

Classification of connected palette diagrams without area and its application to finding relations of formal diffeomorphisms

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Abstract

We have obtained various sufficient conditions of Feynman diagram $\gamma \subset \mathbb{R}^2$ such that the relation $W_{\gamma^*}(f, g) = \text{id}$ of f, g admits solutions of non commuting diffeomorphisms tangent to identity in [6]. Finding relations of formal diffeomorphisms is reduced to finding Feynman diagrams satisfying these sufficient conditions. By the way, Feynman diagrams are obtained from a palette diagram defined in [7]. In this paper we classify all connected palette diagrams without area consisting of four unit weighted squares into 5 types, and apply the classification to finding relations of two formal diffeomorphisms tangent to identity.

1 Introduction

A Feynman diagram in $\mathbb{R}^2_{(x,y)}$ is defined by a polygonal path

$$\gamma = H^{n_1} * V^{n_2} * H^{n_3} * V^{n_4} * \dots * H^{n_{2p-1}} * V^{n_{2p}}, \quad n_1, n_2, \dots, n_{2p} \in \mathbb{Z}^*, \quad (1)$$

consisting of a unit horizontal vector H in x -positive direction, a unit vertical vector V in y -positive direction and their inverse vectors H^{-1}, V^{-1} , where $*$ denotes the composite of paths and H^n stands for the n -fold composite of H . From γ in (1), we obtain a word

$$W_{\gamma^*}(f, g) = f^{(n_1)} \circ g^{(n_2)} \circ f^{(n_3)} \circ g^{(n_4)} \circ \dots \circ f^{(n_{2p-1})} \circ g^{(n_{2p})} \quad (2)$$

of $f, g, f^{(-1)}, g^{(-1)}$ for two holomorphic diffeomorphisms $f, g \in \text{Diff}(\mathbb{C}, 0)$ by substituting H and V in (1) by f and g respectively, where $f^{(m)}$ stands for the m -fold iteration of f , and $\text{Diff}(\mathbb{C}, 0)$ denotes the group of germs of holomorphic diffeomorphisms of $\mathbb{C}$ fixing 0

$$\text{Diff}(\mathbb{C}, 0) = \{f(z) = a_1 z + a_2 z^2 + \dots \mid a_1 \neq 0, a_i \in \mathbb{C}\}$$

and $\circ$ denotes the composite of mappings. The relation of f, g defined for γ in (1) is the equation

$$W_{\gamma^*}(f, g) = \text{id}.$$

It is nothing but a relation in the subgroup of $\text{Diff}(\mathbb{C}, 0)$ generated by two elements f, g . We say an element f of $\text{Diff}(\mathbb{C}, 0)$ (or $\widehat{\text{Diff}}(\mathbb{C}, 0)$) is tangent to identity if $f'(0) = 1$, that is the coefficient a_1 of z is 1. J. Ecalle and B. Vallet [2] constructed various types of relations of two formal diffeomorphisms tangent to identity in the group $\widehat{\text{Diff}}(\mathbb{C}, 0)$ of formal diffeomorphisms. F. Loray [3] investigated those relations in the study of non solvable subgroups of $\text{Diff}(\mathbb{C}, 0)$ from the view point of real and complex codimension one foliation. While, the structure of non solvable sub groups is not well known [3, 4].

A *palette diagram* Γ is defined by a collection

$$\Gamma = \{[S_1, p_1], [S_2, p_2], \dots, [S_n, p_n]\}, \quad (3)$$

where $[S_i, p_i]$, $i = 1, 2, \dots, n$, denotes a unit square with weight p_i , that is a unit square $S = (a, a+1) \times (b, b+1)$, $a, b \in \mathbb{Z}$, in the real 2-plane $\mathbb{R}^2$ which is attached to a non zero integer $p \in \mathbb{Z}^*$. Here we assume $S_1, \dots, S_n$ are distinct with each other. For a palette diagram Γ , we call a collection of n unit squares

$$\tilde{\Gamma} = \{S_1, S_2, \dots, S_n\}$$

the base of Γ . We say a palette diagram (or its base) is *connected* if every square S_i has at least one vertex in common with another square S_j , $j \neq i$. Hence a palette diagram (or its base) is *disconnected* (or *non connected*) if there exists at least one square which has no vertex in common with any other squares. We regard palette diagrams or basis of palette diagrams up to congruence and reflection to be equivalent.

From a palette diagram Γ we can obtain infinitely many (but countable) distinct closed Feynman diagrams $\gamma \subset \mathbb{R}^2$ such that the value of the winding number on the square domains $S_1, S_2, \dots, S_n$ are respectively $p_1, p_2, \dots, p_n \in \mathbb{Z}^*$. Here the winding number $\rho(\gamma) = \rho(\gamma)(x, y)$ at a point $(x, y) \in \mathbb{R}^2$ of a closed path $\gamma \subset \mathbb{R}^2$ is the number that γ winds around (x, y) . And for a closed Feynman diagram γ , there is a palette diagram Γ from which γ is obtained.

