

Weierstrass semigroups on double covers of plane curves of degree 7¹

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Abstract

We study Weierstrass semigroups of ramification points on double covers of plane curves of degree 7. We treat the cases where the Weierstrass semigroups are generated by at most 5 elements and the ramification point is on a total flex

1 Introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \setminus H$ is finite. The cardinality of $\mathbb{N}_0 \setminus H$ is called the *genus* of H , denoted by $g(H)$. In this paper H always stands for a numerical semigroup. A *curve* means a projective non-singular irreducible algebraic curve over an algebraically closed field k of characteristic 0. For a pointed curve (C, P) we set

$$H(P) = \{\alpha \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = \alpha P\},$$

where $k(C)$ is the field of rational functions on C . $H(P)$ is a numerical semigroup of genus $g(C)$ where $g(C)$ is the genus of C . For positive integers $a_1, \dots, a_s$ we denote by $\langle a_1, \dots, a_s \rangle$ the monoid generated by $a_1, \dots, a_s$. Let C be a plane curve of degree 7 and P be a total inflection point of C , i.e., $T_P C = 7P$ where T_P is the tangent line at P on C . Then we have $H(P) = \langle 6, 7 \rangle$. We set

$$d_2(H) = \{h' \in \mathbb{N}_0 \mid 2h' \in H\},$$

which is a numerical semigroup. Let $\pi : \tilde{C} \rightarrow C$ be a double covering of curves with a ramification point $\tilde{P}$. Then we have $d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$. For example, if $C = \mathbb{P}^1$, then we have $H(\tilde{P}) = \langle 2, 2g + 1 \rangle$ and $d_2(H(\tilde{P})) = \mathbb{N}_0$, where $g = g(\tilde{C})$.

We pose the following problem:

DCPHurwitz' Problem. *Let C be a plane curve of degree d and $\pi : \tilde{C} \rightarrow C$ be a double covering with a ramification point $\tilde{P}$. Then determine $H(\tilde{P})$.*

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If $d = 1, 2$, then C is isomorphic to $\mathbb{P}^1$. We have $H(\tilde{P}) = \langle 2, 2g + 1 \rangle$ and $d_2(H(\tilde{P})) = \mathbb{N}_0$, where $g = g(\tilde{C})$. If $d = 3$, then C is isomorphic to an elliptic curve. In this case, we have $H(\tilde{P}) = \langle 4, 6, 4g - 3 \rangle$ or $\langle 4, 6, 4g - 1, 4g + 1 \rangle$ and $d_2(H(\tilde{P})) = \langle 2, 3 \rangle$, where $g = g(\tilde{C})$ (for example, see [6]). For $d \geq 4$ we introduce our previous results. If $d = 4$, then DCPHurwitz' Problem is solved in [5], [2], [3] and [4]. In the case $d = 5$, if $g(\tilde{C}) \geq 15$ and $\text{ord}_{\pi(\tilde{P})}(T_{\pi(\tilde{P})}.C) = 5$ or 4, then DCPHurwitz' Problem is solved in [7]. In the case $d = 6$, if $g(\tilde{C}) \geq 30$ and $\text{ord}_{\pi(\tilde{P})}(T_{\pi(\tilde{P})}.C) = 6$ or 5, then DCPHurwitz' Problem is solved in [8] and [9].

We will investigate the case $d = 7$ where $g(\tilde{C}) \geq 45$ and $\text{ord}_{\pi(\tilde{P})}(T_{\pi(\tilde{P})}.C) = 7$. We get the two results. The first theorem is the following:

Theorem 1.1 *Let H be a numerical semigroup of genus ≥ 45 and $d_2(H) = \langle 6, 7 \rangle$. Assume that H is generated by at most 4 elements. If $H \neq 2\langle 6, 7 \rangle + \langle n, n + 8 \rangle$, then H is attained by a ramification point on a double covers of a plane curve of degree 7. In this case we say that H is DCP7.*

To state the second theorem we prepare some notation. Let H be a numerical semigroup with $d_2(H) = \langle 6, 7 \rangle$ and $g(H) \geq 45$. We set

$$n = \min\{h \in H \mid h \text{ is odd}\}.$$

We denote $30 + \frac{n-1}{2} - g(H)$ by $r(H)$, which is a non-negative inetger less than 16 (see [11]).

