fields $K = \mathbf{Q}(\sqrt{m^2 - 4})$, for the following sets of primes:

$$\begin{aligned} S_A(m) &= \{p \in \mathbf{P} ; p|A_n(m) \text{ for some } n\}, \\ S_B(m) &= \{p \in \mathbf{P} ; p|B_n(m) \text{ for some } n\}, \end{aligned}$$

it is known that $d(S_A(m)) = d(S_B(m)) = \frac{1}{3}$.

3 Theorem

Let

$$\{U_n\} \quad (U_n = mU_{n-1} + U_{n-2}, \quad U_0 = 2, \quad U_1 = m),$$

be a second-order linear recurrence, where we assume that $D = m^2 + 4$ is a prime discriminant of $K = \mathbf{Q}(\sqrt{D})$. Then we have

Theorem 4 *For the sequence $\{U_n\}$ ($U_n = mU_{n-1} + U_{n-2}$, $U_0 = 2$, $U_1 = m$), the set of primes*

$$S_U = \{p \in \mathbf{P} ; p|U_n \text{ for some } n\}$$

has density $d(S_U) = \frac{2}{3}$.

Remark 1 In the case of $m = 1$, the theorem above coincides with Theorem 2. We can prove Theorem 4 by a similar way to Theorem 2.

Acknowledgements The authors would like to express their sincere thanks to Prof. Attila Pethő at Kossuth Lajos University in Debrecen for references [L].

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