The Area and Moment of a palette diagram Γ in (3) (or a closed Feynman diagram γ obtained from Γ) is defined by

$$\text{Area}(\Gamma) = \sum_{i=1}^n p_i, \quad G(\Gamma) = \left(\sum_{i=1}^n p_i \iint_{S_i} x \, dx \wedge dy, \sum_{i=1}^n p_i \iint_{S_i} y \, dx \wedge dy \right)$$

respectively. And we define the polynomial $P_k(\Gamma)(\alpha, \beta)$ of α, β of k -th degree with real coefficients by

$$P_k(\Gamma)(\alpha, \beta) = \sum_{i=1}^n p_i \iint_{S_i} (\alpha x + \beta y)^k \, dx \wedge dy.$$

For a palette diagram Γ in (3), assume $G(\Gamma) \neq 0$ and (α, β) is a vector orthogonal to $G(\Gamma)$. Then we see that $P_k(\Gamma)(\alpha, \beta)$ is a polynomial of degree $k+1$ of $p_1, \dots, p_n$ since α and β are polynomials of degree 1 of $p_1, \dots, p_n$. In this paper we consider palette diagrams without area consisting of four unit weighted squares.

Definition 1.1. If a palette diagram Γ consisting of four unit weighted squares $[S_1, p], [S_2, q], [S_3, r], [S_4, -p - q - r]$ without area has one of the following property (E) or (F) or (G) or (H) or (I), we say Γ has the type (E) or (F) or (G) or (H) or (I).

(E) $G(\Gamma) \neq (0, 0)$, and for a vector (α, β) orthogonal to $G(\Gamma)$ (hence all points on a complex line $(\alpha : \beta)$),

$$P_2(\alpha, \beta) = c(p+q)(p+r)(q+r),$$

where c is 1 or 4,

(F) $G(\Gamma) \neq (0,0)$, and for a vector (α, β) orthogonal to $G(\Gamma)$ (hence all points on a complex line $(\alpha : \beta)$), $P_2(\alpha, \beta)$ has one of $p + q$, $p + r$, $q + r$ as a factor, that is $P_2(\alpha, \beta)$ equals the one of the following three polynomials:

$$c_1 (p + q) p_1(p, q, r), \quad c_2 (p + r) p_2(p, q, r), \quad c_3 (q + r) p_3(p, q, r),$$

where $c_1, c_2, c_3 \neq 0$ are constants and p_1, p_2, p_3 are polynomials of degree 2 of p, q, r ,

(G) $G(\Gamma) \neq (0,0)$, and for a vector (a, b) orthogonal to $G(\Gamma)$ (hence all points on a complex line $(\alpha : \beta)$), $P_2(\alpha, \beta)$ has one of p , q , r as a factor, that is $P_2(\alpha, \beta)$ equals the one of the following three polynomials:

$$c_4 p p_4(p, q, r), \quad c_5 q p_5(p, q, r), \quad c_6 r p_6(p, q, r),$$

where $c_4, c_5, c_6 \neq 0$ are constants and p_4, p_5, p_6 are polynomials of degree 2 of p, q, r ,

(H) $G(\Gamma) \neq (0,0)$, and for a vector (α, β) orthogonal to $G(\Gamma)$ (hence all points on a complex line $(\alpha : \beta)$), $P_2(\alpha, \beta)$ equals a polynomial $p_7(p, q, r)$ of degree 3 of p, q, r other than the above three cases,

(I) $G(\Gamma) \neq (0,0)$, and for a vector (α, β) orthogonal to $G(\Gamma)$ (hence all points on a complex line $(\alpha : \beta)$), $P_k(\alpha, \beta) = 0$ for $k = 2, 3, \dots$

Here $(\alpha : \beta) = \{\lambda(\alpha, \beta) | \lambda \in \mathbb{C}\}$.

In §2, we review my talk at RIMS. In §3, we state the result of classification of all connected palette diagrams consisting of four unit weighted squares into the types (E) ~ (I). One of main results of this paper is the following:

Theorem 1.1. All connected palette diagrams consisting of four unit weighted squares $[S_1, p]$, $[S_2, q]$, $[S_3, r]$, $[S_4, -p - q - r]$ without area and with moment are classified into the above 5 types (E) ~ (I) if p, q, r, s are chosen properly.

In §4 we give the proof of the classification theorem. In §5 we apply the classification to obtaining relations of two formal diffeomorphisms non commute and tangent to identity using theorems already obtained in [5, 8]. For another main result see Theorem 5.3.

2 The substance of my talk at RIMS

Here we state the substance of my talk at RIMS. See [6, 8] for more details.

We consider the non linear ordinary differential equation

$$\frac{dz}{dt} = f(t, z), \quad z(0) = z_0, \quad (4)$$

on the complex domain, where $f(t, z)$ is continuous with regard to a parameter t , holomorphic with regard to a parameter z , and $f(t, 0) = 0$, $\frac{d}{dt}f(t, z)|_{z=0} = 0$. Let $X_t = f(t, z)\partial_z$. It is a holomorphic vector field on $\mathbb{C}$ depending on the time parameter t .