Theorem 1.2 *Let H be a numerical semigroup with $d_2(H) = \langle 6, 7 \rangle$ and $g(H) \geq 45$. Assume that H is generated by 5 elements with $r(H) \leq 6$. Then H is DCP7.*

2 The proof of Theorem 1.1

There are our previous results as follows:

Proposition 2.1 ([11]) *Let $H = 2\langle 6, 7 \rangle + n\mathbb{N}_0$ with an odd integer $n \geq 35$. Then H is DCP7.*

To state the next proposition we need some notation. Let m be the minimum positive integer in H . We set

$$s_i = \min\{h \in H \mid h \equiv i \pmod{m}\}$$

for $i = 1, \dots, m-1$. The set $\{m, s_1, \dots, s_{m-1}\}$ is denoted by $S(H)$, which is called the *standard basis* for H .

Proposition 2.2 ([7]) *Let n be an odd number with $n \geq 35$. We set $H_7 = \langle 6, 7 \rangle$. Let H be a numerical semigroup which is one of the following:*

- (1) $2_7 + \langle n, n + 2t \rangle$ with $t = 35 - l(7 - 1)$ where l is a positive integer with $l \leq 5$ and $n \geq 30 + 1 + 2l$.

(2) $2H_7 + \langle n, n + 2t \rangle$ with $t = s_{7-m} - (7 - 1)$ where m is an integer with $3 \leq m \leq 6$ and $n \geq 30 - 1 + 2m$.

(3) $2H_7 + \langle n, n + 2t \rangle$ with $t = s_{7-m} - 2(7 - 1)$ where m is an integer with $3 \leq m \leq 5$ and $n \geq 30 - 3 + 4m$.

Then H is DCP7.

Remark 2.3 By Propositions 2.1 and 2.2 the remaining numerical semigroups H with $d_2(H) = \langle 6, 7 \rangle$ generated by 4 elements, which we do not know whether H is DCP7 or not, are the following:

(1) $2\langle 6, 7 \rangle + \langle n, n + 20 \rangle$ (2) $2\langle 6, 7 \rangle + \langle n, n + 8 \rangle$ (3) $2\langle 6, 7 \rangle + \langle n, n + 6 \rangle$

First, we will prove that $2\langle 6, 7 \rangle + \langle n, n + 20 \rangle$ is DCP7.

Lemma 2.4 Let (C, P) be a pointed non-singular plane curve of degree 7 and H be a numerical semigroup with $d_2(H) = H(P)$ and $g(H) \geq 45$. Set $n = \min\{h \in H \mid h \text{ is odd}\}$. We note that $g(H) = 30 + \frac{n-1}{2} - r$ with some non-negative integer r . Let $Q_1, \dots, Q_r$ be points of C different from P with $h^0(Q_1 + \dots + Q_r) = 1$. Moreover, assume that H has an expression

$$H = 2d_2(H) + \langle n, n + 2l_1, \dots, n + 2l_s \rangle$$

with positive integers $l_1, \dots, l_s$ such that

$$h^0(K - (l_i - 1)P - Q_1 - \dots - Q_r) = h^0(K - l_i P - Q_1 - \dots - Q_r)$$

where K is a canonical divisor on C . Then there is a double cover $\pi : \tilde{C} \rightarrow C$ with a ramification point $\tilde{P}$ over P satisfying $H(\tilde{P}) = H$.

See [10] for the details of the proof of Lemma 2.4 .

Lemma 2.5 (Cayley-Bacharach) Let C be a non-singular plane curve. Let X_1 and X_2 be two plane curves of degree d and e respectively, meeting in a collection Γ of de points of C with multiplicity. Let Y be a curve of degree $d + e - 3$ such that the intersection $Y.C$ contains all but one point of Γ . Then $Y.C$ contains that remaining point also.

For example, see p. 671 in [1].