From the main theorem (Theorem 8) in [1], we see that the solution of (4) is expressed as

$$z(t) = e^{L \int_0^t -X_i dt} z_0.$$

Here $L \int_0^t -X_i dt$ denotes a formal vector field without constant term and linear term defined by the following recursion formula;

$$L \int_0^t -X_i dt = H_1[t] - H_2[t] + H_3[t] - + \dots, \quad (5)$$

where $H_1[t] = \int_0^t X_i dt = T_0[t]$ and $H_k[t] (k \geq 2)$ is defined by the recursion formula

$$(k+1)H_{k+1}[t] = T_k + \sum_{r=1}^k \left(\frac{1}{2} [H_r, T_{k-r}] + \sum_{\substack{p \geq 1 \\ 2p \leq r}} K_{2p} \sum_{\substack{m_i > 0 \\ m_1 + \dots + m_{2p} = r}} [H_{m_1}, [\dots, [H_{m_{2p}}, T_{k-r}]] \dots] \right), \quad (6)$$

where

$$T_k = T_k[t] = \int \dots \int du_1 du_2 \dots du_{k+1} [\dots [[X_{u_1}, X_{u_2}], \dots], X_{u_{k+1}}] \quad (k \geq 1),$$

$0 \leq u_{k+1} \leq \dots \leq u_2 \leq u_1 \leq t$

and $(-1)^{p-1}(2p)!K_{2p} = B_p$ are Bernoulli numbers. Here $[*,*]$ denotes the Lie bracket of vector fields. And e^X denotes a time one map of a vector field X .

Let $\gamma(t) = (x(t), y(t))$, $0 \leq t \leq 1$, be a piecewise smooth closed path in $\mathbb{R}^2$ with starting point 0 and $a_1(z)\partial_z, a_2(z)\partial_{\bar{z}}$ holomorphic vector fields with $a_1(0) = a_1'(0) = a_2(0) = a_2'(0) = 0$. For $X_t = a_1(z)\partial_z \frac{dx}{dt} + a_2(z)\partial_{\bar{z}} \frac{dy}{dt}$ we calculate the Taylor coefficients $L_2, L_3, L_4, \dots$ of

$$L \int_0^1 X_t dt = H_1[1] + H_2[1] + H_3[1] + \dots = L_2 z^2 + L_3 z^3 + L_4 z^4 + \dots$$

as formal vector field using the above recursion formula. It is nothing but a logarithm function (but formal) of the holonomy mapping of a $\chi(\mathbb{C}, 0)$ -valued connection 1-form $-\omega = -(a_1(z)\partial_z dx + a_2(z)\partial_{\bar{z}} dy)$ on the trivial $(\mathbb{C}, 0)$ bundle over $\mathbb{R}^2$ along γ^* . Here $\chi(\mathbb{C}, 0)$ denotes the Lie algebra of all holomorphic vector fields without constant term, and γ^* is a path obtained by inverting the sign of the velocity and the orientation of γ .

Let

$$\begin{aligned} a_1(z)\partial_z &= (a_{12}z^2 + a_{13}z^3 + \dots + a_{1i}z^i + \dots)\partial_z, \\ a_2(z)\partial_{\bar{z}} &= (a_{22}z^2 + a_{23}z^3 + \dots + a_{2i}z^i + \dots)\partial_{\bar{z}}, \end{aligned}$$

and

$$A_i = \begin{bmatrix} a_{1i} \\ a_{2i} \end{bmatrix}, \quad \text{and} \quad K_i = a_{1i}x + a_{2i}y.$$

Then the results of calculations of Taylor coefficients $L_2, L_3, L_4, \dots$ are the followings.

$L_2 = L_3 = 0$ and

$$\begin{aligned}
 L_4 &= - \iint_D \rho dK_2 \wedge dK_3, \\
 L_5 &= 2 \iint_D \rho K_2 dK_2 \wedge dK_3, \quad \text{mod } \iint_D \rho dx \wedge dy \\
 L_6 &= \iint_D \rho (K_3 - 3K_2^2) dK_2 \wedge dK_3, \quad \text{mod } \iint_D \rho dx \wedge dy, \iint_D \rho K_2 dx \wedge dy \\
 L_7 &= 4 \iint_D \rho (K_2^3 - K_2 K_3) dK_2 \wedge dK_3, \\
 &\text{mod } \iint_D \rho dx \wedge dy, \iint_D \rho K_2 dx \wedge dy, \iint_D \rho (K_3 - 3K_2^2) dx \wedge dy.
 \end{aligned}$$

In general,

$$\begin{aligned}
 L_k &= -\frac{1}{6}(k-7)(k^2-11k+36) \iint_D \rho K_{k-3} dK_2 \wedge dK_3, \\
 &\text{mod } \iint_D \rho dx \wedge dy, \iint_D \rho K_2 dx \wedge dy, \iint_D \rho (K_3 - 3K_2^2) dx \wedge dy, \\
 &\quad R_k(A_2, A_3, \dots, A_{k-4}),
 \end{aligned}$$

for $k \geq 8$. Here D denotes the domain enclosed by γ , $\iint$ the multiple integral over D by the standard measure $dx \wedge dy$, ρ the winding number of γ , and R_k the remainder term. For the proof and further calculations see [6, 8].

In the case γ is a closed Feynman diagram in $\mathbb{R}^2$,

$$\iint_D \rho dx \wedge dy = \text{Area}(\gamma), \quad \iint_D \rho K_2^k dx \wedge dy = P_k(\gamma)(a_{12}, a_{22}).$$

And assume γ has its expression (1), that is

$$\gamma = H^{n_1} * V^{n_2} * H^{n_3} * V^{n_4} * \dots * H^{n_{2p-1}} * V^{n_{2p}}, \quad n_1, n_2, \dots, n_{2p} \in \mathbb{Z}^*,$$

then

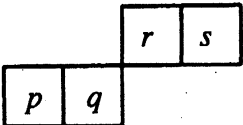
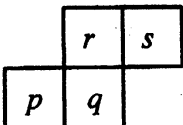
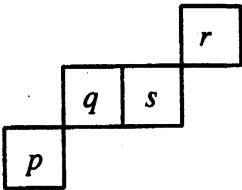
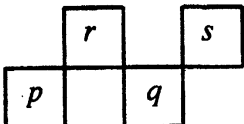
$$e^{L \int_0^1 (a_1(z) \partial_z \frac{dx}{dt} + a_2(z) \partial_z \frac{dy}{dt}) dt} = W_{\gamma^*}(f, g),$$

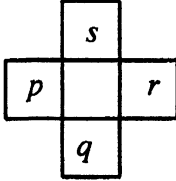
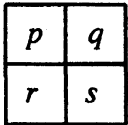
where $f = e^{a_1(z) \partial_z}$, $g = e^{a_2(z) \partial_z}$, and $W_{\gamma^*}(f, g)$ is a word of f, g in (2). Finding sufficient conditions of Feynman diagram γ such that $W_{\gamma^*}(f, g) = \text{id}$ admits solutions of formal diffeomorphisms tangent to identity is equivalent to finding sufficient conditions of γ such that the Lie integral $L \int_0^1 (a_1(z) \partial_z \frac{dx}{dt} + a_2(z) \partial_z \frac{dy}{dt}) dt$ equals to zero vector field, that is its Taylor coefficients are all 0, for properly chosen A_i , $i = 2, 3, \dots$. Some of such conditions are obtained (See Theorem 5.1 for example). Some examples of relations of two formal diffeomorphisms non commute and tangent to identity including a relation constructed by Ecalle and Vallet are also obtained.

3 Result of the classification of connected palette diagrams without area consisting of four unit weighted squares

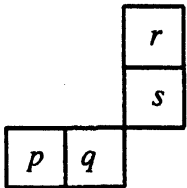
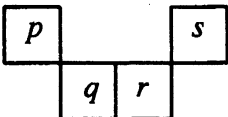
Here we state the classification theorem obtained.

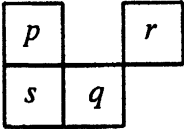
Theorem 3.1. *A connected palette diagram Γ consisting of four unit weighted squares without area with the type (E) equals up to congruence and reflection to the one of the diagrams in the list below, where $p, q, r, p + q + r$ are arbitrary non zero integers and $s = -p - q - r$.*

(E)		
No.	Γ	condition of p, q, r
1		$(3p + 2q + r, p + q) \neq (0, 0)$
2		$(2p + q + r, p + q) \neq (0, 0)$
3		$(-2p - q + r, -p + r) \neq (0, 0)$
4		$(3p + q + 2r, p + q) \neq (0, 0)$

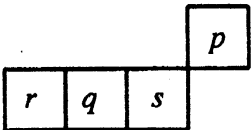
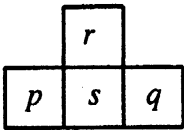
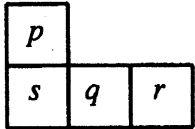
(E)		
No.	Γ	condition of p, q, r
5		$(-p + r, -p - 2q - r) \neq (0, 0)$
6		$(-p - r, p + q) \neq (0, 0)$

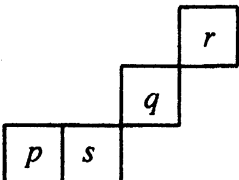
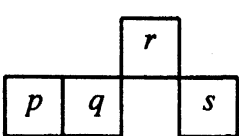
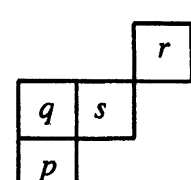
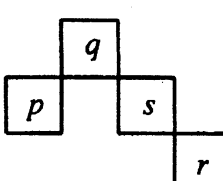
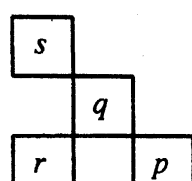
Theorem 3.2. A connected palette diagram Γ consisting of four unit weighted squares without area with the type (F) equals up to congruence and reflection to the one of the diagrams in the list below, where $p, q, r, p + q + r$ are arbitrary non zero integers and $s = -p - q - r$.