Lemma 2.6 The plane curve of degree 7 defined by the equation

$$(yz^2 - x^3) \left(\frac{1}{2}z^4 + ax^4 \right) + (yz^2 + x^3 - 2y^3) \left(\frac{1}{2}z^4 + by^4 \right) = 0$$

is nonsingular for general a and b .

For the proof of Lemma 2.6 see [10].

Proposition 2.7 The numerical semigroup $H = 2\langle 6, 7 \rangle + \langle n, n + 20 \rangle$ is DCP7.

Proof. Let C be the non-singular plane curve of degree 7 in Lemma 2.6. We set $P = (0 : 0 : 1)$. We take six points

$$Q_1 = (1 : 1 : 1), Q_2 = (1 : 1 : \omega), Q_3 = (1 : 1 : \omega^2),$$

$$Q_4 = (1 : -1 : -1), Q_5 = (1 : -1 : -\omega) \text{ and } Q_6 = (1 : -1 : -\omega^2)$$

where ω is a primitive cubic root of unity. Using Lemma 2.5 we can apply Lemma 2.4. Hence, H is DCP7. $\square$

Proposition 2.8 *The numerical semigroup $H = 2\langle 6, 7 \rangle + \langle n, n + 6 \rangle$ is DCP7.*

In this proof we use the plane curve of degree 7 defined by the equation

$$(yz^2 - x^3) \left(\frac{1}{2}z^4 + ax^4 \right) + (yz^3 + x^3z - 2y^4) \left(\frac{1}{2}z^3 + by^3 \right) = 0$$

for general a and b . See [10] for the details of the proof.

By Remark 2.3, Propositions 2.7 and 2.8 we get Theorem 1.1.

3 The proof of Theorem 1.2

Remark 3.1 ([7]) *Any numerical semigroup H with $d_2(H) = \langle 6, 7 \rangle$ and $r(H) \leq 6$ which is generated by 5 elements is DCP7 except the following four semigroups:*

$$2\langle 6, 7 \rangle + \langle n, n + 22, n + 32 \rangle, 2\langle 6, 7 \rangle + \langle n, n + 22, n + 30 \rangle,$$

$$2\langle 6, 7 \rangle + \langle n, n + 16, n + 32 \rangle \text{ and } 2\langle 6, 7 \rangle + \langle n, n + 16, n + 34 \rangle.$$

Proposition 3.2 *$H = 2\langle 6, 7 \rangle + \langle n, n + 16, n + 32 \rangle$ is DCP7.*

Proof. In this case $r(H) = 6$. Let (C, P) be a pointed plane curve of degree 7 with $H(P) = \langle 6, 7 \rangle$. Let L_P and L'_P be distinct lines through P different from T_P . Let us take $Q_1, \dots, Q_4$ such that the four points lie on the line L_P . Let us take Q_5 and Q_6 such that the two points lie on the line L'_P . Let C_4 be a curve of degree 4 with $C_4.C \geq 7P + E_6$ where we set $E_6 = Q_1 + \dots + Q_6$. Then we have $C_4 = T_P L_P C_2$ where C_2 is a conic containing Q_5 and Q_6 . Hence we get

$$h^0(K - 7P - E_6) = h^0(K - 8P - E_6).$$

Moreover, let C'_4 be a curve of degree 4 with $C'_4.C \geq 15P + E_6$. Then we should have $C'_4 = T_P^2 L_P L'_P$, which implies that

$$h^0(K - 15P - E_6) = h^0(K - 16P - E_6) = 1.$$

It follows from Proposition 2.4 that H is DCP7. $\square$

Proposition 3.3 *Let H be one of the following numerical semigroups:*

$$2\langle 6, 7 \rangle + \langle n, n + 22, n + 32 \rangle, 2\langle 6, 7 \rangle + \langle n, n + 22, n + 30 \rangle \text{ and } 2\langle 6, 7 \rangle + \langle n, n + 16, n + 34 \rangle.$$

Then it is DCP7.

See [10] for the proof of Proposition 3.3. By Remark 3.1, Propositions 3.2 and 3.3 we obtain Theorem 1.2.

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