(F)		
No.	Γ	condition of p, q, r
1		$(-2p - q, -p - q + r) \neq (0, 0)$
2		$(3p + 2q + r, q + r) \neq (0, 0)$

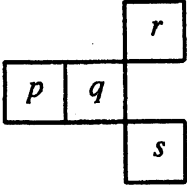
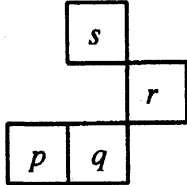
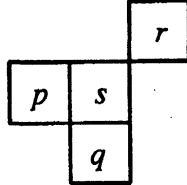
(F)		
No.	Γ	condition of p, q, r
3		$(q + 2r, p + r) \neq (0, 0)$

Theorem 3.3. A connected palette diagram Γ consisting of four unit weighted squares without area with the type (G) equals up to congruence and reflection to the one of the diagrams in the list below, where $p, q, r, p + q + r$ are arbitrary non zero integers and $s = -p - q - r$.

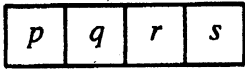
(G)		
No.	Γ	condition of p, q, r
1		$(p - 2q - r, p) \neq (0, 0)$
2		$(-p + q, r) \neq (0, 0)$
3		$(q + 2r, p) \neq (0, 0)$

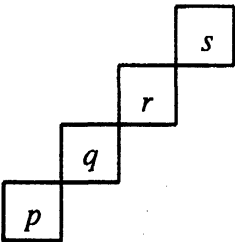
(G)		
No.	Γ	condition of p, q, r
4		$(-p + q + 2r, q + 2r) \neq (0, 0)$
5		$(-3p - 2q - r, r) \neq (0, 0)$
6		$(-p - q + r, -p + r) \neq (0, 0)$
7		$(-2p - q + r, q - r) \neq (0, 0)$
8		$(2p + q, -2p - q - 2r) \neq (0, 0)$

Theorem 3.4. A connected palette diagram Γ consisting of four unit weighted squares without area with the type (H) equals up to congruence and reflection to the one of the diagrams in the list below, where $p, q, r, p + q + r$ are arbitrary non zero integers and $s = -p - q - r$.

(H)		
No.	Γ	condition of p, q, r
1		$(-2p - q, p + q + 2r) \neq (0, 0)$
2		$(-p + r, -2p - 2q - r) \neq (0, 0)$
3		$(-p + r, -q + r) \neq (0, 0)$

Theorem 3.5. A connected palette diagram Γ consisting of four unit weighted squares without area with the type (I) equals up to congruence and reflection to the one of the diagrams in the list below, where $p, q, r, p + q + r$ are arbitrary non zero integers and $s = -p - q - r$.

(I)		
No.	Γ	condition of p, q, r
1		$-3p - 2q - r \neq 0$

(I)		
No.	Γ	condition of p, q, r
2		$-3p - 2q - r \neq 0$

4 Proof of the classification theorem

There exist exactly 22 distinct bases of connected palette diagrams consisting of four unit squares (See [7] for detailed enumeration). To prove the classification theorem (Theorem 3.1 ~ 3.5), we have only to compute $G(\Gamma)$ and $P_2(\alpha, \beta) = \iint_D \rho(\alpha x + \beta y)^2 dx \wedge dy$ for (α, β) orthogonal to $G(\Gamma)$ for 22 palette diagrams consisting of four unit weighted squares attached to non zero integers $p, q, r, s = -p - q - r$.

The moments $G(\Gamma)$ and vectors (α, β) orthogonal to $G(\Gamma)$ of palette diagrams Γ in the lists of Theorem 3.1 ~ 3.5 are followings. The computations are performed for palette diagrams in the usual coordinate system that the horizontal direction is x -direction and the vertical direction is y -direction in the lists.

$$(E-1) \quad G(\Gamma) = (-3p - 2q - r, -p - q), \quad (\alpha, \beta) = (-p - q, 3p + 2q + r).$$

$$(E-2) \quad G(\Gamma) = (-2p - q - r, -p - q), \quad (\alpha, \beta) = (p + q, -2p - q - r).$$

$$(E-3) \quad G(\Gamma) = (-2p - q + r, -p + r), \quad (\alpha, \beta) = (p - r, -2p - q + r).$$

$$(E-4) \quad G(\Gamma) = (-3p - q - 2r, -p - q), \quad (\alpha, \beta) = (p + q, -3p - q - 2r).$$

$$(E-5) \quad G(\Gamma) = (-p + r, -p - 2q - r), \quad (\alpha, \beta) = (-p - 2q - r, p - r).$$

$$(E-6) \quad G(\Gamma) = (-p - r, p + q), \quad (\alpha, \beta) = (p + q, p + r).$$

$$(F-1) \quad G(\Gamma) = (-2p - q, -p - q + r), \quad (\alpha, \beta) = (-p - q + r, 2p + q).$$

$$(F-2) \quad G(\Gamma) = (-3p - 2q - r, -q - r), \quad (\alpha, \beta) = (q + r, -3p - 2q - r).$$

$$(F-3) \quad G(\Gamma) = (q + 2r, p + r), \quad (\alpha, \beta) = (-p - r, q + 2r).$$

$$(G-1) \quad G(\Gamma) = (p - q - 2r, p), \quad (\alpha, \beta) = (-p, p - q - 2r).$$

$$(G-2) \quad G(\Gamma) = (-p + q, r), \quad (\alpha, \beta) = (r, p - q).$$

$$(G-3) \quad G(\Gamma) = (q + 2r, p), \quad (\alpha, \beta) = (-p, q + 2r).$$

$$(G-4) \quad G(\Gamma) = (-p + q + 2r, q + 2r), \quad (\alpha, \beta) = (q + 2r, p - q - 2r).$$

(G-5) $G(\Gamma) = (-3p - 2q - r, r),$	$(\alpha, \beta) = (r, 3p + 2q + r).$
(G-6) $G(\Gamma) = (-p - q + r, -p + r),$	$(\alpha, \beta) = (p - r, -p - q + r).$
(G-7) $G(\Gamma) = (-2p - q + r, q - r),$	$(\alpha, \beta) = (q - r, 2p + q - r).$
(G-8) $G(\Gamma) = (2p + q, -2p - q - 2r),$	$(\alpha, \beta) = (2p + q + 2r, 2p + q).$
(H-1) $G(\Gamma) = (-2p - q, p + q + 2r),$	$(\alpha, \beta) = (p + q + 2r, 2p + q).$
(H-2) $G(\Gamma) = (-p + r, -2p - 2q - r),$	$(\alpha, \beta) = (2p + 2q + r, -p + r).$
(H-3) $G(\Gamma) = (-p + r, -q + r),$	$(\alpha, \beta) = (q - r, -p + r).$
(I-1) $G(\Gamma) = (-3p - 2q - r, 0),$	$(\alpha, \beta) = (0, 1).$
(I-2) $G(\Gamma) = (-3p - 2q - r, -3p - 2q - r),$	$(\alpha, \beta) = (1, -1).$

The conditions of p, q, r in the lists of Theorem 3.1 ~ 3.5 are the conditions that $G(\Gamma) \neq (0, 0)$.

The polynomials $P_2(\alpha, \beta)$ of p, q, r for the above (α, β) orthogonal to $G(\Gamma)$ are the followings.

- (E-1) $P_2(\alpha, \beta) = (p + q)(p + r)(q + r).$
- (E-2) $P_2(\alpha, \beta) = (p + q)(p + r)(q + r).$
- (E-3) $P_2(\alpha, \beta) = (p + q)(p + r)(q + r).$
- (E-4) $P_2(\alpha, \beta) = 4(p + q)(p + r)(q + r).$
- (E-5) $P_2(\alpha, \beta) = 4(p + q)(p + r)(q + r).$
- (E-6) $P_2(\alpha, \beta) = (p + q)(p + r)(q + r).$
- (F-1) $P_2(\alpha, \beta) = (p + r)\{q(q + r) + p(q + 4r)\}.$
- (F-2) $P_2(\alpha, \beta) = (q + r)\{9p^2 + qr + 9p(q + r)\}.$
- (F-3) $P_2(\alpha, \beta) = (p + r)\{q(q + r) + p(q + 4r)\}.$
- (G-1) $P_2(\alpha, \beta) = p\{(q + 2r)^2 + p(q + 4r)\}.$
- (G-2) $P_2(\alpha, \beta) = r\{p^2 + p(-2q + r) + q(q + r)\}.$
- (G-3) $P_2(\alpha, \beta) = p\{(q + 2r)^2 + p(q + 4r)\}.$
- (G-4) $P_2(\alpha, \beta) = p\{(q + 2r)^2 + p(q + 4r)\}.$
- (G-5) $P_2(\alpha, \beta) = r\{9p^2 + 4q(q + r) + 3p(4q + 3r)\}.$
- (G-6) $P_2(\alpha, \beta) = q\{p^2 + p(q - 2r) + r(q + r)\}.$

$$(G-7) \quad P_2(\alpha, \beta) = 4p\{(q-r)^2 + p(q+r)\}.$$

$$(G-8) \quad P_2(\alpha, \beta) = 4r\{4p^2 + 4p(q+r) + q(q+r)\}.$$

$$(H-1) \quad P_2(\alpha, \beta) = 4qr(q+r) + p^2(q+16r) + p(q^2 + 20qr + 16r^2).$$

$$(H-2) \quad P_2(\alpha, \beta) = 4qr(q+r) + p^2(4q+9r) + p(4q^2 + 16qr + 9r^2).$$

$$(H-3) \quad P_2(\alpha, \beta) = p^2(q+r) + qr(q+r) + p(q^2 - 6qr + r^2).$$

$$(I-1) \quad P_k(\alpha, \beta) = 0, k = 2, 3, \dots$$

$$(I-2) \quad P_k(\alpha, \beta) = 0, k = 2, 3, \dots$$

The author used Mathematica for computations of $G(\Gamma)$ and $P_2(\alpha, \beta)$. We note that if we change the position of p, q, r, s attaching to unit squares, the polynomial $P_2(\alpha, \beta)$ should vary. But if we choose it as in the lists of Theorem 3.1 ~ 3.5, we obtain the desired classification. $\square$

5 Application of the classification theorem to finding relations of formal diffeomorphisms

Here we explain the application of the classification in §3 to finding relations of two formal diffeomorphisms. We have obtained in [5, 8] the following theorems for relations of two formal diffeomorphisms in terms of Feynman diagrams.

Theorem 5.1 ([5], Theorem 8.2.). *Let $\gamma \subset \mathbb{R}^2$ be a closed Feynman diagram. Assume $\text{Area}(\gamma) = 0$ and $G(\gamma) \neq 0$. Let $A_2 = (a_{12}, a_{22}) \neq 0$ be orthogonal to $G(\gamma)$, and assume*

$$\iint_D \rho K_2^2 dx \wedge dy \neq 0.$$

Then the relation $W_\gamma(f, g) = 1$ admits formal non commuting solutions f, g such that $f'(0) = g'(0) = 1$, $(f''(0), g''(0)) = A_2$. And the 4-jet of f, g can be arbitrary. If the y-moment $\iint_D \rho y dx \wedge dy$ is not 0, then the Taylor coefficients of f of order ≥ 5 can be arbitrary, and if the x-moment $\iint_D \rho x dx \wedge dy$ is not 0, then the Taylor coefficients of g of order ≥ 5 can be arbitrary.

Theorem 5.2 ([8]). *Let $\gamma \subset \mathbb{R}^2$ be a closed Feynman diagram with $\text{Area}(\gamma) = 0$ and $G(\gamma) \neq 0$. For $A_2 = (a_{12}, a_{22}) \neq 0$ orthogonal to $G(\gamma)$ assume*

$$\iint_D \rho K_2^p dx \wedge dy = 0, \quad p = 2, 3, \dots$$

and $W_\gamma(f, g) = \text{id}$ for $f, g \neq \text{id}$ tangent to identity with $(f''(0), g''(0)) = A_2$. Then f, g commute.

In these theorems ρ denotes the winding number of γ and D the domain enclosed by γ . And $K_2 = a_{12}x + a_{22}y$.

Then we obtain the following lemmas straightforward by Definition 1.1.

Lemma 5.1. Assume a palette diagram Γ consisting of four unit weighted squares without area has one of the following property 1 ~ 4.

1. Γ has the type (E) and non zero integers p, q, r satisfy the condition that $p + q \neq 0$ and $p + r \neq 0$ and $q + r \neq 0$,
2. Γ has the type (F) and non zero integers p, q, r satisfy one of the following conditions that,

$$\begin{aligned} p + q \neq 0 \quad \text{and} \quad p_1(p, q, r) \neq 0, \\ p + r \neq 0 \quad \text{and} \quad p_2(p, q, r) \neq 0, \\ q + r \neq 0 \quad \text{and} \quad p_3(p, q, r) \neq 0, \end{aligned}$$

3. Γ has the type (G) and non zero integers p, q, r satisfy one of the conditions that

$$p_4(p, q, r) \neq 0, \quad p_5(p, q, r) \neq 0, \quad p_6(p, q, r) \neq 0,$$

4. Γ has the type (H) and non zero integers p, q, r satisfy the condition that $p_7(p, q, r) \neq 0$,

where $p_i(p, q, r), i = 1, 2, \dots, 7$, is polynomials in Definition 1.1. Then for all Feynman diagrams γ obtained from Γ , $W_\gamma(f, g) = id$ admits formal solutions $f, g \neq id$ non commute and tangent to identity such that $(f''(0), g''(0)) = (\alpha, \beta)$, where (α, β) is a vector orthogonal to $G(\gamma)$.

Lemma 5.2. Assume a palette diagram Γ consisting of four unit weighted squares without area has the type (I). For a Feynman diagram γ obtained from Γ , assume $W_\gamma(f, g) = id$ for $f, g \neq id$ tangent to identity with $(f''(0), g''(0)) = (\alpha, \beta)$, where (α, β) is a vector orthogonal to $G(\gamma)$. Then f, g commute.

And we obtain the following Theorem 5.3 and 5.4 naturally as the application of the classification theorem form Lemma 5.1 and 5.2.

Theorem 5.3. Assume Γ equals the one of the diagrams in the lists of Theorem 3.1 ~ 3.4 with p, q, r satisfying the following conditions on polynomials for each diagram;

$$\begin{aligned} (E-1) \sim (E-6) \quad & p + q \neq 0 \quad \text{and} \quad p + r \neq 0 \quad \text{and} \quad q + r \neq 0, \\ (F-1) \quad & p + r \neq 0 \quad \text{and} \quad q(q + r) + p(q + 4r) \neq 0, \\ (F-2) \quad & q + r \neq 0 \quad \text{and} \quad 9p^2 + qr + 9p(q + r) \neq 0, \\ (F-3) \quad & p + r \neq 0 \quad \text{and} \quad q(q + r) + p(q + 4r) \neq 0, \\ (G-1) \quad & (q + 2r)^2 + p(q + 4r) \neq 0, \\ (G-2) \quad & p^2 + p(-2q + r) + q(q + r) \neq 0, \\ (G-3) \quad & (q + 2r)^2 + p(q + 4r) \neq 0, \end{aligned}$$

- $(G-4) \quad (q+2r)^2 + p(q+4r) \neq 0,$
 $(G-5) \quad 9p^2 + 4q(q+r) + 3p(4q+3r) \neq 0,$
 $(G-6) \quad p^2 + p(q-2r) + r(q+r) \neq 0,$
 $(G-7) \quad (q-r)^2 + p(q+r) \neq 0,$
 $(G-8) \quad 4p^2 + 4p(q+r) + q(q+r) \neq 0,$
 $(H-1) \sim (H-3) \quad P_2(\alpha, \beta) \neq 0.$

Then for all Feynman diagrams γ obtained from Γ , $W_\gamma(f, g) = id$ admits formal solutions $f, g \neq id$ non commute and tangent to identity such that $(f''(0), g''(0)) = (\alpha, \beta)$, where (α, β) is a vector orthogonal to $G(\gamma)$.

Theorem 5.4. Assume Γ equals the one of two in the list of Theorem 3.5. For a Feynman diagram γ obtained from Γ , assume $W_\gamma(f, g) = id$, $f, g \neq id$ are tangent to identity and $(f''(0), g''(0)) = (\alpha, \beta)$ where (α, β) is a vector orthogonal to $G(\gamma)$. Then f, g commute.

As examples of Theorem 3.3, the following examples shown in [5] are reappeared.

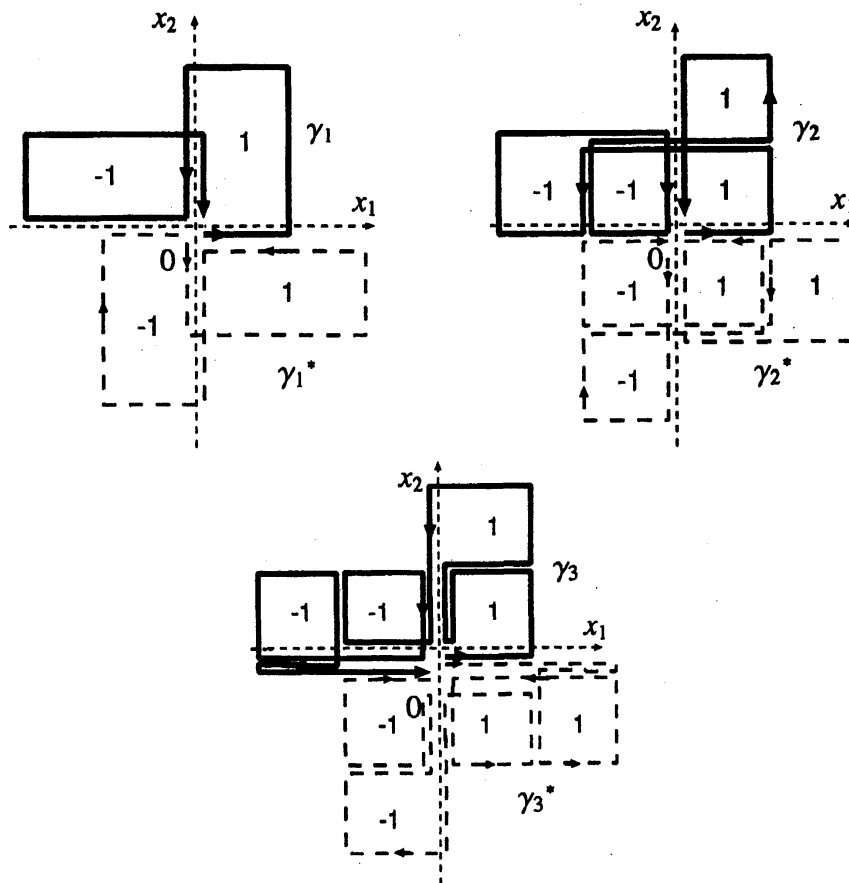


Figure 1: closed Feynman diagram and its dual diagram in $\mathbb{R}^2$

These are Feynman diagrams obtained from a palette diagram (G-3) for $p = 1, q = -1, r = -1, s = 1$. Since

$$(q + 2r, p) = (-3, 1) \neq (0, 0)$$

and

$$(q + 2r)^2 + p(q + 4r) = 9 - 5 = 4 \neq 0,$$

we see by Theorem 3.3 that the relations

$$\begin{aligned} W_{\gamma_1}(f, g) &= \{f^{(-1)}, g^{(-2)}\} \circ \{f^{(2)}, g^{(-1)}\} = \text{id}, \\ W_{\gamma_2}(f, g) &= f \circ g \circ f^{(-2)} \circ \{g, f\} \circ f \circ \{g, f\} \circ \{f^{(-1)}, g^{(-1)}\} \circ g^{(-1)} = \text{id}, \\ W_{\gamma_3}(f, g) &= \{f^{(-1)}, g^{(-1)}\} \circ g \circ \{f^{(-1)}, g^{(-1)}\} \circ g^{(-1)} \circ f^{(-1)} \\ &\quad \circ \{f^{(-1)}, g^{(-1)}\}^{(-1)} \circ f^{(-1)} \circ \{f^{(-1)}, g^{(-1)}\}^{(-1)} \circ f^{(2)} = \text{id} \end{aligned}$$

admit formal solutions $f, g \neq \text{id}$ non commute and tangent to identity such that $(f''(0), g''(0)) = (-1, 3)$, where $\{f, g\} = f^{(-1)} \circ g^{(-1)} \circ f \circ g$.

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